Witten's Proof of Positive Energy Theorem

Yuzhen Cao

Abstract

In this report, we will give a brief sketch of Witten's proof of the Positive Energy Theorem by following [PT82] and [Wit81, section 3].

1 Introduction

In order to state the Positive Energy Theorem, we need to set up the follows, given a 4-manifold N with a Lorentzian metric g of signature $(-,+,+,+)$ and a symmetric energy-momentum tensor field T_{ab} satisfying the Einstein field equation:

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab} \tag{1.1}$$

where G is the gravitational constant, and R_{ab} is the Ricci tensor and R is the scalar curvature. Also equivalently we have the following,

$$R_{ab} = 8\pi G (T_{ab} - \frac{1}{2} g_{ab} T) \tag{1.2}$$

where T is the trace of T_{ab}.
Also there is a complete oriented spacelike hypersurface $M \subset N$ of dimension 3 satisfying the following conditions, M is asymptotically flat, i.e., there is a compact set $K \subset M$ such that $M - K$ is a finite disjoint union of subsets $M_1, ..., M_k \subset M$ which are called the "ends" of M. And each M_i is diffeomorphic to the complement of a contractible compact subset in $\mathbb{R}^3$, and the metric of M_i under these diffeomorphism are of the form:

$$g_{ij} = (\delta_{ij} + a_{ij}) dx^i dx^j$$

with respect to the standard coordinates $\{x_1, x_2, x_3\}$ on $\mathbb{R}^3$, and let $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. where the symmetric tensor a_{ij} satisfies

$$a_{ij} = O(\frac{1}{r}), \qquad \partial_k a_{ij} = O(\frac{1}{r^2}) \quad \text{and} \quad \partial_k \partial_l = O(\frac{1}{r^3})$$

Also we need to impose some conditions on the asymptotic behaviour of the second fundamental form h_{ij} of M in N,

$$h_{ij} = O(\frac{1}{r^2}), \qquad \partial_k h_{ij} = O(\frac{1}{r^3})$$

Also to get the positive energy theorem, we need to impose the so called **dominant energy condition** on the energy-momentum tensor T_{ab}, which can be stated as: for each timelike v_a, we have that $T^{ab} v_a v_b \geq 0$ and $T^{ab} v_a$ is non-spacelike. An equivalent statement can be written as follows: for any orthonormal basis, we have that

$$T^{00} \geq |T^{ab}|, \quad \forall a, b$$

Hence

$$T^{00} \geq (-T_{0i} T^{0i})^{\frac{1}{2}}$$

where i runs from 1 to 3 and a, b run from 0 to 3.
Also the total energy and the total momentum can be defined as [PT82, 1.1]:

$$E_l = \lim_{R \to \infty} \frac{1}{16\pi G} \int_{S_{R,l}} (\partial_j g_{ij} - \partial_i g_{jj}) d\Omega^i \tag{1.3}$$

$$P_{lk} = \lim_{R \to \infty} \frac{1}{8\pi G} \int_{S_{R,l}} (h_{ik} - \delta_{ik} h_{jj}) d\Omega^i \tag{1.4}$$

where $S_{R,l}$ are spheres of radius R in $M_l \subset \mathbb{R}^3$.

Remark. Note, we need $\partial_k g_{ij} = O(\frac{1}{r^2}), h_{ij} = O(\frac{1}{r^2})$ to make the above integral converge, and this is exactly part of the requirements in the definition of asymptotically flat space.

Theorem 1.1 (Positive Energy Theorem). *Under the above setting, and the above notation, one have*

$$E_l - |\mathbf{P}_l| \geq 0$$

where $|\mathbf{P}| = \sqrt{P_{il} P_{jl} \delta_{ij}}$*, on each end* M_l*, if* $E_l = 0$*, for some end* M_l *then* M *has only one* *end and the Ricci tensor of N along M vanishes.*

Remark. The total mass of an end M_l is

$$m_l = \sqrt{E_l^2 - P_{il}P_{jl}\delta_{ij}}$$

Therefore the theorem above means that $m_l \geq 0$ for any l.

