MATH 599 PROBLEM SET 3

DUE TUESDAY MARCH 12

1. Compute the Kretschmann scalar $K = R_{abcd}R^{abcd}$ for the Schwarzschild spacetime, as a function of the Schwarzschild r coordinate.
2. Suppose that a spaceship crosses the horizon of a Schwarzschild black hole.
   (a) Show that the amount of proper time the spaceship spends to reach the singularity $r = 0$ is bounded by a constant multiple of the Schwarzschild radius. Compute the numerical value (e.g., in seconds) of this bound for a typical black hole of 50 solar masses, and for the supermassive black hole *Sagittarius* A^* at the center of the Milky Way.
   (b) Show that in order to avoid the singularity as long as possible, the spaceship must move in a radial direction only.
   (c) What should be the strategy the spaceship captain must use in order to ensure the crew's existence for as long as possible? Should they fire the thrusters at all? If yes, in which direction?
3. Consider metrics of the form
$$ds^2 = -A^2dt^2 + B^2dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
   where $A = A(r)$ and $B = B(r)$ in some given coordinate system.
   (a) Compute its Ricci curvature with respect to the orthonormal frame
$$\omega^t = Adt, \qquad \omega^r = Bdr, \qquad \omega^\theta = rd\theta, \qquad \omega^\phi = r\sin\theta d\phi.$$
   (b) Solve the Einstein-Maxwell field equations for the aforementioned class of metrics, with the electromagnetic field satisfying the ansatz
$$E = C(r)\partial_r, \qquad B = 0,$$
   in the given coordinate system. The resulting class of solutions are called the *Reissner-Nordström* spacetimes.
   (c) Describe the maximally extended Reissner-Nordström spacetimes.
4. Prove Birkhoff's theorem for Reissner-Nordström solutions.

1. Reference material

The following might be useful to verify your computations. For metrics of the form
$$ds^2 = -A^2dt^2 + B^2d\ell^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
with $A = A(t,\ell)$, $B = B(t,\ell)$, and $r = r(t,\ell)$, the Ricci curvature is given by
$$\mathrm{Ric} = -(E+2e)\omega^t\otimes\omega^t + (E+2f)\omega^\ell\otimes\omega^\ell + (e+f+F)(\omega^\theta\otimes\omega^\theta + \omega^\phi\otimes\omega^\phi) - 2H(\omega^t\otimes\omega^\ell + \omega^\ell\otimes\omega^t)$$
where
$$\omega^t = Adt, \qquad \omega^\ell = Bd\ell, \qquad \omega^\theta = rd\theta, \qquad \omega^\phi = r\sin\theta d\phi,$$

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and

$$\begin{aligned}
E &= \frac{AB_{tt} - A_t B_t}{A^3 B} - \frac{A_{\ell\ell} B - A_\ell B_\ell}{AB^3} \\
F &= \frac{1}{r^2}\left(1 + \frac{r_t^2}{A^2} - \frac{r_\ell^2}{B^2}\right) \\
e &= \frac{r_t A_t}{rA^3} + \frac{r_\ell A_\ell}{rAB^2} - \frac{r_{tt}}{rA^2} \\
f &= \frac{r_t B_t}{rA^2 B} + \frac{r_\ell B_\ell}{rB^3} - \frac{r_{\ell\ell}}{rB^2} \\
H &= \frac{r_{t\ell}}{rAB} - \frac{r_t A_\ell}{rA^2 B} - \frac{r_\ell B_t}{rAB^2}.
\end{aligned}$$