FOUNDATION OF EINSTEIN'S THEORY OF GENERAL RELATIVITY FROM A QUANTUM FIELD THEORY PERSPECTIVE

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ABSTRACT. A proof of the fundamental assumption in General Relativity Theory, namely the Equivalence Principle, is given from a purely quantum field theoretic perspective. Indeed, we will observe that this principle is not an independent precept of Nature, but rather a consequence of a $SO(3,1)$-gauge-symmetry for massless spin-two-particles.

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1. Introduction

The experimental results of Eötvös and others [2] show that the inertial mass, $m_i := F/a$, for any force F and corresponding acceleration a, is equal to the gravitational mass, $m_g := F_g/g$. This experimental result is sometimes referred to as the Weak Equivalence Principle (WEP). The WEP has far reaching physical consequences: it is, really, the fundamental assumption of Einstein's General Theory of Relativity. We do though have to be careful and limit our discussion to *small enough* regions in space-time. If the sealed elevator was sufficiently big, the gravitational field would change from place to place inside the elevator, which could be measured. We can therefore restate the WEP as *the laws of freely-falling particles are the same in a gravitational field and in a uniformly accelerated frame, in a small enough region of space-time* (as a quantum interaction region!). Motivated by the equivalence of mass and energy, historically, Einstein postulated an even stronger statement. He postulated that: *at every point in an arbitrary Lorentzian manifold, it is possible to choose a locally inertial coordinate system, such that (within a sufficiently small region of that point) the laws of Nature take the same form as in unaccelerated Cartesian coordinate system.* This is known as the Strong Equivalence Principle (SEP). It implies that at every point in arbitrary strong gravitational field, the laws of Special Relativity hold locally. It is very difficult to imagine theories which respect the WEP but not the SEP. In the modern classical picture, the Equivalence Principle implies that the action of gravity should be attributed to the curvature of space-time: it implies that there is really no such thing as a globally *inertial* frame. One massive object in the Universe is enough to provide a gravitational field, and every frame that we can imagine would be accelerated in this field. There is no such thing as *gravitationally neutral object*, with respect to which we can measure the acceleration due to gravity – i.e. gravity is inescapable. Note that the equivalence of mass and energy implies that this is true for massless particles as well.

In this work, we assume the *Schiff's Conjecture* – see [1] for details – to hold. This conjecture simply suggests that the WEP implies the SEP. Alternatively, this is saying that WEP implies that it is impossible to disentangle the effects of a gravitational field from those of being in a uniformly accelerating frame. For this reason, along the next sections, by *Equivalence Principle*, we aim to mean *Weak Equivalence Principle.*

Yet after a century of close scrutiny, the Equivalence Principle has still remained only a postulate of General Relativity. It cannot be proven from more fundamental principles only in a General Relativity set up. Some of the better literature on General Relativity have drawn attention to this fact, and admit that no explanation can be found as to *why our Universe has a deep and mysterious connection between acceleration and gravity* [4]. One must bear in mind that mass is really nothing more than a vast collection of quantum particles, which interact with each other through forces. Forces are also ultimately the result of quantum particles called bosons, which act like the exchange particles that transmit momentum from one particle to another. Therefore, it is essential for the Equivalence Principle to be understood at a quantum particle level in order for a deeper understanding to emerge.

In the following, we will demonstrate that the Equivalence Principle is basically just an approximation of graviton coupling with arbitrary spin-mass particles in a very soft limit. This idea was first suggested by Weinberg in [9] and [7].

2. Quantum Field Theory: The Spirit and the Tools Needed, Nothing More

In the next sections, the notions of helicity, on-shell conditions and gauge will be discussed. For the love of completeness, a brief introduction for each notion will be given here. To start, let's ask what Quantum Field Theory, our playground, really is. That's *a priori* a pretty deep question that would take us a long time to fully answer rigorously. Withal, roughly, Quantum Field Theory treats particles as excited states – quanta of energy– of their underlying fields (they are morally just *functions* depending on space-time), which are, in a sense, much more fundamental objects than the basic particles. Interactions between particles are described by interaction terms a Lagrangian involving their corresponding fields. Each interaction can be visually represented by Feynman diagrams, which are *formal* computational tools, in the process of relativistic perturbation theory. This is, at least, the spirit of Quantum Field Theory.

First, we introduce the notion of *helicity*. Helicity is used in Quantum Mechanics to express the connection between the direction in which a particle rotates around some axis through that particle – expressed by the spin quantum number – and the direction of propagation of a particle. In Quantum Mechanics, one can measure the spin of some particle by using the spin quantum operator $\hat{S}$ and the momentum being expressed by some vector $\mathbf{p}$. Now, to measure the helicity of some particle, just apply a new operator (called the helicity-operator) onto the particle's wavefunction – namely $\hat{h} := \mathbf{p} \cdot \hat{S}/|\mathbf{p}|$, measuring the components of the spin operator along the direction of momentum. The operator $\hat{h}$ is the scalar product of a vector with an operator. For example, supposing that $\mathbf{p}$ is along the z-axis, then the helicity operator is nothing else then the length of vector $\mathbf{p}$ multiplied with the z-component of the spin operator $\hat{S}$. In other words, this measures the spin along the z-axis. Positive helicity means that the rotation-axis of the spin is in the same direction as the direction in which the particle moves. If helicity is negative, it is the other way around. This discussion is carried in the exact same manner in Quantum Field Theory.

Second, we discuss on-shell conditions. In Special Relativity, energy and momentum are combined into a single entity — the momentum 4-vector. Changing a reference frame will mix up the different components of the vector and one should think of these 4 numbers as one geometric entity. For a particle, these 4 numbers are not independent quantities and satisfy a constraint: $-p^{\mu}p_{\mu} = E^2 - \mathbf{p}^2 = m^2$. These 4 numbers live on a 2-manifold – a hyperboloïd. This manifold is called a shell for evident graph-looking reasons. Since the shape of the shell is dictated by the mass, so this surface is called the *mass-shell*. In Quantum Field Theory, one formulation of theories is with the path integral [6],[5]. The path integral says to sum over all possible paths between the beginning and the end of a process. It turns out, the correct formulation includes including configurations which do not satisfy the mass-shell constraint. When speaking about the intermediate particles in the process, the particles that don't satisfy the mass-shell constraints are said to be off their mass-shell – or simply off-shell.

Third, we describe gauge. To do so, we first define properly what is a *gauge symmetry*. It is simply a symmetry transformation of the action that depends nontrivially on the space-time. It is possible to ask for all physical theories whether such transformations exist or not. For the case of particle mechanics of finitely many degrees of freedom (where the gauge symmetry then only depends on time), it is known that the existence of a gauge symmetry is equivalent to a *constraint* in the Hamiltonian formulation. Such constraints arise from the Legendre transformation of the Lagrangian being noninvertible, therefore the condition

$$\det\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}\partial \dot{q}}\right)=0,$$

with $\mathcal{L}$ being the Lagrangian and $q, \dot{q}$ the generalized coordinates detects the presence of gauge symmetries.

