# NAVIER-STOKES FINAL PROJECT

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These notes are meant to provide a summary of Terrence Tao's paper "A Quantative formulation of the global regularity problem for the periodic Navier-Stokes System"[Tao]. This paper proves a well-known but important result that tells us what a solution to the Navier-Stokes problem would have to look like. It is somewhat hard to write a summary of an already concise and well written summary. So instead I have primarily added detail to many of the proofs with the goal of making it easier for someone whos only background is our class [Schu] to understand (in particular making it easier for me to understand). Be warned that I have likely also added errors in the process.

We recall the definition of the navier-stokes system:

**Definition 1.** Given a smooth pressure term $p : [0, \infty) \times \mathbb{T}^3 \to \mathbb{R}$ and a smooth, divergence free initial condition $u_0 : \mathbb{T}^3 \to \mathbb{R}^3$ then $u : [0, \infty) \times \mathbb{T}^3 \to \mathbb{R}^3$ is a solution to the navier-stokes with initial condition $u_0$ if it satisfies

$$\partial_t u = \Delta u + (u \cdot \nabla u) - \nabla p$$
$$\mathrm{div}(u) = 0$$
$$u(0, x) = u_0(x)$$

Of primary interest is finding a global smooth solution (that is a smooth solution that is defined for all $[0, \infty) \times \mathbb{T}^3$) which is a famous open problem. The result of Tao's paper tells us that we can't solve this problem unless we also show 'explicitly' how to control the $H^1_x(\mathbb{T}^3)$ norm of any smooth solution. In other words any proof of existence needs to have some sort of concrete quantative substance to it. Moreover, we could disprove existence by showing a sequence of solutions $u^{(n)}$ uniformally bounded in the $H^1_x(\mathbb{T}^3)$ norm at time 0 but unbounded at later times $0 \leq T^{(n)} < \infty$.

Formally, the main goal of the paper is to show the following

**Theorem 2.** *The following are equivalent:*

(1) There is a global smooth solution to the navier stokes for smooth initial data
(2) There exists a non decreasing function $F : [0, \infty) \to [0, \infty)$ satisfying
$$||u(T)||_{H^1(\mathbb{T}^3)} \leq F(||u_0||_{H^1(\mathbb{T}^3)})$$
for any smooth solution $u$ (with associated smooth $p$) and any $0 < T < \infty$ for which $u$ is defined.
(3) There exists a non decreasing function $G : [0, \infty) \to [0, \infty)$ satisfying
$$||u(T)||_{H^1(\mathbb{T}^3)} \leq G(||u_0||_{H^1(\mathbb{T}^3)})$$
for any smooth solution $u$ (with associated smooth $p$) and any $0 < T \leq 1$ for which $u$ is defined.

Note part 3 is just a less demanding version of part 2.

We can write $F$ more explictly as a function $F : [0,\infty) \to [0,\infty]$ by

$$F(A) := \sup\{||u||_{C_t^0 H_x^1([0,T]\times\mathbb{T}^3)} : ||u_0||_{H_x^1(\mathbb{T}^3)} \leq A; 0 \leq T < \infty\}$$

so that it is non-decreasing and trivially satisfies the desired inequality $||u(T)||_{H^1(\mathbb{T}^3)} \leq F(||u(0,-)||_{H^1(\mathbb{T}^3)})$. The problem can thus be thought of as showing that this $F$ has range $[0,\infty)$. The rate of growth of $F$ is an interesting open problem. Though we know $\frac{F(A)}{A} \to \infty$ as $A \to \infty$ it is not even known if $F(A) \gtrsim A^{1+\epsilon}$

Before we can prove the main theorem we need to recall some facts. We will see that much of these are a slight tweaking of what was already proven in our notes. First we recall that we can rewrite the navier stokes equation to get

$$\partial_t u = \Delta u + \mathbb{P}(\text{div} u \otimes u)$$

and hence by duhamel formula any solution is of the form

$$u(t) = e^{t\Delta} g + \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\text{div} u \otimes u) d\tau$$

