Constraint Equations and the Well-posedness of Einstein Equations

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1 Introduction

Given a Lorentzian manifold (M, g), it is said to satisfy Einstein's Field Equations if

$$Ric - \frac{1}{2}Sg = T \tag{1}$$

with S the Ricci scalar associated to the metric and T the energy-momentum tensor, which is given by some physical conditions. Although the development of the theory and the elements in the equation are based in tensorial analysis and geometric constructions, the field equations themselves are, after a choice of coordinates with respect to which calculations can be made, a system of non-linear second order PDEs.

When we consider the equations as such, the usual questions about initial conditions and existence and uniqueness of solutions arise. In this case, the questions are harder because, in principle, the nature of M itself should be related to the solution of equation 1.

The groundbreaking result in those questions was made by Yvonne Choquet-Bruhat in [5], who proved local existence and uniqueness for Einstein Field Equations in the vacuum when initial data was given as (Σ, γ, K), Σ a spacelike hypersurface of M, γ a Riemannian metric over Σ and K a symmetric bilinear form over Σ that plays the role of the second fundamental form in the final solution. The result holds only when γ and K toghether satisfy the so called constraint equations, a geometric condition between those quantities and T that emerge from the Gauss-Codazzi equations, which are themselves constraints that couple the first and second fundamental forms over a hypersurface and the geometry of the ambient manifold.

Choquet-Bruhat's result in some sense finds the solution in a manifold that is locally homeomorphic to $\Sigma \times I$, I and interval of the real line. Later, Geroch proved that a globally hyperbolic spacetime (which is the possible development of a solution to Einstein Field Equations) is indeed topologically equivalent to $\Sigma \times I$ ([6]), a result that was strengthened by Bernal and Sánchez for a smooth splitting in [2]. In this setting, a clearer and more meaningful presentation of Choquet-Bruhat's result is possible, with the topology of M already well-established.

In this work we present the necessary geometric background for the existence and uniqueness result for Einstein's Field Equations in section 2 followed by Choquet-Bruhat's existence result in section 3. Then, in section 4 we present the conformal method, a technique to look for solutions of the constraint equations and consequently for candidates to initial data of the Field Equations. Finally, in appendix A we hide some gory calculations necessary for the theory of section 3 that are not easy to find.

2 Geometric Setting

We organize this section based on the approach in [1], but inverting the steps for we judge that is clearer.

2.1 Geometric Constraints: Gauss-Codazzi Equations

In general, if (M^{n+1}, g) is a (pseudo-)Riemannian[1] manifold and Σ is a hypersurface with normal vector field n, one can define the second fundamental form, a symmetric form over Σ, by

$$K(X,Y) := g(D_X n, Y), X, Y \in \mathfrak{X}(\Sigma)$$

with D the affine connection over M.

Now, D over M induces an affine connection ∇ over Σ. If we denote tensors with superscript M to correspond to the connection D and with superscript Σ to correspond to ∇ and R is the $(4,0)$-tensor corresponding to the Riemann curvature tensor, K presents a useful way to relate the R^M restricted to $T\Sigma$ to R^Σ: the Gauss-Codazzi equations. The Gauss equation (tangent part) is:

$$R^M(X,Y,Z,W) = R^\Sigma(X,Y,Z,W) + K(X,W)K(Y,Z) - K(X,Z)K(Y,W) \quad (2)$$

and the Codazzi equation (normal part):

$$R^M(X,Y,n,Z) = \nabla_X(K(Y,Z)) - \nabla_Y(K(X,Z)) \quad (3)$$

in both cases, $X, Y, Z, W \in T\Sigma$.

If we choose a local chart around a point $p \in \Sigma$ that is adapted to Σ, in the sense that over Σ the 0-th coordinate vector is normal to Σ and the others are tangent to Σ, we can write the two equations in coordinates:

$$R^M_{ijkl} = R^\Sigma_{ijkl} + K_{il}K_{jk} - K_{ik}K_{jl} \quad (2)$$

$$R^M_{ij0k} = K_{jk;i} - K_{ik;j} \quad (3)$$

with the indices going from 1 to n and $;i$ corresponding to the covariant derivative with respective to the connection ∇.

With those two equations it is possible to find an expression for the Einstein tensor over Σ as a function of K, and thus to relate the matter information (T) and the geometric information (metric and second fundamental form) over Σ. Those will be the constraint equations. To get there we need to use (2) and (3) to get the Ricci tensor and scalar over Σ. In coordinates, for $i, j, k \in \{1, ..., n\}, \alpha, \beta \in \{0, 1, ..., n\}$:

$$\begin{aligned}
Ric^M_{ij} = (R^\alpha_{i\alpha j})^M &= (R^0_{i0j})^M + (R^k_{ikj})^\Sigma + K^k_k K_{ji} - K^k_j K_{ik} \\
&= (R^0_{i0j})^M + Ric^\Sigma_{ij} + K^k_k K_{ji} - K^k_j K_{ik} \\
Ric^M_{0i} = (R^\alpha_{i\alpha 0})^M &= \\
&= (R^0_{i00})^M + K^k_{i;k} - K^k_{k;i} \\
&= K^k_{i;k} - K^k_{k;i}
\end{aligned}$$

[1] The calculations never involve the signature of g, see section 6.4 in [9].

thus, if S is the Ricci scalar:

$$\begin{aligned} S^M = (Ric^\alpha_\alpha)^M &= (Ric^0_0)^M + (R^{0i}_{0i})^M + (Ric^i_i)^\Sigma + K^k_k K^i_i - K^k_i K^i_k \\ &= (Ric^0_0)^M + (R^{0i}_{0i})^M + S^\Sigma + (K^i_i)^2 - K^k_i K^i_k \\ &= g^{0\beta}\left(Ric^M_{\beta 0} + (R^i_{\beta 0 i})^M\right) + S^\Sigma + (K^i_i)^2 - K^k_i K^i_k \end{aligned}$$

Now, due to our choice of chart, $g_{0i} = 0$ if $i \neq 0$ and hence $g^{00}g_{00} = 1$. So:

$$\begin{aligned} S^M &= g_{00}^{-1}\left(Ric^M_{00} + (R^i_{00i})^M\right) + S^\Sigma + (K^i_i)^2 - K^k_i K^i_k \\ &= 2g_{00}^{-1} Ric^M_{00} + S^\Sigma + (K^i_i)^2 - K^k_i K^i_k \end{aligned}$$