Fourth, let's now see what we mean by *gauge theory*. A gauge theory is any field theory in physics in which some global, continuous symmetry of the theory is promoted to a local symmetry. By doing so, a new field is introduced, namely, the gauge field, which has its own dynamics and couples to the fields (and so to the particles) which have the symmetry. Those particles are then said to be charged under the gauge field. Hence, when physicists are talking about a gauge theory, they just mean a theory with a *gauge field*, such as Yang-Mills theory [5]. The controversy over whether General Relativity is a gauge theory or not stems from the fact that the gauge field of General Relativity "are" the connection coefficients, which are considered non-dynamical fields derived from the metric (unless you are using Palatini's formalism). Maxwellian electrodynamics is the archetypal example of a Yang-Mills gauge theory with gauge group $U(1)$ – it is in fact some modified version of this gauge theory that will be our starting point in the next section. See [3] for further details on gauge symmetries and gauge theories.

Fifth, one essential property of the graviton we shall use below is the value of its spin: gravitons are spin-two-particles. The reason being that gravitation is described by a metric – a symmetric (0,2)-tensor field – modulo a general covariance, which gives locally, in the tangent Minkowski space of any point, a spin–two–representation of the Poincaré group modulo longitudinal directions, which, practically, forces mass zero and helicity two. Gravitational waves also have to be (classically) long range, which again requires (after quantization) massless particles. Thus gravitons (although never observed) should be massless spin-two-particles. Actually, Weinberg proved this results in [8]. Moreover, he proved in the same paper that *canonical minimal self-coupling of a massless spin two field leads classically to Einstein's equations for general relativity.* In other words, that the existence of massless spin-two-particle is the necessary and sufficient property a particle physics theory needs to allow in order for (classical) General Relativity to arise naturally.

Finally, as mentioned above, this discussion will rely on Feynamn's diagrammatic method. The reader must understand that the only way to appreciate fully the mathematical nature of Feynman diagrams, is by taking a course on Quantum Field Theory. In particular, for the gauge theories we are about to consider, the Faddeev-Popov procedure needs, at least, to be introduced [6],[5]. This is usually done in the second course on Quantum Field Theory. Consequently, for the time being, I will take what we call the *Feynman rules for gauge theories* in Quantum Field Theory to be our set of "axioms", even though they are not really since they usually strongly rely on the form of the theory under consideration. That is to say, some, if not a lot, of details from Quantum Field Theory will voluntarily be put under the rug. The first example we will encounter below is a modified version of the theory of Quantum Electrodynamics (QED) – call it MQED for *modified* QED – which will serve as a warm-up for us before looking at its gravitational analogue. The theory is modified in the sense that the interaction with the gauge field will either be from a *bosonic field*, a *fermionic field* or both. The "axioms" of MQED are given in 1.

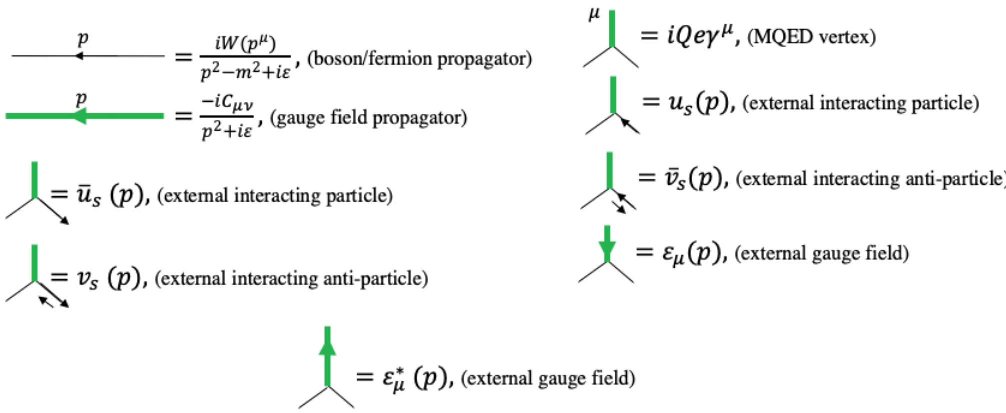

Figure 1. The set of axioms for MQED.

where m is the mass of the interacting particle, Qe is the charge of the boson/fermion, $C_{\mu\nu}$ is some tensor for which the form will not be relevant, $u_s, \overline{u}_s, v_s$ and $\overline{v}_s$ are vectors with four entries with helicity index s and the function W will be associated latter in this work via some function f. The γ^μ are the generators of the Clifford's algebra (usually seen when one studies the Dirac's Relativistic Theory of Electrons), but it will not play any explicit role in the analysis. The diagrams under consideration below will always involve terms like "$u_s\overline{u}_s \times$ (propagators and other stuff)" since there's always an incoming and outgoing particle; in particular, $u_s\overline{u}_s$ is a scalar so we will not include their contributions explicitly in our computations below: they will be hidden in the definition of f. Furthermore, since the computations we will perform will not involve any loops we can safely set $\varepsilon \to 0$ (we are not dealing with integration around propagator poles).

3. Photon's Coupling: A Warm–Up

We shall investigate the consequences of a $SO(3,1)$-gauge-symmetry for the interactions of a photon with other particles (bosons and/or fermions). In particular, we will show that in the soft limit – i.e. at low photon energy – a photon can only couple to charges that are conserved by *all* scattering processes.

3.1. **The Building Block Diagram.**

The construction of this warm-up goes as follows: first, we consider the emission of a very soft photon of four-momentum q^{μ}[1] by other particles (massive or not and with spin or not) as depicted in the interaction diagram below

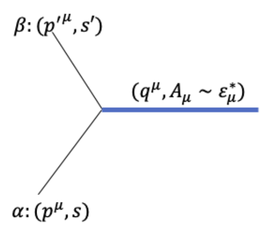

Figure 2. Diagrammatic figure of the photon (blue) emission from the incoming particle of matter α with momentum p^{μ} and spin s. The emergent particle β has momentum p'^{μ} and spin s'. The photon's momentum q^{μ} is soft and its gauge field A_{μ} is associated with polarization four-vector $\varepsilon^{\star}_{\mu}$ depending intrinsically on q.

Without loss of generality, we will assume that the photon helicity is $+1$. Otherwise, the same analysis will apply for helicity of -1. Finally, as shown in 2, the initial and final states of the emitting particle are labelled by the kinematical-spin variables (p^{μ}, s) and $(p^{\mu} - q^{\mu}, s')$, respectively, by conservation of momentum.