2 Spinors and Dirac Operators

To present Witten's proof of theorem 1.1, we shall recall some basic definitions and properties of spinors and Dirac operators in this section. For detail discussion on spin structure and Dirac operators, please refer to [JJ08, section 2.6, 4.3].
First we need to introduce the concept of Clifford algebra,

Definition 2.1 (Clifford Algebra)**.** Let V is a n-dimensional $\mathbb{R}$-space, and let q be a quadratic form on V. And let $\mathfrak{T}(V)$ be the tensor algebra generated by V, and let $I(V)$ be the two-sided algebra generated by elements of the form

$$v \otimes v + q(v)$$

Then the Clifford algebra over V is defined to be $\mathfrak{T}(V)/I(V)$ and we shall denote as $Cl(V)$.

Remark. The multiplication rule in $Cl(V)$ is given by

$$v * w + w * v = -2q(v, w), \qquad \forall \ v, w \in V$$

Remark. Note in this paper, we may take q to be the Minkowski metric on $\mathbb{R}^{3,1}$ and the Riemannian metric on 3-dimensional spaces. Note that in the case when q is the Minkowski metric, the spin group $Spin(3,1)$ is exactly the group $SL(2,\mathbb{C})$.

Then let V be a 2-dimensional vector space over $\mathbb{C}$, then naturally $SL(2,\mathbb{C})$ acts on V. Note V has a invariant symplectic ω which can be viewed as an isomorphism $V \to V^*$. And $\mathbb{R}^{3,1}$ can be viewed as a subspace of $V \otimes \bar{V}$ by the isomorphism

$$x = (-x_0, x_1, x_2, x_3) \mapsto A_x := \begin{bmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{bmatrix}$$

Note that

$$-\det(A_x) = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

which is the Minkowski norm of x.
Then the Dirac spinor space can be viewed as $S = \bar{V} \oplus V^*$, where $\mathbb{R}^{3,1}$ acts on S as follows

$$x * (\xi, \eta) = (x\eta, x^\sigma \xi)$$

where $(\xi, \eta) \in S = \bar{V} \oplus V^*$ and $x^\sigma : \bar{V} \to V^*$ is the σ-adjoint of $x : V^* \to \bar{V}$ since $x \in V \otimes \bar{V}$.
Note that for any $(\xi, \eta) \in S, x \in \mathbb{R}^{3,1}$,

$$\begin{aligned} x * y * (\xi, \eta) &= x * (y\eta, y^\sigma \xi) \\ &= (xy^\sigma \xi, x^\sigma y\eta) \end{aligned}$$

Therefore

$$x * x * (\xi, \eta) = (xx^\sigma \xi, x^\sigma x\eta)$$

Note that $x^\sigma x$ is a scalar since

$$\sigma(\xi, xx^\sigma \xi) = \sigma(x^\sigma \xi, x^\sigma \xi) = 0 \quad \forall \ \xi \in V^*$$

Hence $x^\sigma x = aI_2$, and note that

$$a^2 = (\det x)^2$$

Therefore we can get that

$$x * x = -\|x\|^2 = \det(x)$$

Which is exactly the multiplication rule of the Clifford algebra generated by $\mathbb{R}^{3,1}$.

Now we need to lift those structures to vector bundles. Note that we only need to consider spinor bundles over the spacelike hypersurface M in Witten's proof, then we can consider the principal $SO(3)$, then we can lift the principal $SO(3)$ bundle to a principal $Spin(3)$ bundle, then we can consider the associated vector bundle with fibre $S = \bar{V} \oplus V^*$, where $Spin(3)$ acts on S by the composition:

$$Spin(3) \cong SU(2) \hookrightarrow SL(2, \mathbb{C})$$

for more detailed discussion of spin structure on a Riemannian manifold, please refer to [JJ08, section 2.6].

We will now give a explicit description of the induced connection on the resulting bundle S. Note that any connection on the principal $SO(3)$ bundle of M can be written as

$$d + A$$

with $A \in \mathfrak{so}(n)$, while note that $A \in \mathfrak{so}(n) \in \mathfrak{spin}(n)$. Therefore $d + A$ induced a connection on the principal bundle $Spin(3)$, hence descends to a connection on the bundle S. And in fact, if A is given by the matrix Ω_{ij} in a local orthonormal coordinate chart, we can write $A = \sum_{i<j} \omega_{ij} e^i \wedge e^j$ where $\{e^i\}$ are the coframe of the local coordinate chart, then by [JJ08, 4.4.5, section 4.4], then the corresponding local expression for the induces connection on S is given by [Chr10, 3.1.24, 3.1.27]

$$X * \varphi - \frac{1}{4}\sum_{i,j} \omega_{ij}(X) e^i * e^j * \varphi \qquad \forall \varphi \in \Gamma(S), \quad \forall X \in \Gamma(TM) \tag{2.1}$$