We called any $u$ of that form a *mild solution.*

Moreover it suffices to consider the case where $u$ is mean zero since the mean gets preserved by the equations (and so it can be subtracted off). We let $H_0^s(\mathbb{T}^3)$ be the space of mean zero functions in $H^s(\mathbb{T}^3)$. Note that the usual seminorm

$$|f|_{H^s} := \sum_{k\in\mathbb{Z}^3} |k|^{2s} \hat{f}(k)^2$$

is now a norm when restricted to $H_0^s(\mathbb{T}^3)$ thus we will write $||f||_{H_0^s}$ instead of $|f|_{H^s}$ though this is not to be confused with the norm defined in our notes. It is easy to see why this is now a norm because $|f|_{H^s} = 0$ if and only if $\hat{f}(k) = 0$ for all $k \neq 0$ but $\hat{f}(0)$ *is* the mean so it is also zero, therefore $|f|_{H^s} = 0$ if and only if $f = 0$. (we will sometimes write $H_x^1$ to emphasize we are taking the $H_0^1$ norm with respect to the 'space' variables. Unfortunately this overides the 0 in $H_0^1$...)

In this paper we say a mild solution $u$ is a *strong ($H_0^1$) solution* if $u$ is defined on $[0,T]$ and $u \in X_T^1$ where

$$X_T^1 := C_t^0 H_0^1([0,T]\times\mathbb{T}^3) \cap L_t^2 H_0^2([0,T]\times\mathbb{T}^3)$$

which is a banach space with norm

$$||u||_{X_T^1} := ||u||_{C_t^0 H_0^1([0,T]\times\mathbb{T}^3)} + ||u||_{L_t^2 H_0^2([0,T]\times\mathbb{T}^3)}$$

In other words we say $u$ is a strong solution when we have control over its $H^1$ norm for all $t$ and we have control over the *average* of its $H^2$ norm.

The following important result is key to proving the main theorem

**Lemma 3.** *(Local existence) Let $A > 0$ and set $T := \frac{c}{A^4}$ where c is some constant. If $u_0 \in H_0^1(\mathbb{T}^3)$ with $||u_0||_{H_0^1(\mathbb{T}^3)} \leq A$ then there exists a unique strong solution $u \in X_T^1$ with the bound*

$$||u||_{X_T^1} \lesssim A$$

*Moreover the map $\phi : \{u_0 \in H_0^1(\mathbb{T}^3) : ||u_0||_{H_0^1(\mathbb{T}^3)} \leq A\} \to X_T^1$ is lipschitz continuous and finally $u(t)$ is smooth for all $t > 0$ (note we don't need to assume $u_0$ is smooth).*

The proof of this theorem is similar to the 'local existence in 3 dimensions' we've down in class. Many of the ideas are there but just not written with the space $X_T^1$ in mind. The basic idea for showing existence is to define the map $\Phi : X_T^1 \to X_T^1$ by

$$\Phi(u)(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\text{div}u \otimes u)d\tau$$

and show this has a fixed point by the banach fixed point theorem. This amounts to showing $\Phi$ is a well defined contraction mapping. All the necessary control over norms ultimately arises from an energy estimate which we will prove right now (once again it is reminiscent to something in the notes, namely the 'basic energy estimate')

**Lemma 4.** *(Special case of the energy estimate) For $u_0 \in H_0^1(\mathbb{T}^3), u \in X_T^1$ we have $u(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(u \otimes u)d\tau$ lies in $X_T^1$ with*

$$||u(t)||_{X_T^1} \lesssim c'||u_0||_{H_0^1(\mathbb{T}^3)} + C||\mathbb{P}(divu \otimes u)||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)}$$