Remark 3.1. The latter process cannot happen for finite nonzero q^{μ} if *all* particles are on-shell. Indeed, because $q^{\mu}q_{\mu} = -m_{\gamma}^2 = 0$, this implies $p'^{\mu}p'_{\mu} = -m^2 - 2p^{\mu}q_{\mu} \neq -m^2$ implying that the resulting particle is *not* on-shell in general, which contradicts our assumption. Hence, the time transnational symmetry of our space-time and the on-shell conditions make this process nonphysical by its own – i.e. it can only be a part of the diagrams of the form of 3 (a) or (b),

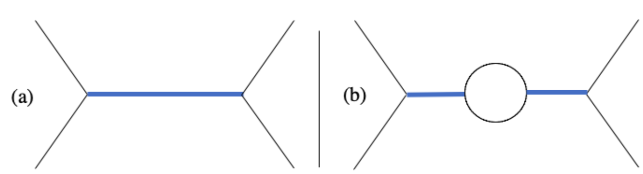

Figure 3. Examples of physical diagrams.

[1] By very soft we mean that the emitted photon has momentum satisfying $q^{\mu} \to 0$, such that if, in any latter computations, we expand in terms of its four-momenta we only keep the term at linear order.

depending of what kind of theory we are sitting in. Therefore, diagram 2, really, is just a building block of our proof and should not be considered as an actual process one could observe in Nature. ⊞

Even though it is not physical, the fact that this process is central in our proof demands us to look at the amplitude for it to happen[2]. In the very soft limit, by conservation of momentum, the initial and final momenta coincide such that we have

$$\mathcal{M}^\mu = \mathcal{M}^\mu(p, p', s, s') = \mathcal{M}^\mu(p, s, s').$$

From here, there are two distinct cases to take into account:

Case 1: If the incoming particle is spinless (e.g. a scalar particle), the amplitude is only a function of momentum

$$\mathcal{M}^\mu = \mathcal{M}^\mu(p^\mu).$$

Hence, for some function $F : \mathbb{R}^{3,1} \to \mathbb{C}$ such that $\xi^\mu \mapsto F(\xi^\mu)$ is a Lorentz scalar, we have

$$\mathcal{M}^\mu = F(p^\mu)p^\mu.$$

If the initial particle is on-shell, then $p^2 = -m^2$, where m is the mass of the scattered particle. Thus, we might define that $F(p^\mu) = g(-m^2)$, such that $g : \mathbb{R} \to \mathbb{R}$ – i.e. the target of this function is a number independent of the kinematics. In a physics mindset, one could simply think of $F(p^\mu)$ as the *charge of the particle* under consideration, modulo some constant. Hence, based on the MQED axioms 1, the amplitude is given by

$$\mathcal{M}^{(0,0,1)}_{\text{tree}} = g(-m^2)p^\mu \varepsilon^\star_\mu(q^\mu),$$

in the soft limit. Note that the label $(0, 0, 1)$ identifies, respectively, that the emitting particle has spin zero, as well as the emerging particle and that the emitted boson has spin one. For completeness, one can show, deriving the Feynman rules carefully from scratch, that $g(-m^2) = 2i(2\pi)^4 e$. Nonetheless, this is not important here – i.e. one could keep $g(-m^2)$ in the expression without changing the conclusion we'll get.

Case 2: In the case where the incoming (emitting) particle has a spin different from zero, the amplitude $\mathcal{M}^\mu$ might be more complicated. *A priori*, it is obscure whether or not the four-vector $\mathcal{M}^\mu$ is just in the p^μ direction. In other words, if new kinds of four-vectors can be constructed using the nonzero spin variables. However, based on the following postulate (that we will fully justify in the next section) this turns out to not be the case.

Postulate 3.2. Even for particles with spin, the amplitude for the emission of a very soft photon does not depend on the original particle's spin s and on the emergent particle's spin s'. Furthermore, the process has nonvanishing amplitude only if the final spin state s' is the same as the initial spin state – i.e. $s = s'$. ⊞

In particular, this postulate implies that the form of the amplitude for the elementary process described above takes the form

$$\mathcal{M}^{(s,s',1)}_{\text{tree}} = 2i(2\pi)^4 e p^\mu \varepsilon^\star_\mu(q^\mu)\delta_{ss'}. \tag{3.1}$$

[2]The word "happen" makes sense at any energy scale only in view of Optical Theorem. See [6] for a very good discussion on the subject.

3.2. **The Generic Scattering Process.** The two cases we just treated are the essence of the analysis for the generic scattering process $\Lambda \to \Lambda'$, 4 (a).

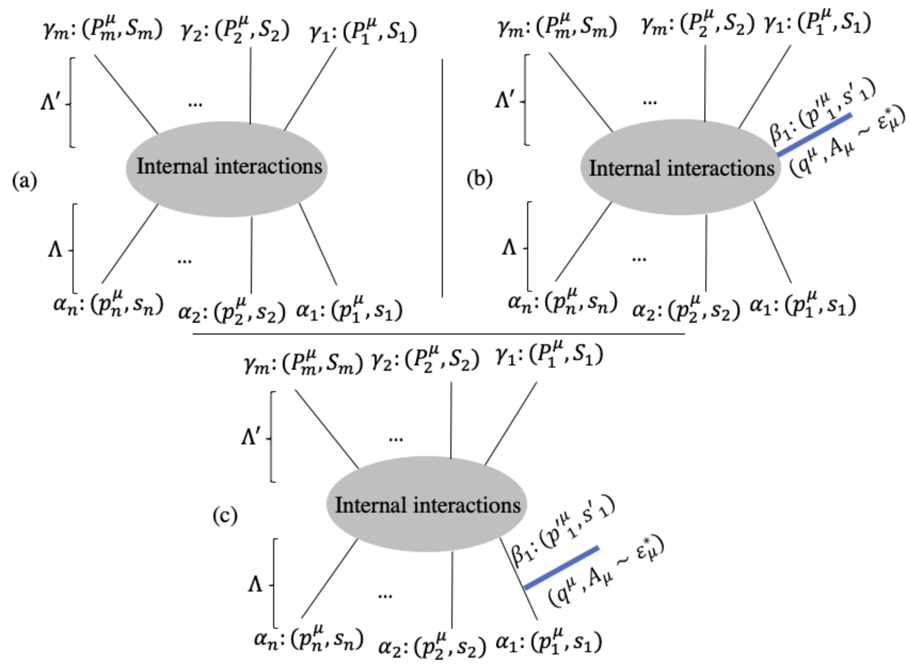

FIGURE 4. (a) The general scattering process under consideration. (b) The same process emitting one soft photon. (c) The same process with one leg emitting a soft photon. Note that $m \neq n$ in general.

Remark 3.3. The blobs present in 4 (a), (b) and (c) represent the *full* scattering amplitude – i.e. the sum of all Feynman diagrams with initial state Λ and final state Λ'. $\boxplus$

In what follows, we shall adopt the notation $\mathcal{M}_{\Lambda\Lambda'}$ in order identify the scattering amplitude depicted in 4 (a). To ensure the generality of the construction, the diagram corresponding to the emission of a very soft photon with momentum q^μ will be the one illustrated in 4 (b). To compute the amplitude related to this diagram, Feynman's perturbative method encourages us to consider all the diagrams that makes up the original diagram with amplitude $\mathcal{M}_{\Lambda\Lambda'}$ and attach the photon line to *all* possible particle lines, external as well as internal.