And the curvature of the above connection can be given as:

$$D_X D_Y \psi - D_Y D_X \psi - D_{[X,Y]}\psi = \frac{-1}{4}\Omega_{ij}(X,Y) e^i * e^j \psi$$

Also S admits an inner product $\langle , \rangle$ which is preserved by the action of e^i and compatible with the induced connection with the action of $e^i, i = 1,2,3$ are all anti-hermitian and e^0 hermitian by [JJ08, Corollary 2.6.3].

Now let D be the connection on N, and let ∇ be the Riemannian connection on M, then we can define the Dirac operator as:

$$\not{\partial}\psi = \sum_{i=1}^{3} e^i * D_i \psi \qquad \forall \psi \in \Gamma(S)$$

where $\{e^i\}$ is an orthonormal coframe of M, and the $*$ denote the Clifford multiplication.

One of the key ingredient of the Witten's proof is the so called Weitzenböck formula of the Dirac operator we defined above,

Lemma 2.1.

$$\not{\partial}^2 = -D_i D_i + \frac{1}{4}(R + 2R_{00} + 2R_{0j}e^0 e^j) - h_{ij}e^i e^0 D_j \tag{2.2}$$

where R is the scalar curvature of N and R_{ij} is the component of the Ricci tensor of N.

Proof. First let $p \in M$ be an arbitrary point, consider a small neighbourhood $U \subset M$ of p, and let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of T_pN with e_0 time-like and normal to M.

Then we can parallel transport $\{e_1, e_2, e_3\}$ to get an orthonormal frame field of $\Gamma(TM, U)$ with $\nabla_i(e_j)|_p = 0$ for any i, j. Also we can extend $\{e_1, e_2, e_3\}$ to an orthonormal frame field of N with $D_0 e_i|_p = 0$.

Then let $h_{ij} = \langle D_i e_0, e_j\rangle$ be the 2nd fundamental form, hence $-h_{ij}e^0 = (D_i e^j)_p, \quad (D_i e^0)_p = -h_{ij}e^j$.

Hence,

$$\begin{aligned}
\not{\partial}^2 &= e^i * D_i(e^j * D_j) = e^i * e^j D_i D_j + e^i * (D_i e^j) D_j \\
&= e^i * e^i D_i D_i + \frac{1}{2}\sum_{i \neq j} e^i * e^j (D_i D_j - D_j D_i) + e^i * (D_i e^j) D_j \\
&= -D_i D_i + \frac{1}{2}\sum_{i \neq j} e^i * e^j * (D_i D_j - D_j D_i) - h_{ij} e^i * e^0 * D_j
\end{aligned}$$

Note that the middle term can be expressed by curvature term, from now, we will always let Greek indices runs from 0 to 3 and Lain indices runs from 1 to 3

$$\frac{1}{2}\sum_{i \neq j} e^i * e^j (D_i D_j - D_j D_i) = \frac{1}{8}\sum_{\alpha,\beta}\sum_{i,j} R_{\alpha\beta ij} e^i * e^j * e^\alpha * e^\beta$$

Then by the calculation in [JJ08, theorem 4.4.1], we can get

$$\frac{1}{2}\sum_{i \neq j} e^i * e^j (D_i D_j - D_j D_i) = \frac{1}{4}(R + 2R_{00} + 2R_{0j}e^0 * e^j)$$

Therefore we can get the following

$$\not{\partial}^2 = -D_i D_i + \frac{1}{4}(R + 2R_{00} + 2R_{0j}e^0 * e^j) - h_{ij}e^i * e^0 D_j$$

$\square$

Remark. Note that

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00}$$
$$R_{0j} = \frac{1}{2} g_{0j} R + 8\pi G T_{0j}$$

Therefore

$$\not{\partial}^2 = -D_i D_i + 4\pi G (T_{00} + T_{0j} e^0 * e^i) - h_{ij} e^i * e^0 D_j$$