So, replacing in $G = Ric - \frac{1}{2}Sg$:

$$\begin{aligned} G^M_{00} &= Ric^M_{00} - \tfrac{1}{2}g_{00}\left(2g_{00}^{-1}Ric^M_{00} + S^\Sigma + (K^i_i)^2 - K^k_i K^i_k\right) \\ &= -\tfrac{1}{2}g_{00}(S^\Sigma + (K^i_i)^2 - K^k_i K^i_k) \\ G^M 0i &= Ric^M_{0i} - \tfrac{1}{2}S^M g_{0i} \\ &= K^k_{i;k} - K^k_{k;i} \end{aligned}$$

This expressions are quite general, but we will be interested particularly in the case (M, g) is a spacetime and (Σ, γ, K), γ the induced metric by g, is a spacelike (γ is a Riemannian metric) hypersurface. In that case g_{00} may be chosen to be -1 and we demand that the metric satisfy Einstein's Field Equation $G = T$. Also, we define $||K||^2_\gamma := K^k_i K^i_k$ for short and we get the Einstein Constraint Equations over Σ:

$$\begin{cases} 2G^M_{00} = S^\Sigma + (tr_\gamma K)^2 - ||K||^2_\gamma = 2T_{00} \\ G^M_{0i} = K^k_{i;k} - K^k_{k;i} = T_{0i} \end{cases} \tag{4}$$

The first equation is usually called "Hamiltonian constraint" while the vector relation given by the second expression is usually called "Momentum constraint".

Notice that the conditions depend only on information of the metric and the second fundamental form over Σ, not anywhere else in the manifold M.

These equations present a necessary condition, that is a constraint, for an initial data set $(\Sigma, \gamma, K, T|_\Sigma)$ to be a possible initial data set for Einstein's Field Equations. Notice that, as for the full equations, the constraints relate matter and geometry, although only in an n-manifold Σ, not over the full spacetime. In section 3 we will show a result first presented by Yvonne Choquet-Bruhat[2] in [5] that proves that the constraints are also a sufficient condition.

2.2 Geometry of Globally Hyperbolic Spacetimes

Since in the globally hyperbolic case the manifold M can be split as $M = \Sigma \times I$, Σ a 3-manifold, $I \subset \mathbb{R}$ an interval, it can be folliated by spacelike hypersurfaces diffeomorphic to Σ.

In order to do the calculations for this section, let's structure that by a diffeomorphism $i : \Sigma \times I \to M$ satisfying $i(\Sigma \times \{s\}) := i_s(\Sigma) := \Sigma_s$ is a spacelike hypersurface of M. Denote n_s the unitary future-directed timelike normal vector field over Σ_s and γ_s the Riemannian metric induced on Σ_s by g.

[2] As a clarifying note it is useful to say that her surename by the date of the publication of the original paper was Fourès-Bruhat, which justifies the different name in the bibliography.

Now, i is a diffeomorphism, so if we define local coordinates $\mathbf{x}$ over $U \subset \Sigma$, they can be extended to $(\mathbf{x}, s)$ over $U \times I \subset M$ with the identification through i, as local coordinates on M. Let $\{\partial_s, \partial_\alpha\}_{\alpha \in \{1,2,3\}}$ be the associated coordinate basis. Note that, in this case, ∂_s doesn't have to be timelike, since i is an arbitrary folliation, but it can be decomposed over Σ_s as

$$\partial_s := N_s n_s + X_s$$

with N_s a scalar field and X_s a tangent vector field. In this case N_s is called the *lapse function* and X_s the *shift vector* associated to the folliation. This allows one to write the metric g with respect to this coordinates as

$$g = -N^2 ds \otimes ds + \gamma_{ij}(dx^i + X^i ds) \otimes (dx^j + X^j ds). \tag{5}$$

What will be important in the next section is that if we are given a slice (Σ, γ), a Riemannian manifold, N_s a one-parameter family of non-vanishing scalar fields and X_s of vector fields over Σ, one can reverse engineer the previous reasoning by setting g as in 5 a metric in $\Sigma \times I$ and in that case you get that the normal field over Σ_s becomes

$$n_s = \frac{1}{N_s}(\partial_s - X_s)$$

And then you can write the second fundamental form in those coordinates as:

$$K_{ij} = \frac{1}{2N}\left(\partial_s \gamma_{ij} - \mathcal{L}_X(\gamma_{ij})\right)$$

and if X, N, γ_{ij} (as functions) and K are prescribed, one can write down the functions

$$\partial_s \gamma_{ij} = 2NK_{ij} + \mathcal{L}_X(\gamma_{ij})$$

3 Well-Posedness of the Initial Value Problem

In this setting we are able to present the proof of existence and uniqueness of a solution of Einstein's equation in the vacuum ($T = 0$) if the initial condition satisfies the constraint equations. In more precise terms, the result is that there is a local solution to the following problem:

Theorem 1 (Choquet-Bruhat)**.** *There is (M, g) a Lorentzian manifold satisfying:*

$$\begin{cases} Ric = 0, \ in \ M; \\ \Sigma \subset M \ is \ a \ spacelike \ hypersurface; \\ g = \gamma \ in \ T\Sigma; \\ K(X,Y) = g(D_X n, Y), \forall Y, X \in \ \mathfrak{X}(\Sigma) \end{cases}$$

with D the Levi-Cività connection in M if K, γ satisfy the constraint equations:

$$\begin{cases} 2G^M_{00} = S^\Sigma + (tr_\gamma K)^2 - ||K||^2_\gamma = 0 \\ G^M 0 i^M = K^k_{i;k} - K^k_{k;i} = 0 \end{cases} .$$

The result was proved for the first time in [5] and relies on the local existence and uniqueness of solutions for the following initial value problem on a Lorentzian manifold (M, g):

$$\begin{cases} g^{\alpha\beta} T^{i_1 \dots i_k}_{,\alpha\beta} + H^{i_1 \dots i_k} = 0, \; in \; M \\ T^{i_1 \dots i_k} = T^{i_1 \dots i_k}_{initial}, \; given \; in \; \Sigma; \\ T^{i_1 \dots i_k}_{,0} = \partial T^{i_1 \dots i_k}, \; given \; in \; \Sigma \end{cases} \tag{6}$$

with Σ a spacelike hypersurface, $H^{i_1 \dots i_k}$ a polynomial of the components of g, g^{-1}, T and their first derivatives.

In fact, most of Choquet-Bruhat's paper (chapters I-III) is devoted to building the corresponding theory of existence and uniqueness for hyperbolic PDEs necessary in the final result for Einstein's equations. In the following we will assume that fact as given, and present the final result following the ideas of [1].