For reasons that will become clear soon, we will first focus on the diagrammatic contributions for which the photon line is attached to an external line 4 (c). It is easy to write down the corresponding amplitude, $\mathcal{M}$, for this process; we only need to multiply the amplitude for the elementary process $\mathcal{M}^{(s,s',1)}_{\text{tree}}$ by the propagator connecting it to the rest of the $\Lambda \to \Lambda'$ process, which itself contributes to the amplitude by an overall factor of $\mathcal{M}_{\Lambda\Lambda'}$ –

i.e. using on-shell conditions and the fact the photon can be taken to be a massless particle

$$\begin{aligned}\mathcal{M}_1 &= \mathcal{M}_{\Lambda\Lambda'}\frac{-i}{(2\pi)^4}\frac{1}{(p_1-q)^2-m_1^2}\mathcal{M}_{\text{tree}}^{(s_1,s_1',1)}(e_1,p_1^\mu)\\ &= \mathcal{M}_{\Lambda\Lambda'}\frac{2e_1p_1^\mu\varepsilon_\mu^\star(q^\mu)}{(p_1-q)^2-m_1^2}\delta_{s_1s_1'}\\ &= -\mathcal{M}_{\Lambda\Lambda'}\frac{e_1p_1^\mu\varepsilon_\mu^\star(q^\mu)}{p_1\cdot q}\delta_{s_1s_1'},\end{aligned}$$

where m_1 and e_1 are the emitting particle's mass and charge respectively and where $p_1\cdot q$ means that we are taking the *Minkowskian*-dot-product of the two four-vectors – i.e. $p_1\cdot q := p_1^\mu q_\mu$, unless specified otherwise. One must note that we get a similar contribution if the photon is initially attached to a different external leg. Indeed, if we attach it to any the i^{th}-incoming external one, we just make the identification $1\to i$ [3] to pass from $\mathcal{M}_1$ to $\mathcal{M}_i$, while, if we attach the photon to an outgoing external leg, we still just make a similar identification up to the minor modification $p_1^\mu \to -p_j'^\mu$ [4]. We pick an overall minus sign because initial momentum for the outgoing particle (the one appearing in the propagator) is $p_j'^\mu + q^\mu$ and not $p_i^\mu - q^\mu$ as it was the case for the incoming particle. This invites us to define the following function of k

$$\eta_k := \begin{cases} +1 & \text{if photon couples with the } k^{\text{th}} \text{ outgoing particle,}\\ -1 & \text{if photon couples with the } k^{\text{th}} \text{ incoming particle,}\end{cases}$$

enabling us to to write down an abstract expression for the *full* amplitude of the $\Lambda\to\Lambda'$ process, namely

$$\mathcal{M} = \mathcal{M}_{\Lambda\Lambda'}\sum_k \frac{\eta_k e_k p_k^\mu}{p_k\cdot q}\varepsilon_\mu^\star(q^\mu)\delta_{s_ks_k'} + (\text{internal photon coupling contributions}).$$

The latter contributions from the internal pieces are not singular in the soft-limit $q^\mu\to 0$. Indeed, since virtual particles (particles arising from internal interactions) are by definition off-shell (they do not need to satisfy any specific equations of motion); the propagator is not of the form $p_k\cdot q$ and, consequently, does not diverge as it did for the external couplings. This means that dominant terms are those coming from external propagators and not from the internal propagators. Hence, in the soft-limit, we will only consider the singular part coming from the on-shell propagators (external couplings) and ignore the finite part coming from the off-shell propagators. This is not an approximation, if one wonders. At the tree-level, the coupling constant at linear order multiplies the finite internal contributions. However, since we are using perturbative methods, this constant must go to zero, making this finite part (formally) vanish. Finally, we write

$$\mathcal{M} = \mathcal{M}_{\Lambda\Lambda'}\sum_k \frac{\eta_k e_k p_k^\mu}{p_k\cdot q}\varepsilon_\mu^\star(q^\mu)\delta_{s_ks_k'}, \quad (\text{as } \ q^\mu\to 0). \tag{3.2}$$

Remark 3.4. We will discuss at the end of this section why we didn't take $\mathcal{M}_{\Lambda\Lambda'}$ to depend on k. This is not totally obvious and deserves its own section. ⊞

[3] Where i runs from 1 to n, where n is the number of incoming particles.

[4] Where j runs from 1 to m, where m is the number of outgoing particles.

3.3. **Gauge Invariance.** We appeal to the $SO(3,1)$-gauge-symmetry in order to make use of the redundancy of the emitted photon with its polarization four-vector; indeed, under (the Lorentz invariant) Lorenz gauge, $\varepsilon_\mu(q)$ and $\varepsilon_\mu(q) + \ell q_\mu$, for any $\ell \in \mathbb{C}$,[5] describe the same photon state in the sense that they share the same amplitude. Consequently, the fact that

$$\mathcal{M}^\mu \varepsilon^\star_\mu(q^\mu) = \mathcal{M}^\mu(\varepsilon^\star_\mu(q^\mu) + \ell q_\mu),$$

implies the gauge fixing

$$\mathcal{M}^\mu q_\mu = 0.$$

In particular, it follows from this that

$$\begin{aligned} 0 = \mathcal{M}^\mu q_\mu &= \mathcal{M}_{\Lambda\Lambda'} \sum_k \frac{\eta_k e_k}{p_k \cdot q} p_k^\mu q_\mu \\ &= \mathcal{M}_{\Lambda\Lambda'} \sum_k \eta_k e_k. \end{aligned}$$

Since $\mathbb{R}$ is a field, we either conclude that $\mathcal{M}_{\Lambda\Lambda'} = 0$ or $\sum_k \eta_k e_k = 0$. It is easy to see that the first option is absurd; indeed, if $\mathcal{M}_{\Lambda\Lambda'} = 0$, no scattering process is happening and so this whole discussion has no purpose. Hence, the latter option must be the good one. That is to say

$$\sum_{\text{Incoming}} e_k = \sum_{\text{Outgoing}} e_k.$$

In more informal language, this is saying that the charge is conserved trough processes emitting soft-photon, establishing what we wanted to show.