Then let $\mathscr{R} = (R + 2R_{00} + 2R_{0j} e^0 * e^j) \in \mathrm{End}(S)$, and act on $\forall \psi \in \Gamma(S)$ and then take inner product, we have

$$\langle \psi, \not{\partial}^2 \psi \rangle = -\langle \psi, D_i D_i \psi \rangle + \frac{1}{4} \langle \psi, \mathscr{R} \psi \rangle - \langle \psi, h_{ij} e^i * e^0 D_j \psi \rangle \tag{2.3}$$

Then note by 2.1

$$D_i = \nabla_i - \frac{1}{2} h_{ij} e^j * e^0$$

And in order to evaluate the integral of 2.3 on M, we need the following to perform integration by parts.

$$d(< \phi, D_i \psi > \iota_{e_i} \mu) = (\langle D_i + \frac{1}{2} h_{ij} e^j * e^0 \phi, D_i \psi \rangle + \langle \phi, D_i D_i \psi + \frac{1}{2} h_{ij} e^j * e^0 D_i \psi \rangle \ ) \iota_{e_i} \mu \tag{2.4}$$
$$= (\langle D_i \phi, D_i \psi \rangle + \langle \phi, D_i D_i \psi \rangle + \langle \phi, h_{ij} e^j * e^0 D_i \psi \rangle) \mu \tag{2.5}$$

where $\mu = e^1 \wedge e^2 \wedge e^3$ is the volume form on M and $\iota_{e_i} \mu$ denote the contraction of μ by e_i.
Now note that

$$(-\langle \psi, D_i D_i \psi \rangle + \frac{1}{4} \langle \psi, \mathscr{R} * \psi \rangle - \langle \psi, h_{ij} e^i * e^0 D_j \psi \rangle) \mu = \langle D_i \psi, D_i \psi \rangle \mu + \frac{1}{4} \langle \psi, \mathscr{R} \psi \rangle \mu$$
$$- d(< \psi, D_i \psi > \iota_{e_i} \mu)$$

And in Witten's proof, we only need to consider spinor fields ψ along M satisfying $\not{\partial} \psi = 0$, then we can get the the following integral form of the **Weitzenböck formula**,

$$\int_M \langle D\psi, D\psi \rangle + \langle \psi, \frac{1}{4} \mathscr{R} * \psi \rangle = \int_{\partial M} \langle \psi, D_i \psi \rangle \iota_{e_i} \mu \tag{2.6}$$

3 Proof of Positive Theorem

Note that if we consider the special case, where $\not{\partial}\psi = 0$ along M, then the left hand side of 2.6 is non-negative, since

$$< \psi, \frac{1}{4}\mathscr{R} * \psi >= 4\pi G\langle \psi, (T_{00} + T_{0i}e^0 e^i)\psi \rangle$$

and the matrix $T_{00} + \sum_j T_{0j}e^0 * e^j$ is a semi-positive definition matrix, the hermitian matrix $\sum_j T_{0j}e^0 * e^j$ has eigenvalues equal to plus or minus the magnitude of the momentum flux, so the eigenvalues of $T_{00} + \sum_j T_{0j}e^0 * e^j$ are all nonnegative by the dominant energy condition, [Wit81].
And if ψ is asymptotically constant in a proper way, then the right hand side of 2.6 depends only on h_{ij} and g_{ij}, and Witten noticed that the only invariant of the $O(\frac{1}{r})$ part of g_{ij} and h_{ij} are the total energy and momentum defined in the first section [Wit81], therefore the integral on the right hand side of 2.6 must contain information of E and P and direct calculation shows that the integral is actually the energy-momentum integral.
In conclusion, [PT82] gave the following theorem,

Theorem 3.1. *[PT82, theorem 4.1] Let $\{\psi_{0l}\}_{l=1}^{k}$ be constant spinors defined in the asymptotic ends $\{M_l\}_{l=1}^{k}$, then there exists a unique, smooth spinor ψ on M that satisfies:*

1. $\not{\partial}\psi = 0$

2. $$\lim_{r\to\infty} r|\psi - \psi_{0l}| = 0$$

   in each end M_l.