The proof of the existence theorem 1 involves a lot of technical unhelpful manipulations of coordinates that are hidden in appendix A, but in broad strokes we have the following:

Step 1. If $F(g)$ is a function of the components of g and g^{-1}, but does not involve any of its derivatives and we define the vector field W such that its covariant form has components

$$W_\alpha = -g^{\rho\lambda} \left(g_{\lambda\alpha,\rho} - \frac{1}{2} g_{\rho\lambda,\alpha} \right) + F(g)$$

the equation

$$Ric + \frac{1}{2} \mathcal{L}_W g = 0 \tag{7}$$

is hyperbolic in the sense that it satisfies the conditions of system 6. Notice that W is a function of g and if g is a solution to equation 7 for which W vanishes identically, then the equation becomes

$$Ric = 0$$

that is, g is a solution for Einstein's Field Equations in the vacuum.

Step 2. If W satisfies equation 7, then we can call $A := \frac{1}{2} \mathcal{L}_W g$ and we have:

$$\begin{cases} A = -Ric \\ tr_g A = -S \\ Ric - \frac{1}{2} S g = -\left(A - \frac{1}{2} (tr_g A) g \right) \end{cases}$$

so, since the Einstein tensor is divergence-free, we have from the last identity above that

$$D_\mu \left(A - \frac{1}{2} (tr_g A) g \right)^{\mu\nu} = 0$$

and hence W satisfies the hyperbolic equation:

$$\frac{1}{2} g^{\mu\lambda} W^\nu_{,\lambda\mu} + P(W) = 0 \tag{8}$$

with $P(W)$ a polynomial of the components of W, their first derivatives, which components involve the components of g and g^{-1} and their first derivatives and such that $P(0) = 0$. So, if we have an initial condition $W^\alpha = W_0^\alpha = 0$ along Σ, uniqueness gives that $W \equiv 0$ is the only solution for the initial value problem given by equation 8 with null initial condition.

Step 3. Given (Σ, γ, K) satisfying the constraint equations for the vacuum, it is possible to find $g|_\Sigma$ and $\partial_0 g|_\Sigma$, such that ($N \neq 0$ and $X \in \mathfrak{X}(\Sigma)$ are given by the choices of g_{0a} according to equation 5):

$$\begin{cases} g \text{ Lorentzian metric} \\ g_{ij}|_\Sigma = \gamma_{ij} \\ \partial_0 g_{ij}|_\Sigma = 2NK_{ij} + \mathcal{L}_X(\gamma_{ij}) \\ W|_\Sigma = 0 \\ W_{,0}|_\Sigma = 0 \end{cases}$$

First, let's show that we only have to guarantee $W|_\Sigma \equiv 0$, because then the constraint condition guarantees that $W_{,0}|_\Sigma \equiv 0$.

Theorem 2. *In the setting of this step, assume $g^{00} \neq 0$ and $g|_\Sigma$ is such that $W|_\Sigma \equiv 0$, then $W_{,0}|_\Sigma \equiv 0$.*

Proof. If $W \equiv 0$, $(D_a W)^b = W^b_{,a}$, for the other terms vanish, so, by equation 7 we have:

$$Ric_{\alpha\beta} + \frac{1}{2}\left(g_{\beta\mu} W^\mu_{,\alpha} + g_{\alpha\mu} W^\mu_{,\beta}\right) = 0$$

Now, if $W \equiv 0$ along Σ, since the spatial coordinates are tangent to Σ, $W^\alpha_{,i} \equiv 0$ for any $i \neq 0$, so the only non-trivial components of the Ricci tensor along Σ are:

$$\begin{cases} Ric_{0\alpha} = -\frac{1}{2}\left(g_{\alpha\mu} W^\mu_{,0}\right), \alpha \neq 0 \\ Ric_{00} = -\left(g_{0\mu} W^\mu_{,0}\right), \alpha = 0 \end{cases}$$

Also

$$\begin{aligned} Ric^\alpha_\alpha &= -\tfrac{1}{2}\left(W^\alpha_{,\alpha} + W^\alpha_{,\alpha}\right) \\ &= -W^0_{,0} \end{aligned}$$

Finally, by the constraint equations, $G^0_\alpha = 0$. But

$$\begin{aligned} G^0_\alpha &= Ric^0_\alpha - \tfrac{1}{2}(Ric^\alpha_\alpha) g^0_\alpha \\ &= g^{0\mu} Ric_{\mu\alpha} + \tfrac{1}{2} W^0_{,0} \delta^0_\alpha \end{aligned}$$

If $\alpha \neq 0$:

$$0 = G^0_\alpha = -\frac{1}{2} g^{00} g_{\alpha\nu} W^\nu_{,0}$$

and

$$\begin{aligned} 0 = G^0_0 &= g^{0\mu} Ric_{0\mu} + \tfrac{1}{2} W^0_{,0} \\ &= -\tfrac{1}{2} g^{00} g_{0\nu} W^\nu_{,0} - \tfrac{1}{2} g^{0\mu} g_{\mu\nu} W^\nu_{,0} + \tfrac{1}{2} W^0_{,0} \\ &= -\tfrac{1}{2} g^{00} g_{0\nu} W^\nu_{,0} - \tfrac{1}{2} \delta^0_\nu W^\nu_{,0} + \tfrac{1}{2} W^0_{,0} \\ &= -\tfrac{1}{2} g^{00} g_{0\nu} W^\nu_{,0} \end{aligned}$$

So, if $g^{00} \neq 0$ we the following system for the components $W^\nu_{,0}$:

$$0 = g_{\alpha\nu} W^\nu_{,0}$$

Multiplying by the inverse of g in both sides we get that all time derivatives of the components of W vanish identically. ☐

So the only thing we need to do is to be able to solve the system $W_\alpha = 0, \forall\alpha$ by choosing the components of g and $\partial_0 g$ along Σ in a way that agrees with the initial condition on γ and K.

Notice that there is no restriction to the choice of $\partial_0 g_{0\alpha}$ along Σ, so it is a matter of solving the system of $n+1$ equations

$$0 = -g^{\rho\lambda}\left(g_{\lambda\alpha,\rho} - \frac{1}{2}g_{\rho\lambda,\alpha}\right) + F(g)$$

for $(g_{0\alpha}, g_{0\alpha,0})$. Given the high degree of underdetermination of the system, one can choose $g|_\Sigma$ freely (only assuming $g^{00} \neq 0$) and then solve the resulting linear system above with $F(g) \equiv 0$ to get:

$$\begin{cases} g_{0i,0} = \frac{1}{g^{00}}\left(\frac{1}{2}g^{\rho\lambda}g_{\rho\lambda,i} - \sum_{\rho^2+\lambda^2\neq 0} g^{\rho\lambda}g_{\lambda i,\rho}\right), i \neq 0 \\ g_{00,0} = -\frac{2}{g_{00}}\sum_{\rho^2+\lambda^2\neq 0} g^{\rho\lambda}\left(g_{\lambda\alpha,\rho} - \frac{1}{2}g_{\rho\lambda,\alpha}\right) \end{cases}$$

then W vanishes identically over Σ and we get the existence result for Einstein's Field Equations as a consequence.