3.4. **Some Justifications on the Assumptions Made: Part I.** As a closure for this section, we will motivate Postulate 3.2. To do so, we look at the most general form the building block diagram amplitude can take, namely

$$\mathcal{M}^{(s,s',1)}_{\text{tree}} = \mathcal{M}^\mu_{\text{tree}}(p, s, s') \varepsilon^\star_\mu(q^\mu).$$

We shall now go through the same computation we did above for the amplitude of $\Lambda \to \Lambda' + \gamma$ process and analyse the changes that would occur if we were to use the previous fully general vertex amplitude. Considering this general vertex, we have essentially two new things to worry about: (i) the vertex amplitude is obviously different, (ii) *a priori*, the emergent particle from the vertex carries an arbitrary spin state s'_i or s'_j over which we now have to sum. For simplicity, let's first suppose that the photon couples with the particle (p_1^μ, s_1, m_1). The amplitude related to this $\Lambda \to \Lambda' + \gamma$ process is simply

$$\mathcal{M}_{\Lambda \to \Lambda' + \gamma} = \sum_{s'_1} \mathcal{M}_{\Lambda\Lambda'}(s'_1, \ldots) \frac{\mathcal{M}^\mu_{\text{tree}}(p_1, s_1, s'_1)}{-2 p_1 \cdot q} \varepsilon^\star_\mu(q^\mu),$$

such that, for the same reason as we already encountered, the amplitude for the whole process – the sum of all possible Feynman diagrams modulo the internal couplings – is

$$\mathcal{M} = \sum_k \sum_{s'_k} \mathcal{M}_{\Lambda\Lambda'}(\ldots, s'_k, \ldots) \frac{\eta_k \mathcal{M}^\mu_{\text{tree}}(p_k, s_k, s'_k)}{-2 p_k \cdot q} \varepsilon^\star_\mu(q^\mu).$$

[5]Under the Lorenz gauge, the gauge field transformation $A_\mu \to A_\mu + \partial_\mu \Theta$ reduces to the harmonic equation $\partial^2 \Theta = 0$. In the soft limit, for any $\ell \in \mathbb{C}$, $\Theta = \ell \exp(iq \cdot x)$ is a solution. In terms of the polarization $\varepsilon^\star_\mu = A_\mu \exp(-iq \cdot x)$, this yields to $\varepsilon^\star_\mu \to \varepsilon^\star_\mu + \ell q_\mu$.

As above, imposing the $SO(3.1)$-gauge-symmetry yields to

$$\sum_k \sum_{s'_k} \mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...) \frac{\eta_k \mathcal{M}^\mu_{\text{tree}}(p_k, s_k, s'_k)}{-2p_k \cdot q} q_\mu = 0.$$

We notice that the latter expression needs to vanish for all light-like four-momenta q^μ, q^μ being the photon four-momentum. But by the gauge-redundancy, it must also vanish for an arbitrary four-vector q^μ – not necessarily just light-like. This is only possible if the direction of each individual $\mathcal{M}^\mu_{\text{tree}}(p_k, s_k, s'_k)$ is along p^μ_k. Indeed, for any given external particle labelled by k, we can choose (go in a frame where) the four-vector q^μ to be orthogonal to p^μ_k, but *not necessarily* orthogonal to the other external four-momenta. Consequently, the k^{th} term in the sum is divergent unless

$$\mathcal{M}^\mu_{\text{tree}}(p_k, s_k, s'_k) q_\mu = 0 \Leftrightarrow q_\mu p^\mu_k = 0.$$

In this manner, the sum argument is finite for each k and the sums can eventually converge to zero. If we write the tree-level amplitude in terms of the charge function $e_k := e_k(p_k, s_k, s'_k)$, we find

$$\sum_k \sum_{s'_k} \mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...) \eta_k e_k(p_k, s_k, s'_k) = 0, \tag{3.3}$$

which holds for all processes $\Lambda \to \Lambda' + \text{soft}\text{–}\gamma$. Now, the new thing with this equation, as opposed to the equation for the amplitude in the last section, is that, together with the constraint on the way the photon couples to other particles, we are also constraining the amplitude for the original process $\Lambda \to \Lambda'$ since it depends on s'_k. In particular, this constraint is such that it depends on the way the photon couples to other particles – i.e. on the charge functions. Although, at least at lowest order in perturbation theory, causality imposes that the *original* process $\Lambda \to \Lambda'$ knows nothing about the photon and its couplings and the only constraint it satisfies is four-momentum conservation. Thus, there is no way of satisfying (3.3) unless $e_k(p_k, s_k, s'_k) = e_k(p_k)\delta_{s_k, s'_k}$ [6]. This allows us to erase the sum over the spin

$$\sum_k \mathcal{M}_{\Lambda\Lambda'}(..., s_k, ...) \eta_k e_k(p_k, s_k) = \mathcal{M}_{\Lambda\Lambda'} \sum_k \eta_k e_k(p_k, s_k) = 0. \tag{3.4}$$

This simply means that for any process with non-vanishing amplitude, the momentum and the spin is conserved by the charge after soft-emission. There is one final trick we can use to close this discussion. Indeed, if the process $\Lambda \to \Lambda'$ has a non-vanishing amplitude – as it needs to –, the same process where we change, say the value of the spin s_1 to S_1, and let everything else unchanged in (3.4), we pass from

$$- e_1(p_1, s_1) + \sum_{k>1} \eta_k e_k(p_k, s_k) = 0, \tag{3.5}$$

to

$$- e_1(p_1, S_1) + \sum_{k>1} \eta_k e_k(p_k, s_k) = 0. \tag{3.6}$$

In particular, we can set (3.5) and (3.6) equal to each other and we immediately see that

$$e_1(p_1, s_1) = e_1(p_1, S_1).$$

[6] This could also have been seen by carefully listing all the possible (p^μ, s, s')-variables dependence for the amplitude in this limit and see that the only nonvanishing possibility is $\mathcal{M} \sim \mathbf{s} \cdot \mathbf{s}' \sim \delta_{ss'}$.

However, there's nothing particular about $i = 1$. That is, we see that the more general statement

$$e_i(p_i, s_i) = e_i(p_i, S_i),$$

is equally true. This is nothing but the manifestation of the fact that the charge of the emitting particle cannot depend on its initial and final spin-states. Since the charge is by definition a scalar quantity, it has no explicit dependence on the *four-vector* momentum and we can write

$$e_k(p_k, s_k) \equiv e_k.$$

This completes the motivation of the postulate and, thereupon, the proof.

4. Graviton's Coupling: A Proof of the Equivalence Principle

In this section, with minor modifications to the previous analysis, we will apply the same reasoning to the soft emission of a graviton, the gravity boson. The main difference is that the constraint will be, somehow, much stronger. The set of Feynman rules (our "axioms") for gravity follows the form of the one discussed for MQED, but where one-index structures are promoted to two indices structures. This generalization pursues from the development of Linearized General Relativity (see chapter 18 of [4] for instance): in particular, $\mathcal{M}^{\mu} \to \mathcal{M}^{\mu\nu}$, $A_{\mu} \to h_{\mu\nu}$ and $\varepsilon^{\star}_{\mu} \to \varepsilon^{\star}_{\mu\nu}$. The discussion we are about to start will treat the amplitudes abstractly, so we shall not care about anything more than its gravitational coupling dependency and its tensorial structure.

4.1. **The Principle of Equivalence.** Let α denotes the incoming particle flavor and let β denotes the outgoing particle flavour. We will consider the emission of a very soft graviton[7], as depicted in 5.