*for some constatns $c', C$.*

*Proof.* By taking a limit we can assume everything is smooth and hence we can assume that $u$ solves $\partial_t u = -\Delta u + \mathbb{P}(\text{div}u \otimes u)$

then (ignoring some constants arising from the precise definition of the norm) we get

$$\begin{aligned}
\partial_t ||u(t)||^2_{H_0^1} &= \partial_t \int |\nabla u|^2 \\
&= -\int \nabla u \cdot \nabla \partial_t u \\
&= \int \nabla u \cdot \nabla \Delta u - \int \nabla u \cdot \nabla \mathbb{P}(\text{div}u \otimes u) \\
&= \int \partial_i u_j \partial_i \partial_k^2 u_j + \int \nabla^2 u \cdot \mathbb{P}(\text{div}u \otimes u) \\
&= -\int \partial_i^2 u_j \partial_k^2 u_j - \int \nabla^2 u \cdot \mathbb{P}(\text{div}u \otimes u) \\
&\leq -||u(t)||^2_{H_0^2(\mathbb{T}^3)} + a||u(t)||^2_{H_0^2(\mathbb{T}^3)} + a'||\mathbb{P}(\text{div}u \otimes u)||^2_{L^2(\mathbb{T}^3)}, \ a > 0, a' > 0
\end{aligned}$$

where the last line follows from generalized youngs inequality. by choosing the constants of the generalized youngs inequality appropriately we can ensure that we get

$$\partial_t ||u(t)||^2_{H_0^1} \leq c'||u(t)||^2_{H_0^2(\mathbb{T}^3)} + c''||\mathbb{P}(\text{div}u \otimes u)||^2_{L^2(\mathbb{T}^3)}, \ c' > 0, c'' > 0$$

So by integration we get

$$||u(t)||^2_{H_0^1} - ||u_0||^2_{H_0^1} \leq c'||u||^2_{L_t^2 H_0^2([0,T]\times\mathbb{T}^3)} + c''||\mathbb{P}(\text{div}u \otimes u)||^2_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)}$$

for all $t \in [0,T]$ hence we can replace $||u(t)||^2_{H_0^1}$ with $||u||_{C_t^0 H_0^1([0,T]\times\mathbb{T}^3)}$ and the result follows by appropriate algebraic manipulations. □

Now we can show some of the details of Lemma 3

*Proof. (part of Lemma 3)* We will show the the first major part of the proof to get a basic idea of the computations involved. Like we mentioned above define the map $\Phi : X_T^1 \to X_T^1$ by

$$\Phi(u)(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\text{div}u \otimes u)d\tau$$

We would like to show this map has the claimed domain. To this end recall some basic facts, firstly for $f \in L^2[0,T]$ we have

$$\begin{aligned} ||f||_{L^2[0,T]} &= \sqrt{\int_0^T |f|^2 dt} = \sqrt{\int_0^T |f|^2 \cdot 1 dt} \\ &\leq \sqrt{(\int_0^T |f|^4 dt)^{\frac{1}{2}}(\int_0^T 1^2 dt)^{\frac{1}{2}}} \\ &= T^{\frac{1}{4}}(\int_0^T |f|^4 dt)^{\frac{1}{4}} = T^{\frac{1}{4}}||f||_{L^4[0,T]} \end{aligned}$$

(this is just an instance of the well known proof that $L^p[0,T] \subset L^q[0,T]$ for $p \leq q$). We also recall (without proof) another fundamental result; a special case of Sobolev's inequality which says for $f : \mathbb{T}^3 \to \mathbb{R}$ smooth we have

$$||f||_{L^6(\mathbb{T}^3)} \lesssim ||\nabla f||_{L^2(\mathbb{T}^3)}$$

(see [Shko]). We begin with our energy estimate

$$\begin{aligned} ||\Phi(u)||_{X_T^1} &\lesssim ||u_0||_{H_0^1(\mathbb{T}^3)} + ||\mathbb{P}(\text{div}u \otimes u)||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \\ &\leq A + ||\mathbb{P}(\text{div}u \otimes u)||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \end{aligned}$$

To control $||\mathbb{P}(\text{div}u \otimes u)||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)}$ We note that $\mathbb{P}(\text{div}u \otimes u)$ is made up of terms of the form $\partial_i u_j u_i$ (when $\text{div}u = 0$) whereas $|u||\nabla u|$ is made up of terms of the from $u_i \delta_j u_k$ thus we can conclude

$$||\mathbb{P}(\text{div}u \otimes u)||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \lesssim |||u||\nabla u|||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)}$$

using our first fact

$$|||u||\nabla u|||_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \leq T^{\frac{1}{4}}|||u||\nabla u|||_{L_t^4 L_x^2([0,T]\times\mathbb{T}^3)} = c^{\frac{1}{4}}A^{-1}|||u||\nabla u|||_{L_t^4 L_x^2([0,T]\times\mathbb{T}^3)}$$

