MATH 599 PROBLEM SET 2

DUE THURSDAY FEBRUARY 14

1. By only relying on Postulate 1 (i.e., existence of inertial frames such that time is homogenous, and space is homogenous, isotropic, and Euclidean) from class, and for simplicity assuming that spacetime is 2 dimensional, show that any (sufficiently smooth) transformation between two inertial frames is of the form
$$\begin{cases} x' = \frac{x+Vt}{\sqrt{1-kV^2}} \\ t' = \frac{t+kVx}{\sqrt{1-kV^2}}, \end{cases}$$
where k is a constant, and V is the relative speed of the two frames.
2. In Newton's theory of gravitation, the *field equation* for the gravitational (scalar) potential ϕ is given by
$$\Delta\phi = 4\pi\rho,$$
where ρ is mass density, and the *equation of motion* for point particles is
$$\ddot{x} + \nabla\phi = 0.$$
Show that this theory is invariant under Galilean transformations, that is, the theory retains its form, regardless of the inertial frame we use, provided that the inertial frames are related to each other by Galilean transformations.
3. Let M be a pseudo-Riemannian manifold, and let $X, Y, Z, W \in \mathfrak{X}(M)$. Prove the following identities.
   (a) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$.
   (b) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.
4. Let local coordinates (t, r, θ, ϕ) be given on a 4-manifold M, and consider the form
$$ds^2 = -e^{2a}dt^2 + e^{2b}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
for the metric, where $a = a(r)$ and $b = b(r)$ are functions of r only. To work through this exercise, you might need to use additional resources on geodesics in an exterior Schwarzschild spacetime.
   (a) Compute the Levi-Civita connection, the Riemann and Ricci tensors for this metric.
   (b) Find all functions a and b such that the metric satisfies the Einstein field equations in vacuum. In the following we fix these forms for a and b.
   (c) Derive the geodesic equation, and show that all geodesics tangent to the plane $\theta = \frac{\pi}{2}$ stay in the same plane.
   (d) Compute the bending of light rays near the Sun. Compare your results with experimental data.
   (e) Derive a formula for the perihelion advance of Mercury. Compare your results with experimental data.

Date: Winter 2019.