3. $$4\pi G\sum_{l=1}^{k}(E_l\langle\psi_{0l},\psi_{0l}\rangle + \langle\psi_{0l}, P_{lk}dx^0 * dx^k * \psi_{0l}\rangle) = \int_{\partial M}\langle\psi, D_i\psi\rangle\iota_{e_i}\mu \tag{3.1}$$
   $$= \int_M(\langle D\psi, D\psi\rangle + \langle\psi, \frac{1}{4}\mathscr{R} * \psi\rangle)\mu \geq 0 \tag{3.2}$$

   where $\{x^\alpha\}$ are basis of $\mathbb{R}^{3,1}$.

Proof. To keep the simplicity of this report,we will not present the proof the statement (1), (2), but actually the existence of the prescribed spinor fields is an important step of the proof, and we can see that the statement (3) of this theorem is crucial. In the rest of this report, we will show that the statement (3) holds following the calculation in [PT82, section 4].

Write $\psi = \psi_0 + \psi_1$, where ψ_0 is the given constant spinor defined in the asymptotic ends.

Then note that

$$\nabla_i \psi = \nabla_i \psi + e^i * \partial\!\!\!/ \psi = (\delta_{ij} + e^i * e^j *)\nabla_j \psi = \frac{1}{2}[e^i, e^j] * \nabla_j \psi$$

where $[e^i, e^j] = e^i * e^j - e^j * e^i$, then we have the following

$$\int_{\partial M} \langle \psi, D_i \psi \rangle \iota_{e_i} \mu = \frac{1}{2}\int_M d(\langle \psi_0, [e^i, e^j] * D_j \psi_0 \rangle \iota_{e_i} \mu) + \frac{1}{2}\int_M d\eta$$

where

$$\eta = (\langle \psi_1, [e^i, e^j] * D_j \psi \rangle + \langle \psi_0, [e^i, e^j] * D_j \psi_1 \rangle)\iota_{e_i} \mu)$$

before the calculation of the integral, we need the following estimate of the integral of the ψ_1 part, which is integration of the $O(\frac{1}{r^3})$ part of ψ,and this goes to 0 under the asymptotic hypothesis, for the detailed proof, the readers may refer to the original paper [PT82, section 4]

Lemma 3.2.

$$\int_M d\eta = 0$$

Then we have that

$$\int_M (\langle D\psi, D\psi \rangle + \langle \psi, \frac{1}{4}\mathscr{R} * \psi \rangle)\mu = \frac{1}{2}\lim_{r\to\infty}\sum_l \int_{\partial M_l} \langle \psi_0, [e^i, e^j] * D_j \psi_0 \rangle \iota_{e_i} \mu$$

Then note that the ends M_l are all diffeomorphic to $\mathbb{R}^3 - K_l$ where K_l are compact sets, then we can pull back the connection to $\mathbb{R}^3 - K_l$ for each l, then we have the following:

$$D_j \psi_0 = \nabla_j \psi_0 - \frac{1}{2}h_{ji} dx^i * dx^0 \psi_0 = -\frac{1}{4}\Gamma^l_{jk} dx^k * dx^l \psi_0 - \frac{1}{2}h_{jk} dx^k * dx^0 * \psi_0$$

where Γ^k_{jl} is the connection on M_l with respect to the coordinate chart $\{x^1, x_2, x_3\}$.

Then by the hypothesis, we have the following since we only need to calculate the $O(\frac{1}{r^2})$ part of the integrand, and $|dx^i - e^i| = O(\frac{1}{r})$ and integral of this part will go to 0 when $r \to \infty$

$$\frac{1}{2}\int_{\partial M_l} \langle \psi_0, [e^i, e^j] * D_j\psi_0 \rangle \iota_{e_i}\mu = \frac{1}{2}\int \langle \psi_0, [dx^i, dx^j] * D_j\psi_0 \rangle \iota_{dx_i}\mu$$

And we have that

$$\begin{aligned}
\langle \psi_0, [dx^i, dx^j] * (\frac{-1}{4}\Gamma^l_{jk} dx^k * dx^l * \psi_0 - \frac{1}{2} h_{jk} dx^k * dx^0 * \psi_0) \rangle &= -\frac{1}{4}\langle \psi_0, [dx^i, dx^j] * \Gamma^l_{jk} dx^k * dx^l * \psi_0 \rangle \\
&\quad - \frac{1}{2}\langle \psi_0, h_{jk}[dx^i, dx^j] * dx^k * dx^0 * \psi_0 \rangle \\
&= \frac{1}{4}\langle \psi_0, (\partial_j g_{ij} - \partial_i g_{jj})\psi_0 \rangle \\
&\quad + \frac{1}{2}\langle \psi_0, (h_{ik} - \delta_{ik} h_{jj}) * dx^0 * dx^k * \psi_0) \rangle
\end{aligned}$$

Therefore we finished the proof.