Now we apply holders inequality (twice) and basic properties of suprenum

$$
\begin{aligned}
&|||u||\nabla u|||_{L_t^4L_x^2([0,T]\times\mathbb{T}^3)} \\
&= \left(\int_0^T (\int_{\mathbb{T}^3} |u|^2|\nabla u|^2 dx)^2 dt\right)^{\frac{1}{4}} = \left(\int_0^T (\int_{\mathbb{T}^3} |u|^2|\nabla u||\nabla u| dx)^2 dt\right)^{\frac{1}{4}} \\
&\leq \left(\int_0^T (\int_{\mathbb{T}^3} |u|^{\frac{12}{5}}|\nabla u|^{\frac{6}{5}} dx)^{\frac{5}{3}} (\int_{\mathbb{T}^3} |\nabla u|^6 dx)^{\frac{1}{3}} dt\right)^{\frac{1}{4}}, \text{ holder with } p = \frac{6}{5}, q = 6 \\
&\leq (\int_0^T (\int_{\mathbb{T}^3} |\nabla u|^6 dx)^{\frac{1}{3}} dt)^{\frac{1}{4}} \cdot \sup_{0\leq t\leq T} (\int_{\mathbb{T}^3} |u|^{\frac{12}{5}}|\nabla u|^6 dx)^{\frac{5}{12}} \\
&= ||\nabla u||_{L_t^2L_x^6}^{\frac{1}{2}} \cdot \sup_{0\leq t\leq T} (\int_{\mathbb{T}^3} |u|^{\frac{12}{5}}|\nabla u|^6 dx)^{\frac{5}{12}} \\
&\leq ||\nabla u||_{L_t^2L_x^6}^{\frac{1}{2}} \cdot \sup_{0\leq t\leq T} (\int_{\mathbb{T}^3} |u|^6 dx)^{\frac{1}{6}} \cdot (\int_{\mathbb{T}^3} |\nabla u|^2)^{\frac{1}{4}}, \text{ hold with } p = \frac{5}{2}, q = \frac{5}{3} \\
&\leq ||\nabla u||_{L_t^2L_x^6}^{\frac{1}{2}} \cdot ||u||_{L_t^\infty L_x^6} \cdot ||\nabla u||_{L_t^\infty L_x^2}^{\frac{1}{2}}
\end{aligned}
$$

Next, using the sobolev inequality mentioned we have

$$
\begin{aligned}
&||\nabla u||_{L_t^2L_x^6}^{\frac{1}{2}} \cdot ||u||_{L_t^\infty L_x^6} \cdot ||\nabla u||_{L_t^\infty L_x^2}^{\frac{1}{2}} \\
&\lesssim ||\nabla^2 u||_{L_t^2L_x^2}^{\frac{1}{2}} \cdot ||\nabla u||_{L_t^\infty L_x^2} \cdot ||\nabla u||_{L_t^\infty L_x^2}^{\frac{1}{2}} \\
&= ||u||_{L_t^2H_x^2}^{\frac{1}{2}} ||u||_{L_t^\infty H_x^1}^{\frac{3}{2}} \leq ||u||_{X_T^1}^{\frac{1}{2}} ||u||_{X_T^1}^{\frac{3}{2}} = ||u||_{X_T^1}^2
\end{aligned}
$$

So we've shown that

$$||\Phi(u)||_{X_T^1} \lesssim A + c^{\frac{1}{4}} A^{-1} ||u||_{X_T^1}^2$$

and so (if $c$ is small enough) there is a constant $C$ so that $\Phi$ maps the ball of radius $CA$ to itself. □