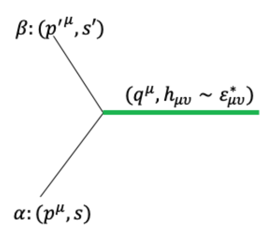

Figure 5. Analogue of 2 but where the emitted particle is now a graviton (green).

Same remarks than in the last section apply to this diagram. In addition, as argued above, since the amplitude is by definition a Lorentz scalar and since the graviton is a spin-two-particle, we have

$$\mathcal{M} = \mathcal{M}^{\mu\nu}\varepsilon^{\star}_{\mu\nu}(q^{\mu}),$$

where $\varepsilon^{\star}_{\mu\nu}(q^{\mu})$ is the polarization tensor for the outgoing graviton and the tensor amplitude $\mathcal{M}^{\mu\nu}$ depends on the kinematical variables of the other particles interacting, namely variables in the set $\{p^{\mu}, s, p^{\mu} - q^{\mu}, s'\}$. In the very soft limit $q^{\mu} \to 0$, the dependence reduces to

$$\mathcal{M}^{\mu\nu} = \mathcal{M}^{\mu\nu}(p, s, s').$$

Now, as in the photon case, if the emitting particle is *spinless*, then $\mathcal{M}^{\mu\nu}$ can only depend on symmetric combinations of the four-momentum p^{μ} matching the indices of the amplitude tensor. The only such possibility is clearly

$$\begin{aligned}\mathcal{M}^{\mu\nu} &= p^{\mu}p^{\nu}F(p^{\rho}) \\ &= p^{\mu}p^{\nu}g(-m^2),\end{aligned}$$

where in the last line we used the on-shell condition at the vertex. Again, $F(p^{\rho})$ is a constant which only depends on the particle species, but not on its momentum-spin state. For latter

[7] Note that this interaction is allowed at lowest order expansion of the Einstein-Hilbert action. See the example coming next page.

convenience, we relabel $F(p^\rho) = 2i(2\pi)^4 f(-m^2)$. Hence, in the soft limit, if the emitting particle is spinless, we have

$$\mathcal{M}^{(0,0,2)}_{\text{tree}} = 2i(2\pi)^4 f(-m^2) p^\mu p^\nu \varepsilon^\star_{\mu\nu}(q^\mu). \tag{4.1}$$

Remark 4.1. We can work out a particular example showing that (4.1) is indeed the form one should expect for graviton emission. For the love of simplicity, this example treats a scalar field ϕ with spin $s = 0$ and mass m coupled with gravity. The action for gravity and *minimally* coupled scalar ϕ in General Relativity is

$$\begin{aligned} S &= S_{\text{gravity}} + S_{\text{matter}} \\ &= \frac{1}{16\pi G}\int \mathrm{d}^4\xi \sqrt{-g} g_{\mu\nu} R^{\mu\nu} + \int \mathrm{d}^4\xi \sqrt{-g}\left(-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \end{aligned}$$

One gets nontrivial contributions to the amplitude of 5 only from interactions that involve two ϕ-fields (the incoming and the outgoing ones) and one $h_{\mu\nu}$ (to create the outgoing graviton). These only can come from the matter action, which in a soft/weak–field–limit is expanded to the linear power in $h_{\mu\nu}$,

$$\begin{aligned} S_{\text{matter}}[\phi, g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}] &= S_{\text{matter}}[\phi, \eta_{\mu\nu}] + \int \mathrm{d}^4\xi \left.\frac{\delta S_\phi}{\delta g_{\mu\nu}(\xi)}\right|_{\eta_{\mu\nu}} h_{\mu\nu}(\xi) + \mathcal{O}((\eta_{\mu\nu}h^{\mu\nu})^2) \\ &\equiv S^{\text{flat}}_{\text{matter}} + \frac{1}{2}\int \mathrm{d}^4\xi h_{\mu\nu} T^{\mu\nu}_{\text{flat}} + \mathcal{O}(h^2), \end{aligned}$$

where the stress-energy tensor is easily seen to be given, in the flat background, by

$$T^{\mu\nu}_{\text{flat}} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}((\partial\phi)^2 + m^2\phi^2).$$

In the limit where $q^\mu \to 0$, we know that this four-quantity won't appear in the expression of the amplitude. Furthermore, the derivatives hitting the scalar field will take down a four-momentum $\pm ip^\beta$ from the ϕ-plane-wave expansion. Moreover, since the stress-energy tensor is symmetric, for some constant θ, the amplitude for 5 reads as

$$\mathcal{M} = i(2\pi)^4 \frac{1}{2}[2(ip^{(\mu})(-ip^{\nu)})]\varepsilon^\star_{\mu\nu}(q^\mu) + \theta\eta^{\mu\nu}\varepsilon^\star_{\mu\nu}(q^\mu).$$

However, we know from Linearized General Relativity that $h_{\mu\nu}$ is traceless and so is the polarization. The last term then vanishes. As desired, we recover the form of (4.1) with gravitational coupling $f(-m^2) = 1/2$. ⊞

For the case where the spin of the emitting particle is *not* zero, the amplitude tensor can depend on the spin variables s and s' and its tensorial structure needs not just to rely on $p^\mu p^\nu$, but perhaps on some kind of spin tensor $\Sigma^{\mu\nu}$ too. Again, like in the photon case, we will *postulate* that, in the soft limit, $\mathcal{M}^{\mu\nu}$ has no spin dependence and that this amplitude tensor is nonvanishing only if the initial spin is the same as the final one. In this respect, we have

$$\mathcal{M}^{(s,s',2)}_{\text{tree}} = 2i(2\pi)^4 f(p^\mu) p^\mu p^\nu \varepsilon^\star_{\mu\nu}(q^\mu)\delta_{ss'}, \quad q^\mu \to 0. \tag{4.2}$$

Obviously, we will discuss the validity of these assumptions (postulate) in the closure of this section.

Now, we will follow precisely the same steps than in the last section: we first consider a generic scattering process $\Lambda \to \Lambda'$, as depicted in 6 (a)

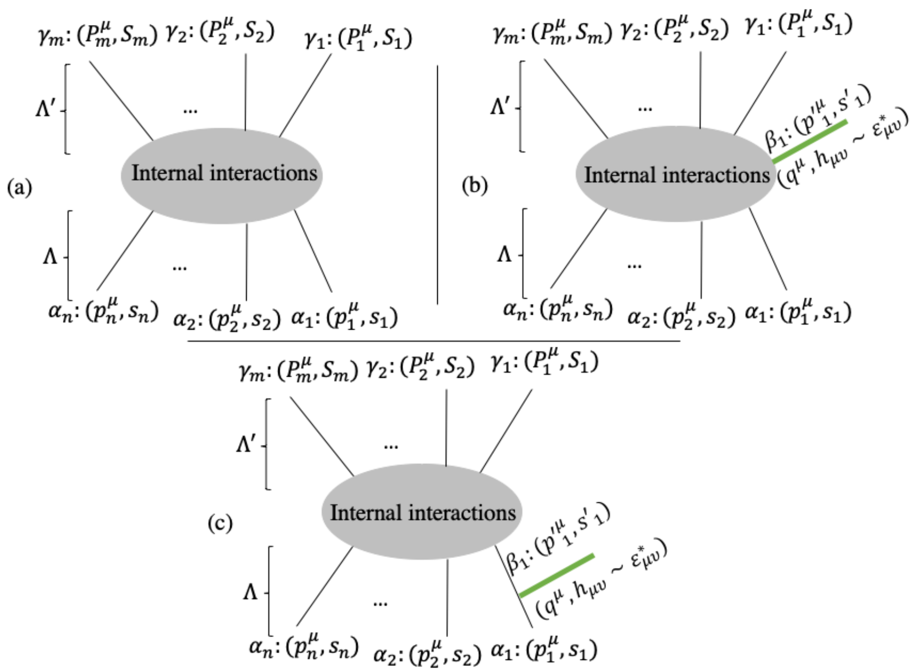

FIGURE 6. Analogues of 4 but where the emitted particle is a graviton.