In the same work Choquet-Bruhat proves a uniqueness result for the theorem in a similar way, but in the sense that if there is a metric $\tilde{g}$ that also solves the initial value problem for the same initial data on Σ, there is an isometric change of variables on M that is the identity over Σ taking one solution to the other.

Another question related to this problem is that of maximality of the solution. Indeed, there is a maximal solution for each initial value problem (in the sense that other solutions may be isometrically imbedded inside the larger one) as was proved in [3].

Finally, the result concerns only the vacuum case of the field equations. In order to be able to answer the question in the presence of matter, one needs a model for the tensor T, and then the question of existence depends also on the mathematical conditions that determine the energy momentum tensor, being impossible to answer in general. A good point to start covering that direction of the existence theory for the field equations with matter is by looking at the references given in section 2.4 of [12].

Example (Minkowski space)

Let's see how those considerations play out in the case of the flat Minkowski metric in $\mathbb{R}^4$. In that case we can adopt as initial condition $(\Sigma, \gamma) = (\mathbb{R}^3, \eta)$, where η is the Euclidean metric in $\mathbb{R}^3$. In that case $S^\Sigma \equiv 0$, the musical isomorphisms are trivial and the covariant derivative is the normal derivative on the space. In the vacuum case the constraint equations for K become:

$$\begin{cases} (K^k_k)^2 - K^i_j K^j_i = 0 \\ K^k_{i,k} - K^k_{k,i} = 0 \end{cases}$$

if we assume, for example, that the mean curvature of $\mathbb{R}^3$ in our problem is constant and equal to 0, we will get $\sum_{i,j}(K^i_j)^2 = 0$, so $K \equiv 0$ and, by uniqueness, the only solution to Einstein's equation is Minkowski space.

Now, if we follow the steps of the proof of Choquet-Bruhat's theorem for the initial data $(\gamma = \eta, K = 0)$, we have to choose a Lorentzian metric $g_{\alpha\beta}$ over Σ

such that $g_{ij} = \delta_{ij}$ and $\partial_0 g_{\alpha\beta}$ satisfying

$$\begin{cases} \partial_0 g_{ij} = \mathcal{L}_X(g_{ij}) \\ g^{\rho\lambda}\left(g_{\lambda\alpha,\rho} - \frac{1}{2}g_{\rho\lambda,\alpha}\right) = 0 \end{cases}$$

But for any constant vector field X, since γ_{ij} is constant over g, $\mathcal{L}_X\gamma_{ij} = 0$ and we can take $\partial_0 g \equiv 0$ over Σ. Assuming X constant amounts to assuming all the coefficients $g_{\alpha\beta}$ constant, acording to expression 5 for the metric g in a neighborhood of Σ. Summing up, we can solve the initial value problem

$$\begin{cases} Ric = 0, \ in \ M; \\ g_{\alpha\beta} \equiv const., \ on \ \Sigma; \\ \partial_0 g_{\alpha\beta} \equiv 0, \ on \ \Sigma \end{cases}$$

the solution to this initial value problem is, as we know, Minkowski space with metric $g_{\alpha\beta}$ constant and equal to the values at Σ. The different possible choices of $g_{0\alpha}$ are different choices of charts that folliate $\mathbb{R}^4$ with copies of Σ along more or less tilted lines parallel to $X + N\partial_t$ with $N = \sqrt{-g_{00}}$, $X^i = g_{0i}$.

In the opposite direction, if we want $(\mathbb{R}^3, \eta)$ to have a prescribed constant mean curvatura τ inside an Einsteinian spacetime M, for example, we can choose

$$K = \frac{\tau}{3}dx^i \otimes dx^i + \frac{\tau}{\sqrt{3}}\left(dx^1 \otimes dx^2 + dx^2 \otimes dx^1\right)$$

and then

$$\left(K^k_k\right)^2 = K^i_j K^j_i = \tau^2$$

and the pair (η, K) satisfies the constraint equations, so there should be a spacetime M such that (Σ, η, K) is a hypersurface of M satisfying Einstein's equations "bent" with constant mean curvature τ.

4 Conformal Method

Since the constraint equations are a necessary and sufficient condition for initial data for Einstein's Field Equations, an interesting question becomes how to find (γ, K) over Σ satisfying the constraint equations. One prolific way to solve this problem is the conformal method.

The geometric idea behind the method is very simple. First, one fixes an arbitrary Riemannian metric h over Σ. Then, one assumes $\gamma = u^{\frac{4}{n-2}}h$, n the dimension of Σ^3, is a metric conformal to h over Σ and look at the resulting system of PDEs for u obtained by plugging this form of γ in the constraint equations. We will present the calculations involved in the following as presented in section 1.3 of [7].

First, let's assume T is given as the energy-momentum tensor of a scalar field $\psi : M \to \mathbb{R}$, given by

$$T_{ij} = \partial_i\psi\partial_j\psi - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi g_{ij} - V(\psi)g_{ij}$$

[3] In this section the theory can be easily developed without assuming the spacetime is 3+1-dimensional. To get results for the usual theory of General Relativity, simply assume $n = 3$ throughout.

with $V(\psi)$ a scalar potential. Now, calling $\pi := \partial_0\psi$, $\gamma^{ip}\gamma^{js}K_{pj}K_{is} := ||K||^2_\gamma$, the constraint equations become:

$$\begin{cases} S^\Sigma_\gamma + (tr_\gamma K)^2 - ||K||^2_\gamma = 2\pi^2 - g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi g_{00} - 2V(\psi)g_{00} \\ K^k_{i;k} - K^k_{k;i} = \pi\partial_i\psi - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi g_{0i} - V(\psi)g_{0i} \end{cases} \tag{9}$$

Now we want to turn that into an equation only over Σ, so we will assume we are in a coordinate chart in which we can write

$$g = -dt^2 + \gamma$$

with γ a Riemannian metric over Σ as before. Then the equation becomes (since $g_{00} = -1$, $g_{0i} = 0$):

$$\begin{cases} S^\Sigma_\gamma + (tr_\gamma K)^2 - ||K||^2_\gamma = \pi^2 + \gamma^{kl}\partial_k\psi\partial_l\psi + 2V(\psi) \\ K^k_{i;k} - K^k_{k;i} = \pi\partial_i\psi \end{cases} \tag{10}$$