We are now ready to prove most of the main theorem

*Proof.* *($1 \Leftarrow 2$ in Theorem 2)* From 3 we have a smooth solution up to time $T := \frac{c}{||u_0||_{H_0^1}^4}$ and moreover from our assumption we have

$$||u(T)||_{H_x^1(\mathbb{T}^3)} \leq F(||u_0||_{H_x^1(\mathbb{T}^3)})$$

so by plugging in $u(T)$ as our *initial* time we can extend this solution by time $\frac{c}{F(||u_0||_{H_x^1(\mathbb{T}^3)})^4}$ and once again at the end of this time we have the same bound $F(||u_0||_{H_x^1(\mathbb{T}^3)})$. Iterating process we can move forward by intervals of $\frac{c}{F(||u_0||_{Hx^1(\mathbb{T}^3)})^4}$ adinfinum and thus get a global smooth solution. (It's worth contrasting this with the control over $||u(T)||_{H_x^1(\mathbb{T}^3)}$ we get just from Lemma 3 directly which at each step $n$ only allows one to move forward by $\frac{c}{Q^{4n}||u_0||_{H_x^1}^{4n}}, Q > 0$ and hence may converge to some blow up time) □

*Proof.* *($3 \Rightarrow 2$ of Theorem 2)* Let $u : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ be a smooth solution for $0 < T < \infty$. Let $\epsilon > 0$ be a small number depending on $||u_0||_{H_x^1(\mathbb{T}^3)}$ which will be chosen later. To simplify equations let $E = ||u_0||_{H_x^1(\mathbb{T}^3)}$, our *goal* is to gain uniform control over $||u(t)||_{H^1_x(\mathbb{T}^3)}$ in terms of $E$ from time 0 to time $T$, i.e. we want to show:

$$||u||_{C^0_t H^1_x([0,T],\mathbb{T}^3)} \lesssim_E 1$$

Case 1: $T < \frac{1}{\epsilon^2}$ having fixed epislon, we just need to iterate our assumption: we know that from our assumption we get control up to time 1. if $T < 1$ we are done, else we can gain (somewhat worse) control up to time 2 etc... in the end we get

$$||u||_{C^0_t H^1_x([0,T],\mathbb{T}^3)} \lesssim_{E,\epsilon} 1$$

so as long as $\epsilon$ only depends on $E$ (which we will see it does) we are done.

Case 2: $T \geq \frac{1}{\epsilon^2}$. As usual we begin with an energy estimate. In this case it is the 'basic energy estimate' from our notes.

$$\begin{aligned}
\partial_t ||u||^2_{L^2_x(\mathbb{T}^3)} = \partial_t \int |u|^2 dx &= 2\int u\cdot \partial_t u \\
&= 2\int u \cdot \Delta u + 2\int u\cdot(u\cdot\nabla u) - 2\int u\cdot\nabla p \\
&= 2\int u_i \partial_j^2 u_i + 2\int u_i\partial_j(u_j u_i) - 2\int u_i \delta_i p \\
&= -2\int (\partial_j u_i)^2 + 2\int u_i(d_j u_j)u_i + 2\int u_i u_j \partial_j u_i + 2\int \partial_i u_i p \\
&= -2\int |\nabla u|^2 + 0 - 2\int u_i \partial_j(u_j u_i) + 0 \\
&= -2\int |\nabla u|^2 = -2||\nabla u||^2_{L^2_x(\mathbb{T}^3)}
\end{aligned}$$

(where the second last line follows from the fact that div$u = 0$ and the last from the fact that we've shown $2\int u_i\partial_j(u_j u_i) = -2\int u_i \partial_j(u_j u_i)$)

By integration and taking a sup over all t we thus get

$$||u||_{L^\infty_t L^2_x([0,T]\times\mathbb{T}^3)} + ||\nabla u||_{L^2_t L^2_x([0,T]\times\mathbb{T}^3)} \lesssim ||u_0||_{L^2_x(\mathbb{T}^3)} \leq ||u_0||_{H^1_x(\mathbb{T}^3)}$$