and the same process where a soft graviton is emitted 6 (b). The amplitude for the latter process is the sum over all diagrams that make up the former where we also attached a graviton line in all possible places. The only ingredient which differs from the photon case is our new vertex 5, for which the contribution is mathematically given by (4.2). In particular, all the propagators are the same as for the last section and for $q^\mu \to 0$ we can, obviously, still ignore diagrams where we attach the graviton to an internal particle line. Consequently, in a similar fashion, we just need to compute 6 (c). The sum still runs over all external legs and the function η_k is defined exactly in the same way as before. Moreover, the $f_k(-m_k^2)$'s are the individual gravitational coupling constants which may, in principle, depend on the species of the particles interacting.

The same $SO(3,1)$-gauge-symmetry than in the previous section, it still manifests itself from the gauge transformation[8]

$$\varepsilon^\star_{\mu\nu} \to \varepsilon^\star_{\mu\nu} + 2q_{(\mu}\Theta_{\nu)},$$

for every constant four-vector Θ_ν. Since this transformation is one from a state to itself we must have

$$\mathcal{M}_{\Lambda\Lambda'} \sum_k \eta_k f_k(-m_k^2) p_k^\nu \Theta_\nu = 0, \ \forall\ \Theta_\nu \ \Rightarrow \mathcal{M}_{\Lambda\Lambda'} \sum_k \eta_k f_k(-m_k^2) p_k^\nu = 0,$$

[8]Under the Lorenz gauge $\partial^\mu h_{\mu\nu} = 0$, the Green's function method applied to the linearized Einstein's equations yields to the harmonic equation $\partial^2 h_{\mu\nu} = 0$, which admits plane-wave solution $h_{\mu\nu} = \varepsilon^\star_{\mu\nu} \exp(iq{\cdot}x)$, where $\varepsilon^\star_{\mu\nu}$ is the gravitational wave polarization. Consequently, the Lorentz-gauge-symmetry $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\Theta_{\nu)}$, for any Θ_ν, now reads as $\varepsilon^\star_{\mu\nu} \to \varepsilon^\star_{\mu\nu} + 2q_{(\mu}\Theta_{\nu)}$.

implying that we either have $\mathcal{M}_{\Lambda\Lambda'} = 0$ or $\sum_k \eta_k f_k(-m_k^2)p_k^\mu = 0$, because $\mathbb{R}$ is an algebraic field. As before, the first possibility is absurd physically. This yields to the conclusion

$$F_{\mathrm{I}}^\mu := \sum_{\text{Incoming}} f_k(-m_k^2)p_k^\mu = \sum_{\text{Outgoing}} f_k(-m_k^2)p_k^\mu =: F_{\mathrm{O}}^\mu.$$

However, the only such four-vector quantity depending on the four-momenta that is conserved by all nontrivial scattering processes – meaning that, at least, all individual momenta change – is the total four-momentum.

Remark 4.2. A good way to see this is the only possible conclusion one can make is by looking at the special case where $m = n = 2$ in the Center-Of-Mass-frame (COM-frame). Indeed, in this frame, by definition, the conservation of momentum indicates that $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2' = 0$. The conservation of F^μ, on the other hand, indicates that $f_1\mathbf{p}_1 + f_2\mathbf{p}_2 = f_1\mathbf{p}_1' + f_2\mathbf{p}_2'$. Combining these two equations gives $(f_1 - f_2)(\mathbf{p}_1 - \mathbf{p}_1') = 0$. But the scattering process is assumed to be nontrivial so $\mathbf{p}_1 - \mathbf{p}_1' \neq 0$. This implies that $f_1 = f_2$. ⊞

Therefore, we observe that this imposes the new and really strong condition on the way gravity can couple

$$F_{\mathrm{I,O}}^\mu = f P_{\mathrm{Tot}}^\mu,$$

or, in a more explicit expression,

$$\boxed{f_k(-m_k^2) = f, \ \forall\ k.} \tag{4.3}$$

The punchline we get from (4.3) is that all the particles must share the same gravitational coupling. This is nothing but the Equivalence Principle: at low energies, gravity couples to all form of matter with the same strength, regardless of their *mass* and spin. In other words, we can say that gravitational interactions are insensitive to the particle species they are coupling to. Especially, at our energy scale, this shows that there's nothing special about the gravitational mass as defined in the introduction section – i.e. at energy scales where the quantum nature of gravity is not effective, we do not need to adapt our definition of mass.

4.1.1. *Some General Remarks on Last Subsections.* The proof of the Equivalence Principle we did in the last subsection was completely general. Indeed, the particles under consideration may be massless and carry arbitrary spin. We concluded from this that the Equivalence Principle was basically just a theorem in Quantum Field Theory. Also, something that deserves to be noted is that the soft coupling to another graviton must be the same as for other particles – low energy gravitational interactions cannot distinguish between any matter and gravity itself. Curious!

4.2. **Some Justifications on the Assumptions Made: Part II.**

4.2.1. *Some Caveats?* Before going into the justification of the postulate, we should point out a possible caveat in our previous reasoning. Suppose that particle species can be divided into two (or more) subgroups Q and P that do not interact with each other as pictured in 7. That is, particles belonging to one subgroup have nontrivial interactions within the same subgroup only. In any scattering process we can decompose the initial and the final overall states as $\Lambda = \Lambda_Q + \Lambda_P$ and $\Lambda' = \Lambda_Q' + \Lambda_P'$. The scattering amplitude then splits in a similar

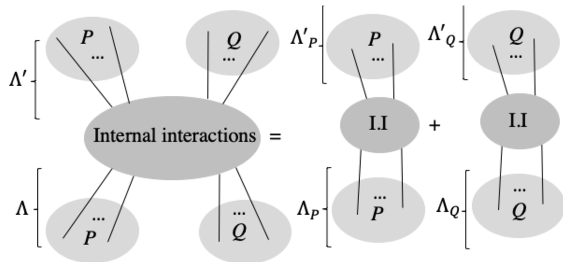

FIGURE 7. Separable interactions.