Now we apply the conformal method. Assume ψ, π scalar fields, h a Riemannian metric and K a symmetric bilinear form are all given over Σ and consider $\gamma = u^{\frac{4}{n-2}}h$ satisfies equation 10. To get an equation for u we must see how each component of the equation changes under conformal transformations:

From the theory related to the Yamabe problem (see for example section 1.1 of [7]), we get that

$$S^\Sigma_\gamma = u^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_h u + S^\Sigma_h u\right)$$

Also

$$\gamma_{ij} = u^{\frac{4}{n-2}}h_{ij} \iff \gamma^{ij} = u^{-\frac{4}{n-2}}h^{ij}$$

So

$$\gamma^{kl}\partial_k\psi\partial_l\psi = u^{-\frac{4}{n-2}}\left(h^{kl}\partial_k\psi\partial_l\psi\right)$$

and

$$||K||^2_\gamma = K^i_j K^j_i = \gamma^{ip}\gamma^{jk}K_{jp}K_{ik} = u^{-\frac{4}{n-2}}u^{-\frac{4}{n-2}}h^{ip}h^{jk}K_{jp}K_{ik} = u^{-\frac{8}{n-2}}||K||^2_h$$

So we can already write the Hamiltonian constraint:

$$\begin{aligned} u^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_h u + S^\Sigma_h u\right) &= \pi^2 + \left(u^{-\frac{4}{n-2}}\left(h^{kl}\partial_k\psi\partial_l\psi\right)\right) + 2V(\psi) \\ &\quad + u^{-\frac{8}{n-2}}||K||^2_h - (tr_\gamma K)^2 \end{aligned}$$

Notice we didn't write the change of $tr_\gamma K := \tau$, the mean curvature of Σ in M. The reason is that we are going to consider that as an initial condition of our final system.

With that assumption, the momentum constraint becomes

$$K^k_{i;k} = \pi\partial_i\psi + \tau_{;i} = \pi\partial_i\psi + \partial_i\tau$$

To get the momentum constraint we will have to find the relation between the Christoffel symbols $\tilde{\Gamma}^k_{ij}$ associated to γ and the Γ^k_{ij}, Christoffel symbols associated to h:

$$\begin{aligned}
\tilde{\Gamma}^k_{ij} &= \tfrac{1}{2}\gamma^{kp}(\gamma_{ip,j} + \gamma_{jp,i} - \gamma_{ij,p}) \\
&= \tfrac{1}{2}u^{-\frac{4}{n-2}}h^{kp}((u^{\frac{4}{n-2}}h_{ip})_{,j} + (u^{\frac{4}{n-2}}h_{jp})_{,i} - (u^{\frac{4}{n-2}}h_{ij})_{,p}) \\
&= \tfrac{1}{2}h^{kp}(h_{ip,j} + h_{jp,i} - h_{ij,p}) \\
&\quad + \tfrac{1}{2}u^{-\frac{4}{n-2}}h^{kp}\left(u^{\frac{4}{n-2}}_{,j}h_{ip} + u^{\frac{4}{n-2}}_{,i}h_{jp} - u^{\frac{4}{n-2}}_{,p}h_{ij}\right) \\
&= \Gamma^k_{ij} + \tfrac{4u^{-1}}{2(n-2)}\left(u_{,j}\delta^k_i + u_{,i}\delta^k_j - h^{kp}u_{,p}h_{ij}\right) \\
&= \Gamma^k_{ij} + \tfrac{2u^{-1}}{(n-2)}\left(u_{,j}\delta^k_i + u_{,i}\delta^k_j\right) - \tfrac{2u^{-1}}{(n-2)}h^{kp}u_{,p}h_{ij}
\end{aligned}$$

So

$$\begin{aligned}
(K^k_{i;k})_\gamma &= \gamma^{kp}K_{ip;k} \\
&= \gamma^{kp}\left(K_{ip,k} - \tilde{\Gamma}^s_{ki}K_{sp} - \tilde{\Gamma}^s_{kp}K_{is}\right) \\
&= u^{-\frac{4}{n-2}}h^{kp}\Big((K_{ip;k})_h - \tfrac{2u^{-1}}{(n-2)}K_{sp}\left(u_{,k}\delta^s_i + u_{,i}\delta^s_k\right) \\
&\quad + \tfrac{2u^{-1}}{(n-2)}h^{sl}u_{,l}h_{ik}K_{sp} - \tfrac{2u^{-1}}{(n-2)}K_{is}\left(u_{,k}\delta^s_p + u_{,p}\delta^s_k\right) \\
&\quad + \tfrac{2u^{-1}}{(n-2)}h^{sl}u_{,l}h_{kp}K_{is}\Big) \\
&= u^{-\frac{4}{n-2}}(K^k_{i;k})_h - \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}K^k_i u_{,k} - \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}\left(K^k_k\right)_h u_{,i} \\
&\quad + \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}u_{,l}\delta^p_i K^l_p - \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}K^k_i u_{,k} - \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}K^p_i u_{,p} \\
&\quad + \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}h^k_k u_{,l}K^l_i \\
&= u^{-\frac{4}{n-2}}(K^k_{i;k})_h + (n-2)\tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}K^k_i u_{,k} - \tfrac{2u^{-\frac{2+n}{n-2}}}{(n-2)}\left(K^k_k\right)_h u_{,i} \\
&= u^{-\frac{4}{n-2}}(K^k_{i;k})_h + 2u^{-\frac{2+n}{n-2}}K^k_i u_{,k} - \tfrac{2u^{-1}}{(n-2)}\tau u_{,i}
\end{aligned}$$

Again, we want to treat τ as a given function, although it is a quantity of the target manifold.

Now, in order to get a nicer expression for $K^k_{i;k}$, let's split K into a trace-free and an expansion part with respect to γ in the following way

$$K_{ij} = u^{-2}P_{ij} + \frac{\tau}{n}\gamma_{ij} \iff P^i_j = u^2\left(K^i_j - \frac{\tau}{n}u^{\frac{4}{n-2}}\delta^i_j\right)$$

where $u^{-2}P$ is trace-free with respect to γ.