□

To prove the rigidity part of theorem 1.1, we need the following lemma,

Lemma 3.3. *[PT82, Lemma 4.3] Suppose that ψ and $\{\psi_i\}$ are smooth spinor fields along M with $D\psi = 0$ and $D\psi_i = 0$ for any i, then*

1. *If $\lim_{x\to\infty} \psi(x) = 0$, where this limit is taken along some path in one asymptotic end M_l, then $\psi = 0$.*
2. *if $\{\psi_i\}$ are linearly independent in some end M_l, then they are linearly independent everywhere on M.*

Proof. 1. Let $|\psi| = \langle \psi, \psi \rangle$, then we have that following:

$$d|\psi|^2 = \langle -h_{ij} e^j * e^0 * \psi, \psi \rangle$$

Then we have that

$$2|\psi||d|\psi| \le |h_{ij}||\psi|^2$$

Note that $|h_{ij}| = O(\frac{1}{r^2})$ by the asymptotic assumption of M, then

$$|d \ln |\psi|| \le \frac{C}{r^2}$$

for some constant C on the complement of the zero set of ψ.

Then by integrating, we have that

$$|\psi(x)| \geq |\psi(x_0)| \exp(C(\frac{1}{|x|} - \frac{1}{|x_0|}))$$

And passing to the limit, we get that

$$0 \geq |\psi(x_0)|$$

Therefore $\psi(x_0) = 0$ for any $x_0 \in M$.

2. Suppose that there are constants c_i, such that $\psi = \sum_i c_i \psi_i$, vanishes at some point $x_1 \in M$, note that $\nabla \psi = 0$, and note that

$$|\psi(x_1)| \geq |\psi(x_0)| \exp(C(\frac{1}{|x|} - \frac{1}{|x_0|}))$$

Therefore contradicts with the hypothesis.

□

Now we end this report by taking the proof of Positive theorem from [PT82, section 4]

Proof of theorem1.1. 1. Let $P_{i,l}$ $\quad i = 1,2,3$ be the components of the total momentum of the end $M_l \subset \mathbb{R}^3$, let $\{\psi_l\}_{l=1}^k$ be constant spinor fields on the asymptotic ends with $\psi_l = 0$ on each ends except M_1, and ψ_1 is an eigen-spinor of $P_{i1} dx^i * dx^0$ with eigenvalue $-|P|$, then the theorem gives a harmonic spinor ψ which is asymptotically $\{\psi_l\}$, then by the theorem 3.1, we have that $E_l \geq |P_l|$.

2. Now suppose that the energy of M_l is 0, then choose a basis of constant spinors $\{\psi_\alpha\}$ of constant spinors, and and let $\psi_l^\alpha = \psi_\alpha$ on M_1 and 0 for all other ends M_l. Then let ϕ^α be the solutions constructed in theorem 3.1, note that the boundary term vanishes hence $D\phi^\alpha = 0$ and $\phi^\alpha \to 0$ uniformly on each end except M_1, which contradicts with the lemma 3.3a, unless M_1 is the only end of M.

   And note that $\{\psi^\alpha\}$ are linearly independent on M_1, they are linearly independent everywhere by lemma 3.3, moreover, $D\psi^a = 0$, so we have

$$R_{\alpha\beta ij} = 0$$

Therefore $T_{\alpha\beta} = 0$ by the dominant energy condition, therefore we know that all component of the energy momentum tensor vanishes. Thus N is flat along M.

□

References

[Chr10] PT Chrusciel. Lectures on energy in general relativity. *Krakow, March April*, 2010.

[JJ08] Jürgen Jost and Jèurgen Jost. *Riemannian geometry and geometric analysis*, volume 42005. Springer, 2008.

[PT82] Thomas Parker and Clifford Henry Taubes. On witten's proof of the positive energy theorem. *Communications in Mathematical Physics*, 84(2):223–238, 1982.

[Wit81] Edward Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.