manner as in 7 and besides total four-momentum conservation, we also have that the total four momentum within each subset is conserved – i.e.

$$P^{\mu}_{\Lambda_Q} = P^{\mu}_{\Lambda'_Q} \text{ and } P^{\mu}_{\Lambda_P} = P^{\mu}_{\Lambda'_P}.$$

As a consequence, we can split the four-vector introduced in the last subsection, namely F^{μ}, as $F^{\mu} = F^{\mu}_P + F^{\mu}_Q$ and each of the two contributions must be *separately* conserved. This means that

$$F^{\mu}_P = f_P P^{\mu}_{\text{tot},P} \text{ and } F^{\mu}_Q = f_Q P^{\mu}_{\text{tot},Q}.$$

Phrased differently, this is saying that subsystems of the Universe that do not interact with each other can have different gravitational coupling constants. However (!), if both $\mathcal{M}_{\Lambda_Q \Lambda'_Q}$ and $\mathcal{M}_{\Lambda_P \Lambda'_P}$ are nonzero, that is, if they both have a chance to happen, then subgroups do interact with gravity (they either have mass or energy...) and therefore they are interacting with each other, at least, in diagrams like 8 in which the momenta p^{μ}_Q and p^{μ}_P are no more *separately* conserved. Hence, this means that we are back to the case where $f = f_P = f_Q$.

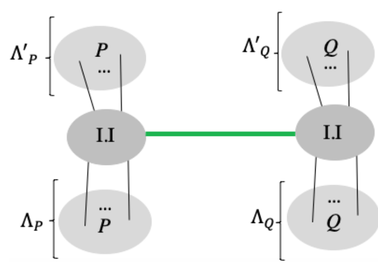

FIGURE 8. The two subgroups interact gravitationally together.

There is one other possibility for a caveat. It happens if one of the two subgroups, say P, does not interact with gravity *at all*. In such a case, one would have $f_P = 0$. This means that we have to decide to which subgroup gravitons *themselves* belong to. But, the isolated subset $\mathcal{L}_{\text{no gravity}} \subset \mathcal{L}$ of our theory is irrelevant to our gravitational analysis. Therefore, we can safely ignore such exotic subgroups, non-interacting gravitationally.

4.2.2. *Justifications.* We are about to justify the postulate in an analogous way we proceeded for the photon. The most generic elementary vertex amplitude is, still in the soft limit, given by

$$\mathcal{M} = \mathcal{M}^{\mu\nu}(p, s, s')\varepsilon^{\star}_{\mu\nu}(q^{\rho}).$$

Then, the amplitude for a process like 6 (c), ignoring the contributions coming from attaching the graviton to internal particles line, is

$$\mathcal{M} = \sum_k \sum_{s'_k} \mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...) \frac{\eta_k \mathcal{M}^{\mu\nu}_{\text{tree}}(p_k, s_k, s'_k)}{-2p_k \cdot q} \varepsilon^\star_{\mu\nu}(q^\mu),$$

where we recall that $\mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...)$ is the amplitude for the original $\Lambda \to \Lambda'$ process with the k^{th}-spin replaced by s'_k, k being the label used to identify the external leg interacting by itself with gravity. Again, invoking the $SO(3,1)$-gauge-symmetry, we must have

$$\sum_k \sum_{s'_k} \mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...) \frac{\eta_k \mathcal{M}^{\mu\nu}_{\text{tree}}(p_k, s_k, s'_k)}{p_k \cdot q} q_\mu \Theta_\nu = 0 : \quad \Theta_\nu \in \mathbb{R}^{3,1}, \text{ nontrivial.} \tag{4.4}$$

Recall that the amplitude is a Lorentz scalar, so it is the same in any inertial frame imaginable. Extrapolating (4.4) to generic momenta q^μ (not necessarily null: $q_\mu q^\mu \neq 0$) for any given k, we can boost such that q^μ is orthogonal to p^μ_k and not orthogonal to the other momenta. The k^{th} denominator consequently vanishes and the resulting divergence can only be canceled if the k^{th} numerator also vanishes – i.e. if

$$\mathcal{M}^{\mu\nu}_{\text{tree}}(p_k, s_k, s'_k) q_\mu = 0, \ \forall\ q^\mu : q \cdot p_k = 0. \tag{4.5}$$

The reason motivating this trick is that, in the boosted frame, we easily see that (4.5) can cancel *exactly* the diverging denominator only if the amplitude tensor takes the form

$$\mathcal{M}^{\mu\nu}_{\text{tree}}(p_k, s_k, s'_k) = p^\mu_k p^\nu_k f_k(p^\mu_k, s_k, s'_k),$$

carrying the desired tensorial structure. Consequently, we are left with

$$\sum_k \sum_{s'_k} \mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...) \eta_k f_k(p_k, s_k, s'_k) p^\nu_k = 0.$$

This equation rather than constraining only the graviton couplings imposes a linear relation between different scattering amplitude in the original system – *before* taking into account gravity. A relation that eventually depends on how the various particles *are going* to couple with gravity. At least, at lowest order in the gravitational coupling, this makes no causal sense; the value of $\mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...)$ for different k changes[9] depending on how gravity will couple. Again, the only mathematical way out of this, is that the gravitational couplings are diagonal in the spin-space – i.e. $f_k(p^\mu_k, s_k, s'_k) = f_k(p^\mu_k, s_k)\delta_{s_k, s'_k}$. In particular, this fixes s'_k for each k. Therefore, we are getting an overall $\mathcal{M}_{\Lambda\Lambda'}$ out of the sum over the external particles and we are left with

$$\mathcal{M}_{\Lambda\Lambda'} \sum_k \eta_k f_k(p_k, s_k) p^\nu_k = 0,$$

stressing out that either the original amplitude $\mathcal{M}_{\Lambda\Lambda'}$ is vanishing or the four-vector

$$\sum_k f_k(p_k, s_k) p^\nu_k,$$

[9] For example, take $k \neq k'$ to label two distinct outgoing particles. Then, $\mathcal{M}_{\Lambda\Lambda'}(..., s'_k, ...) \neq \mathcal{M}_{\Lambda\Lambda'}(..., s'_{k'}, ...)$. But the $\Lambda \to \Lambda'$ process is in the past light cones of these two events and so the value of $\mathcal{M}_{\Lambda\Lambda'}$ must be independent of how gravity couples with the two external legs. Otherwise, we reach a clear causal inconsistency.

has to be conserved in the process. The first option being nonphysical and having already argued that the only physical quantity that can be associated with the momentum four-vector is the total four-momentum, we get back (4.3) and so the conclusion following from it: the Einstein's Equivalence Principle.

Remark 4.3. Following the idea we developed in the closure section for the soft photon emission – explicitly, by sending $s_i \to S_i$ and comparing the resulting equations –, one can show that the gravitational couplings are also independent of the incoming particle spin. ⊞

This completes the proof.

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