Then, taking the divergence with respect to h we have

$$\begin{aligned}
\left(K^k_{i;k}\right)_h &= \left((u^{-2}P)^k_{i;k}\right)_h + \left(\left(\tfrac{\tau u^{\frac{4}{n-2}}}{n}h\right)^k_{i;k}\right)_h \\
&= u^{-2}\left(P^k_{i;k}\right)_h + \left(u^{-2}\right)_{,k}P^k_i + \left(\tfrac{\tau u^{\frac{4}{n-2}}}{n}\right)_{,i} \\
&= u^{-2}\left(P^k_{i;k}\right)_h - 2u^{-3}u_{,k}P^k_i + \tfrac{\tau_{,i}}{n}u^{\frac{4}{n-2}} + \tfrac{4}{n(n-2)}\tau u_{,i}u^{\frac{4}{n-2}-1}
\end{aligned}$$

so

$$\begin{aligned}
u^{-\frac{4}{n-2}}\left(K^k_{i;k}\right)_h &= u^{-\frac{2n}{n-2}}\left(P^k_{i;k}\right)_h - 2u^{-\frac{2+n}{n-2}-2}u_{,k}P^k_i \\
&\quad + \tfrac{\tau_{,i}}{n} + \tfrac{4}{n(n-2)}\tau u_{,i}u^{-1}
\end{aligned}$$

Replacing in the expression for the left hand side of the momentum constraint:

$$\begin{aligned}
\left(K^k_{i;k}\right)_\gamma &= u^{-\frac{2n}{n-2}}\left(P^k_{i;k}\right)_h - 2u^{-\frac{2+n}{n-2}-2}u_{,k}u^2\left(K^k_i - \tfrac{\tau}{n}u^{\frac{4}{n-2}}\delta^k_i\right) \\
&\quad + \tfrac{\tau_{,i}}{n} + \tfrac{4}{n(n-2)}\tau u_{,i}u^{-1} + 2u^{-\frac{2+n}{n-2}}K^k_i u_{,k} - \tfrac{2u^{-1}}{(n-2)}\tau u_{,i} \\
&= u^{-\frac{2n}{n-2}}\left(P^k_{i;k}\right)_h + \tau u^{-1}u_{,i}\tfrac{2}{n} + \tfrac{\tau_{,i}}{n} + \tfrac{4}{n(n-2)}\tau u_{,i}u^{-1} - \tfrac{2}{(n-2)}u^{-1}\tau u_{,i} \\
&= u^{-\frac{2n}{n-2}}\left(P^k_{i;k}\right)_h + \tfrac{\tau_{,i}}{n}
\end{aligned}$$

with the magic cancelations justifying the choices of the exponents. Finally, replacing on the expression of the momentum constraint we get

$$u^{-\frac{2n}{n-2}}\left(P^k_{i;k}\right)_h = \pi\partial_i\psi + \frac{(n-1)}{n}\partial_i\tau$$

or, equivalently

$$\left(P^k_{i;k}\right)_h = u^{\frac{2n}{n-2}}\left(\pi\partial_i\psi + \frac{(n-1)}{n}\partial_i\tau\right)$$

Notice that, by now, we have replaced the initial condition K by its trace-free part $u^{-2}P$ and its trace τ. It can be proved that only the divergent-free portion of P is relevant. Indeed, write

$$P = \sigma + H$$

with σ trace-free, divergent-free. Then the divergence of P is the divergence of H.

Now, let's introduce the conformal Killing operator for $X \in \mathfrak{X}(\Sigma)$:

$$(\mathcal{L}_h X)_{ij} = (\nabla_j X)_i + (\nabla_i X)_j - \frac{2}{n}\nabla_k X^k h_{ij}$$

and the conformal Laplacian operator:

$$(\Delta_{h,conf.}X)_i = \nabla_k(\mathcal{L}_h X)^k_i$$

Then it is known that the equation

$$(\Delta_{h,conf.}X) = Y$$

can be solved if Y is orthogonal to all the solutions Z to $\mathcal{L}_h Z = 0$, called conformal Killing fields. On the other hand, it can also be proven that the vector equivalent to the divergence of P (that is, equivalent to the divergence of H) is indeed orthogonal to all conformal Killing fields, so there is $X \in \mathfrak{X}(\Sigma)$ such that $\mathcal{L}_h(X) = H$.

We finally find our desired form of the momentum constraint:

$$(\Delta_{h,conf.}X) = u^{\frac{2n}{n-2}}\left(\pi\partial_i\psi + \frac{(n-1)}{n}\partial_i\tau\right)$$

which has a solution as soon as the vector with coordinates $\pi\partial_i\psi + \frac{(n-1)}{n}\partial_i\tau$ is orthogonal to all conformal Killing fields.

So in the end we get a system of PDEs giving the conformal form of the constraint equations. As a last resort to decoupling the system we define the artificial variable $\tilde{\pi} = u^{\frac{2n}{n-2}}\pi$ and the system becomes:

$$\begin{cases} u^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_h u + S_h^{\Sigma} u\right) & = & u^{-\frac{4}{n-2}}\left(\left(h^{kl}\partial_k\psi\partial_l\psi\right) + \tilde{\pi}^2\right) + 2V(\psi) \\ & & +u^{-\frac{8}{n-2}}||K||_h^2 - \tau^2 \\ (\Delta_{h,conf.}X)_i = \tilde{\pi}\partial_i\psi + u^{\frac{2n}{n-2}}\frac{(n-1)}{n}\partial_i\tau & & \end{cases} \tag{11}$$

with $K = u^{-2}(\sigma + \mathcal{L}_h X) + \frac{\tau}{n}u^{\frac{4}{n-2}}h$.

The initial data for the system is $(h, \sigma, \psi, \tilde{\pi}, \tau, V)$, all defined over Σ, and it should be solved for X and u. If u is found we get our metric γ satisfying the constraint equations along with K.

If τ is constant, the system can be decoupled and the Hamiltonian constraint becomes an elliptic PDE called Lichnerowicz equation. There is a wide range of results concerning the existence of positive solutions to the equation in a variety of settings, usually under some conditions on the initial data, for example on compact manifolds ([8]), on compact manifolds with boundary ([10]), among others. There are also some results on existence and stability of the whole system, such as for assymptotically flat manifolds ([4]) as well as stability results such as ([11]).