Where the last inequality just the observation that $\sum_{k\in\mathbb{Z}^3} |\hat{u_0}(k)|^2 \leq \sum_{k\in\mathbb{Z}^3} k^2|\hat{u_0}(k)|^2$ since the mean of $u_0$ is $\hat{f}(0) = 0$ (or just a special case of poincare's inequality). Now recalling our definition of $E$ we rewrite this as

$$||u||_{L^\infty_t L^2_x([0,T]\times\mathbb{T}^3)} + ||\nabla u||_{L^2_t L^2_x([0,T]\times\mathbb{T}^3)} \lesssim_E 1$$

in particular $||\nabla u||_{L^2_t L^2_x([0,T]\times\mathbb{T}^3)} \lesssim_E 1$ and hence there exists a $0 \leq T' \leq \frac{1}{\epsilon^2}$ so that

$$||\nabla u(T')||_{L^2_x(\mathbb{T}^3)} \lesssim_E \epsilon$$

(since otherwise we would have $||\nabla u(T')||_{L^2_t L^2_x([0,\frac{1}{\epsilon^2}],\mathbb{T}^3)} \gtrsim_E ||\epsilon||_{L^2_t([0,\frac{1}{\epsilon^2}])} = \epsilon^2 \cdot \frac{1}{\epsilon^2} = 1$). So

$$||u(T')||_{H^1_x(\mathbb{T}^3)} \lesssim_E \epsilon$$

we will show that having this arbitrary control allows us to extend our control to an arbitrary time. Lets start by extending our control to $\min(T'+1, T)$ we will just assume $T'+1 < T$ as the other case holds by identical computations. By triangle inequality

$$||u(T'+1)||_{H^1_x(\mathbb{T}^3)} \leq ||e^{\Delta}u(T')||_{H^1_x(\mathbb{T}^3)} + ||u(T'+1) - e^{\Delta}u(T')||_{H^1_x(\mathbb{T}^3)}$$
$$= ||e^{\Delta}u(T')||_{H^1_x(\mathbb{T}^3)} + ||\int_{T'}^{T'+1} e^{(T'+1-\tau)\Delta}\mathbb{P}(\text{div}u \otimes u)dt||_{H^1_x(\mathbb{T}^3)}$$

The first term is bounded by

$$||e^{\Delta}u(T')||_{H^1_x(\mathbb{T}^3)} = \sum_{k\in\mathbb{Z}^3} |k|^2 e^{-|k|^2}\widehat{u(T')}(k) \leq e^{-1}\sum_{k\in\mathbb{Z}^3}|k|^2\widehat{u(T')}(k) = e^{-1}||u(T')||_{H^1_x(\mathbb{T}^3)}$$

To bound the second term is more involved. If $\epsilon$ is small enough we can apply Lemma 3 with initial condition $u(T')$ to get

$$||u||_{C^0_t H^1_x([T',T'+1]\times\mathbb{T}^3)} + ||u||_{L^2_t H^2_x([T',T'+1]\times\mathbb{T}^3)} \lesssim ||u(T')||^2_{H^1_x(\mathbb{T}^3)}$$

In fact the details of the proof of that lemma give that

$$||\mathbb{P}(\text{div}u\otimes u)||_{L^2_t L^2_x([T',T'+1]\times\mathbb{T}^3)} \lesssim_E ||u(T')||^2_{H^1_x(\mathbb{T}^3)}$$

We also recall from our notes that any $t \in [0,\infty)$ and $s \geq 1$ with $f \in H^s(\mathbb{T}^3)$ we get

$$||e^{t\Delta}f||_{H^s(\mathbb{T}^3)} \lesssim (1+t^{-\kappa})||f||_{H^{s-1}(\mathbb{T}^3)}, 0<\kappa<1$$

combining this we have

$$||\int_{T'}^{T'+1} e^{(T'+1-\tau)\Delta}\mathbb{P}(\text{div}u\otimes u)dt||_{H^1_x(\mathbb{T}^3)} \leq \int_{T'}^{T'+1} ||e^{(T'+1-\tau)\Delta}\mathbb{P}(\text{div}u\otimes u)||_{H^1_x(\mathbb{T}^3)}dt$$
$$\lesssim \int_{T'}^{T'+1}(1+(T'+1-\tau)^{-\kappa})||\mathbb{P}(\text{div}u\otimes u)||_{L^2_x(\mathbb{T}^3)}dt$$
$$\lesssim ||\mathbb{P}(\text{div}u\otimes u)||_{L^2_x(\mathbb{T}^3)}$$
$$\lesssim_E ||u(T')||^2_{H^1_x(\mathbb{T}^3)}$$

