MATH 599 PROBLEM SET 1

DUE FRIDAY FEBRUARY 24

In the following, M and N will always be smooth manifolds.

1. Let $A \in \mathbb{R}^{n\times n}$ be an invertible, symmetric matrix. Then in each of the following cases, show that S is a smooth manifold, and determine the dimension of S. Show also that S is compact if A is positive definite.
   (a) $S = \{x \in \mathbb{R}^n : x^\mathsf{T} Ax = 1\} \subset \mathbb{R}^n$.
   (b) $S = \{X \in \mathbb{R}^{n\times n} : X^\mathsf{T} AX = A\} \subset \mathbb{R}^{n\times n}$.
2. Show that each of the following sets admits a canonical smooth structure.
   (a) Tangent bundle $TM = \{(p, v) : p \in M,\ v \in T_pM\}$.
   (b) Cotangent bundle $T^*M = \{(p, \alpha) : p \in M,\ \alpha \in T_p^*M\}$.
   (c) Frame bundle $FM = \{(p, v_1, \dots, v_n) : p \in M,\ \{v_1, \dots, v_n\}$ is an ordered basis of $T_pM\}$. Here n is the dimension of M.
3. Let $M \to N$ be an embedding, and let $X, Y \in \mathfrak{X}(N)$ be vector fields tangent to M. Then show that $[X, Y]$ is also tangent to M.
4. Let g be a covariant 2-tensor on M, and let $X \in \mathfrak{X}(M)$. Compute the Lie derivative $\mathfrak{L}_X g$ in local coordinates.
5. Let $X, Y \in \mathfrak{X}(M)$. Prove the following for Lie derivatives acting on differential forms (focusing on reasonably nontrivial special cases is acceptable).
   (a) $\mathfrak{L}_X(\alpha \wedge \beta) = \mathfrak{L}_X\alpha \wedge \beta + \alpha \wedge \mathfrak{L}_X\beta$
   (b) $\mathfrak{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathfrak{L}_X$
   (c) $\mathfrak{L}_{[X,Y]} = [\mathfrak{L}_X, \mathfrak{L}_Y] := \mathfrak{L}_X \circ \mathfrak{L}_Y - \mathfrak{L}_Y \circ \mathfrak{L}_X$
   (d) $\mathfrak{L}_X = i_X \circ \mathrm{d} + \mathrm{d} \circ i_X$
6. Let $\phi : M \to N$ be a smooth map. Prove the following for the pull-back map (focusing on reasonably nontrivial special cases is acceptable).
   (a) $\phi^*(\alpha \wedge \beta) = \phi^*(\alpha) \wedge \phi^*(\beta)$ for any differential forms α and β on M.
   (b) $\phi^*(\mathrm{d}\alpha) = \mathrm{d}(\phi^*\alpha)$ for any differential form α on M.
7. Prove that the torsion tensor of a connection on M vanishes at $p \in M$ if and only if there exists a local coordinate system in a neighbourhood of p for which the Christoffel symbols of the connection vanish at p.
8. Consider the unit sphere $S^2 \subset \mathbb{R}^3$ with a chart on which the usual spherical coordinates (ϕ, θ) are well defined. Let $\{\partial_\phi, \partial_\theta\}$ be the coordinate frame field corresponding to (ϕ, θ).
   (a) Compute the commutator $[\partial_\phi, \partial_\theta]$.
   (b) Compute the Christoffel symbols of the natural connection on S^2 induced by the embedding $S^2 \subset \mathbb{R}^3$.
   (c) Write down the equation of geodesics, and describe the geodesics.
   (d) Compute all components of the Riemann curvature tensor with respect to $\{\partial_\phi, \partial_\theta\}$.
   (e) Compute the Ricci tensor and the scalar curvature.

Date: Winter 2017.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.