Example (Minkowski space) - Second version

Let's use the conformal method to try to find a metric γ that is conformally flat in $\mathbb{R}^3$, that is $h = \eta$ is the Euclidean metric, and that is a possible initial condition for the Einstein equations in the vacuum case, that is $V = \psi = \tilde{\pi} \equiv 0$, with constant mean curvature τ. Say $\gamma = u^{\frac{4}{n-2}}\eta$. In this case the momentum constraint becomes:

$$(\Delta_{\eta,conf.}X)_i = 0 \iff \partial_k(\mathcal{L}_\eta X)_i^k = 0$$

which has a trivial solution $\mathcal{L}_\eta X = 0$. Now, this can be solved, for example $X \equiv 0$ is a trivial solution, but all the information we need to find u is $\mathcal{L}_\eta X$, since:

$$K = u^{-2}P + \frac{\tau}{3}u^{\frac{4}{n-2}}\eta = u^{-2}(\sigma + \mathcal{L}_\eta X) + \frac{\tau}{3}u^{\frac{4}{n-2}}\eta$$

and we can write the Hamiltonian constraint, since $S_\eta \equiv 0, \Delta_\eta = \Delta$:

$$8\Delta u = u^{-3}\left|\left|u^{-2}\sigma + \frac{\tau}{3}\eta\right|\right|_\eta^2 - u^5\tau^2$$

Notice that σ is still to be prescribed. For example, even if we make $\tau = 0$, in which case $K = u^{-2}\sigma$ is trace-free, the solutions to

$$8\Delta u = u^{-7}\sigma_j^i\sigma_i^j$$

are not constants if $\sigma \neq 0$, which does NOT contradict what we discussed in the last example because the metric that must satisfy the constraint equations is not h, but γ. Indeed, that shows that if the second fundamental form is trace-free but non-zero, the Euclidean metric (which would correspond to a constant solution $u \equiv C$), but some deformation of it.

On the other hand, if $\sigma = 0, \tau = 0$ the solutions are harmonic functions, but we are looking for strictly positive solutions $u > 0$, since the conformal factor should not vanish, so the only harmonic solutions bounded from below in the entire $\mathbb{R}^3$ are constant functions. That is more or less a converse to what we discussed in the previous example: if $K \equiv 0$, $S^{\Sigma} \equiv 0$, so the only conformally flat Riemannian metric with zero mean curvature over $\mathbb{R}^3$ is the constant metric.

Another thing we can assume is that K is shear-free, that is $\sigma \equiv 0$. In that case the Lichnerowicz equation becomes

$$8\Delta u = \tau^2 \left(\frac{u^{-3}}{9} - u^5 \right)$$

which solution gives conformally flat metrics with constant mean curvature τ inside their respective spacetime solutions to Einstein Field Equations.

A Appendix: Calculations for Theorem 1

In this section we present the technical manipulations behind the steps of the proof of theorem 1 in section 3.

In a coordinate chart we have that:

$$\begin{aligned} Ric_{\alpha\beta} &= \Gamma^{\rho}_{\beta\alpha,\rho} - \Gamma^{\rho}_{\rho\beta,\alpha} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\rho}_{\alpha\lambda}\Gamma^{\lambda}_{\rho\beta} \\ &= \tfrac{1}{2} g^{\rho\mu} \left(g_{\beta\mu,\alpha\rho} + g_{\alpha\mu,\beta\rho} - g_{\alpha\beta,\rho\mu} \right) \\ &\quad - \tfrac{1}{2} g^{\rho\mu} \left(g_{\rho\mu,\beta\alpha} + g_{\beta\mu,\rho\alpha} - g_{\rho\beta,\mu\alpha} \right) \\ &\quad + \tfrac{1}{2} g^{\rho\mu}_{,\rho} \left(g_{\beta\mu,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu} \right) \\ &\quad - \tfrac{1}{2} g^{\rho\mu}_{,\alpha} \left(g_{\rho\mu,\beta} + g_{\beta\mu,\rho} - g_{\rho\beta,\mu} \right) \\ &\quad + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\rho}_{\alpha\lambda}\Gamma^{\lambda}_{\rho\beta} \end{aligned}$$

If we group all the terms involving zeroth and first derivatives of the metric only in $L_{\alpha\beta}$ we get:

$$Ric_{\alpha\beta} = -\tfrac{1}{2} g^{\rho\mu} g_{\alpha\beta,\rho\mu} + \tfrac{1}{2} g^{\rho\mu} \left(g_{\alpha\mu,\beta\rho} - g_{\rho\mu,\beta\alpha} + g_{\rho\beta,\mu\alpha} \right) + L_{\alpha\beta}$$

so, to prove that equation 7 is indeed hyperbolic we only have to prove that

$$\frac{1}{2}(\mathcal{L}_W g)_{\alpha\beta} = -\frac{1}{2} g^{\rho\mu} \left(g_{\alpha\mu,\beta\rho} - g_{\rho\mu,\beta\alpha} + g_{\rho\beta,\mu\alpha} \right) + H_{\alpha\beta}$$

with $H_{\alpha\beta}$ a polynomial depending only on the components of g and g^{-1} as well as in their first derivatives. In that case we will have that equation 7 becomes:

$$Ric + \frac{1}{2}\mathcal{L}_W g = -\frac{1}{2} g^{\rho\mu} g_{\alpha\beta,\rho\mu} + H_{\alpha\beta} + L_{\alpha\beta} = 0$$

Indeed

$$\frac{1}{2}(\mathcal{L}_W g)_{\alpha\beta} = \frac{1}{2} \left(g_{\beta\mu}(D_\alpha W)^\mu + g_{\alpha\mu}(D_\beta W)^\mu \right)$$

and

$$\begin{cases} (D_\alpha W)^\mu = W^\mu_{,\alpha} + \Gamma^\mu_{\alpha\nu} W^\nu \\ W^\mu = g^{\mu\nu} W_\nu = -g^{\mu\nu} g^{\rho\lambda} \left(g_{\lambda\nu,\rho} - \tfrac{1}{2} g_{\rho\lambda,\nu} \right) + g^{\mu\nu} F(g) \end{cases}$$

so the terms in $(\mathcal{L}_W g)_{\alpha\beta}$ that do not involve derivatives of W and that involve $F(g)$ do not have second order derivatives, let's group those into $\tilde{H}_{\alpha\beta}$. Then

$$\begin{aligned}\tfrac{1}{2}(\mathcal{L}_W g)_{\alpha\beta} &= -\tfrac{1}{2}\left(g_{\beta\mu}\left(g^{\mu\nu}g^{\rho\lambda}\left(g_{\lambda\nu,\rho}-\tfrac{1}{2}g_{\rho\lambda,\nu}\right)\right)_{,\alpha}\right)\\ &\quad -\tfrac{1}{2}\left(g_{\alpha\mu}\left(g^{\mu\nu}g^{\rho\lambda}\left(g_{\lambda\nu,\rho}-\tfrac{1}{2}g_{\rho\lambda,\nu}\right)\right)_{,\beta}\right)\\ &\quad +\tilde{H}_{\alpha\beta}\end{aligned}$$