So for some constant $C_E$ depending on $E$ we've shown

$$||u(T'+1)||_{H^1_x(\mathbb{T}^3)} \leq e^{-1}||u(T')||_{H^1_x(\mathbb{T}^3)} + C_E||u(T')||^2_{H^1_x(\mathbb{T}^3)}$$

but since $||u(T')||_{H^1_x(\mathbb{T}^3)} \lesssim_E \epsilon$, if $\epsilon$ is small enough (depending on $E$) we get

$$e^{-1}||u(T')||_{H^1_x(\mathbb{T}^3)} + C_E||u(T')||^2_{H^1_x(\mathbb{T}^3)} \leq ||u(T')||_{H^1_x(\mathbb{T}^3)}$$

so

$$||u(T'+1)||_{H^1_x(\mathbb{T}^3)} \leq ||u(T')||_{H^1_x(\mathbb{T}^3)}$$

We can control up from $T'$ to $T'+1$ just using by Lemma 3 (since we get $||u||_{C^0_t H^1_x([T',T'+1]\times\mathbb{T}^3)} \lesssim ||u(T')||^2_{H^1_x(\mathbb{T}^3)}$), but now with the above inequality we can iterate this procedure to get control from $T'$ to $T$ i.e.

$$||u||_{C^0_t H^1_x([T',T],\mathbb{T}^3)} \lesssim_E 1$$

and finally since $T' \leq \frac{1}{\epsilon^2}$ we can gain control from 0 to $T'$ by case 1,

$$||u||_{C^0_t H^1_x([0,T],\mathbb{T}^3)} \lesssim_E 1$$

□

for the last stage of the theorem we need to recall some basic notions of convergence and a technical lemma relating them

**Definition 5.** Given an element $u \in H^1(\mathbb{T}^3)$ and sequence $\{u^{(n)}\}_{n\in\mathbb{N}} \subset H^1(\mathbb{T}^3)$, we say $u^{(n)}$ *converges weakly to* $u$ *in* $H^1(\mathbb{T}^3)$ provided that $u^{(n)}$ converges weakly to $u$ in $L^2(\mathbb{T}^3)$ and $\nabla u^{(n)}$ converges weakly to $\nabla u$ in $L^2(\mathbb{T}^3)$ (in the usual sense viewing $L^2(\mathbb{T}^3)$ as the dual banach space to itself). It's easy to see that weak convergences preserves the mean providing a notion of weak convergence in $H^1_0(\mathbb{T}^3)$ [Shko].

**Lemma 6.** *As in Lemma 3 we let $A > 0$ and set $T := \frac{c}{A^4}$. If $u_0 \in H^1_0(\mathbb{T}^3)$ with $||u_0||_{H^1_0(\mathbb{T}^3)} \leq A$ and $u \in X^1_T$ is the corresponding strong solution from Lemma 3 and if $\{u_0^{(n)}\}_{n\in\mathbb{N}} \subset H^1_0(\mathbb{T}^3)$ is a sequence with $||u_0^{(n)}||_{H^1_0(\mathbb{T}^3)} \leq A$ which converges weakly to $u_0$ in $H^1_0(\mathbb{T}^3)$ then for any $0 < \epsilon < T$, $u^{(n)}$ converges strongly in $C^0_t H^1_0([\epsilon, T]\times\mathbb{T}^3)$ to $u$.*

*Proof. (very quick glimpse)* Let $v^{(n)} := u^{(n)} - u$ so

$$v^{(n)}(t) = e^{t\Delta}v^{(n)}(0) + \int_0^t e^{(t-\tau)\Delta}(\mathbb{P}(\text{div}(v^{(n)} \otimes u)) + \mathbb{P}(\text{div}(u^{(n)} \otimes v^{(n)}))d\tau$$

We can thus bound $v^{(n)}(t)$ in a similar manner to the computations done in Lemma 3. In particular the energy estimate can be made more general (as it is done in Tao's paper). In fact, the proof of this more general result is virtually identical to what we proved here. In this case it gives us

$$||v^{(n)}||_{C^0_t L^2_s} + ||v^{(n)}||_{L^2_t H^1_0} \lesssim ||v^{(n)}(0)||_{L^2_x} + ||\mathbb{P}(\text{div}(v^{(n)} \otimes u)) + \mathbb{P}(\text{div}(u^{(n)} \otimes v^{(n)})||_{L^2_t H^{-1}_0}$$

and from here we go through some similar (though messier) computations to get $v^{(n)}$ converging strongly in $C^0_t H^1_0([\epsilon, T] \times \mathbb{T}^3)$ to 0. □

We can now finish the proof of Theorem 2

*Proof. (1 ⇒ 3 of Theorem 2)* Suppose that property 3 was false. We have there is some $A > 0$ so that *for all* non decreasing functions $G : [0,\infty) \to [0,\infty)$ and any smooth mean zero solution $u^{(g)} : [0, T^{(g)}] \times \mathbb{T}^3 \to \mathbb{R}^3$ (with associated $p^{(g)} : [0, T^{(g)}] \times \mathbb{T}^3 \to \mathbb{R}$), $0 < T^{(g)} \leq 1$ with $||u_0^{(g)}||_{H^1(\mathbb{T}^3)} \leq A$ so that

$$||u^{(g)}(T^{(g)})||_{H^1_0(\mathbb{T}^3)} > G(||u_0^{(g)}||_{H^1_0(\mathbb{T}^3)})$$

That is to say for some $A$, if the initial condition is bounded by $A$ this is *not* enough to control later times. Hence we can find a sequence of smooth mean zero solutions $u^{(n)} : [0, T^{(n)}] \times \mathbb{T}^3 \to \mathbb{R}^3$,$0 < T^{(g)} \leq 1$ with $||u_0^{(n)}|| \leq A$ for which

$$\lim_{n\to\infty} ||u^{(n)}(T^{(n)})||_{H^1_0(\mathbb{T}^3)} = \infty$$

by passing to a subsequence we may assume

$$\lim_{n\to\infty} T^{(n)} = T$$

where $0 \leq T \leq 1$. Moreover if $T = 0$ then for $T^{(n)}$ suffienciently small we can just apply lemma 3 to control $||u^{(n)}(T^{(n)})||_{H^1_0(\mathbb{T}^3)}$ which is a contradiction thus we may assume $T > 0$.

Now $||u_0^{(n)}||_{L^2_x(\mathbb{T}^3)} \leq ||u_0^{(n)}||_{H^1_x(\mathbb{T}^3)} = ||\nabla u_0^{(n)}||_{L^2_x(\mathbb{T}^3)} \leq A$ so using the Banach-Alaoglu theorem twice we can pass to a subsequence so that both $u^{(n)}$ and $\nabla u_0^{(n)}$ converge weakly in $L^2(\mathbb{T}^3)$. Therefore by definition $u_0^{(n)}$ converges weakly to some $u_0$ in $H^1_0(\mathbb{T}^3)$. Thus by Lemma 3 we can extend $u_0$ to $[0, T']$ for some $T'$ and moreover $u(t)$ for $t > T'$ is smooth, therefore by our assumption applied to the smooth initial data $u(T')$ we can extend $u_0$ to a global smooth solution $u$. In particular

$$A' := ||u(t)||_{C_t^0 H_x^1([0,2T]\times\mathbb{T}^3)} < \infty$$

So if we let $A'' = \max\{A', A\}$ then we have $||u_0^{(n)}||_{H_0^1(\mathbb{T}^3)}$ and $||u_0||_{H_0^1(\mathbb{T}^3)} \leq A''$ so we can iterate Lemma 6 to get $u^{(n)}$ converges strongly to $u$ in $C_t^0 H_0^1([\frac{T}{2}, 2T] \times \mathbb{T}^3)$. In particular for $n$ large since $T^{(n)} \in (\frac{T}{2}, 2T)$ so we have

$$||u^{(n)}(T^{(n)}) - u(T^{(n)})||_{H_0^1(\mathbb{T}^3)} \to 0$$

and so

$$||u^{(n)}(T^{(n)})||_{H_0^1(\mathbb{T}^3)} \leq 2A'$$

which is a contradiction. □

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