but $g_{\beta\mu}g^{\mu\nu}=\delta^\nu_\beta$. Also, the two terms of the type

$$g_{\beta\mu}(g^{\mu\nu}g^{\rho\lambda})_{,\alpha}\left(g_{\lambda\nu,\rho}-\frac{1}{2}g_{\rho\lambda,\nu}\right)$$

do not involve second derivatives of the metric, so they can be grouped with $\tilde{H}_{\alpha\beta}$ in a term $H_{\alpha\beta}$ that only involve lower order terms. Then:

$$\begin{aligned}\tfrac{1}{2}(\mathcal{L}_W g)_{\alpha\beta} &= -\tfrac{1}{2}g^{\rho\lambda}\left(g_{\lambda\beta,\rho\alpha}-\tfrac{1}{2}g_{\rho\lambda,\beta\alpha}\right)-\tfrac{1}{2}g^{\rho\lambda}\left(g_{\lambda\alpha,\beta\rho}-\tfrac{1}{2}g_{\rho\lambda,\alpha\beta}\right)+H_{\alpha\beta}\\ &= -\tfrac{1}{2}g^{\rho\lambda}\left(g_{\lambda\beta,\rho\alpha}+g_{\lambda\alpha,\rho\beta}-g_{\rho\lambda,\alpha\beta}\right)+H_{\alpha\beta}\end{aligned}$$

Renaming the indices $\lambda\to\rho$ and $\rho\to\mu$ in the first and third factors and $\lambda\to\mu$ in the second we get the expected expression.

*

We have

$$\begin{cases} A=\tfrac{1}{2}\mathcal{L}_W g\\ D_\mu\left(A-\tfrac{1}{2}(tr_g A)g\right)^{\mu\nu}=0\end{cases}$$

Since $(\mathcal{L}_W g)_{\alpha\beta}=(D_\alpha W)_\beta+(D_\beta W)_\alpha$:

$$\begin{cases} A^{\mu\nu}=\tfrac{1}{2}\left(g^{\mu\lambda}(W^\nu_{,\lambda}+\Gamma^\nu_{\lambda\rho}W^\rho)+g^{\nu\lambda}(W^\mu_{,\lambda}+\Gamma^\mu_{\lambda\rho}W^\rho)\right)\\ (D_\mu A)^{\mu\nu}=A^{\mu\nu}_{,\mu}+\Gamma^\mu_{\mu\lambda}A^{\lambda\nu}+\Gamma^\nu_{\mu\lambda}A^{\mu\lambda}\\ \left(D_\mu(A^\lambda_\lambda g)\right)^{\mu\nu}=A^\lambda_{\lambda,\mu}g^{\mu\nu}+A^\lambda_\lambda g^{\mu\nu}_{,\mu}+\Gamma^\mu_{\mu\rho}A^\lambda_\lambda g^{\rho\nu}+\Gamma^\nu_{\mu\rho}A^\lambda_\lambda g^{\mu\rho}\\ A^\lambda_\lambda=\left(W^\lambda_{,\lambda}+\Gamma^\lambda_{\lambda\rho}W^\rho\right)\end{cases}$$

So, since $A^{\mu\nu}$ is a polynomial on the components of W and their first derivatives which coefficients themselves are polynomials on the components of g, g^{-1} and their first derivatives and that vanishes at $W\equiv 0$, we can write

$$D_\mu\left(A-\frac{1}{2}(tr_g A)g\right)^{\mu\nu}=A^{\mu\nu}_{,\mu}-\frac{1}{2}A^\lambda_{\lambda,\mu}g^{\mu\nu}+\tilde{P}(W)$$

with $\tilde{P}$ a polynomial satisfying said conditions. Also, when we do take the derivatives of A we can reduce the expression to

$$D_\mu\left(A-\frac{1}{2}(tr_g A)g\right)^{\mu\nu}=\frac{1}{2}g^{\mu\lambda}W^\nu_{,\lambda\mu}+\frac{1}{2}g^{\nu\lambda}W^\mu_{,\lambda\mu}-\frac{1}{2}g^{\mu\nu}W^\lambda_{,\lambda\mu}+P(W)$$

with P satisfying the same properties as $\tilde{P}$. If in the third factor we rename the indices $\lambda\to\mu$, $\mu\to\lambda$ we see that it cancels the second factor and we finally get

$$D_\mu\left(A-\frac{1}{2}(tr_g A)g\right)^{\mu\nu}=\frac{1}{2}g^{\mu\lambda}W^\nu_{,\lambda\mu}+P(W)=0$$

as desired.

References

[1] Robert BARTNIK and Jim ISENBERG. The constraint equations. In Piotr T. CHRUŚCIEL and Helmut FRIEDRICH, editors, *The Einstein Equations and the Large Scale Behavior of Gravitational Fields*, pages 1–38. Birkhšauser, Basel, 2004.

[2] Antonio N. BERNAL and Miguel SÁNCHEZ. On smooth cauchy hypersurfaces and geroch's splitting theorem. *Communications in Mathematical Physics*, 243:461–470, 2003.

[3] Yvonne CHOQUET-BRUHAT and Robert GEROCH. Global aspects of the cauchy problem in general relativity. *Communications in Mathematical Physics*, 14(4):329–335, 1969.

[4] James DILTS, Jim ISENBERG, Rafe MAZZEO, and Caleb MEIER. Non-cmc solutions of the einstein constraint equations on assymptotically euclidean manifolds. *Classical and Quantum Gravity*, 31(6):1–10, 2014.

[5] Yvonne FOURÈS-BRUHAT. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Mathematica*, 88, 1952.

[6] R. GEROCH. Domain of dependence. *Journal of Mathematical Physics*, 11:437–449, 1970.

[7] Emmanuel HEBEY. *Compactness and Stability for Nonlinear Elliptic Equations*. 2014.

[8] Emmanuel HEBEY, Frank PACARD, and Daniel POLLACK. A variational analysis of einstein-scalar field lichnerowicz equations on compact riemannian manifolds. *Communications in Mathematical Physics*, 278:117–132, 2008.

[9] Noel J. HICKS. *Notes on Differential Geometry*. 1965.

[10] Michael HOLST and Gantumur TOSGTGEREL. The lichnerowicz equation on compact manifolds with boundary. *Classical and Quantum Gravity*, 30(20), 2013.

[11] Bruno PREMOSELLI. Stability and instability of the einstein-lichnerowicz constraint system. *International Mathematics Research Notices*, 2016:1951–2025, 2016.

[12] Allan D. RENDALL. Local and global existence theorems for the einstein equations. *Living Reiviews in Relativity*, 1998.