MATH 595: Topics in Analysis
Introduction to the Navier-Stokes Equations

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Fall 2014, Mcgill University
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Chapter 1

Introduction

In this course, we will study the Navier-Stokes equations for a viscous incompressible newtonian fluid under neglected temperature effects given by

$$\rho\left(\partial_t u + u \cdot \nabla u\right) = -\nabla p + \nu \Delta u + f$$

and

$$\nabla u = 0.$$

We start with some nomenclature. We refer to the above system of two equations as the Navier-Stokes equations. The top equation can be called the momentum equation, and the incompressibilty condition $\nabla u = 0$ is a special case of what we call the continuity equation. The function $u : \Omega \times [0, T) \to \mathbb{R}^n$ is associated to the flow velocity, while $f : \Omega \times [0, T) \to \mathbb{R}^n$ represents the external force (also called the body forces), $\rho \in \mathbb{R}$ the fluid density and $p : \Omega \times [0, T) \to \mathbb{R}$ the pressure. The images of u and p are respectively called the velocity field and the pressure field. Throughout our investigation, we will generally assume that $\rho = 1$.

An open mathematical question is the determination of a well-posed boundary value problem. It was proven for $n = 2$ that the equations must be completed, for $t > 0$ and $x \in \Omega$, by the initial condition

$$u(x, 0) = u_0(x),$$

where u_0 is given, and

$$u(x, t)\big|_{\partial\Omega} = \phi(x, t)$$

with ϕ given. This is also believed to be true for higher dimensional velocity field. In this course, we will favor the case $u\big|_{\partial\Omega} = 0$.

Chapter 2

The Heat Equation

2.1 The Homogeneous Heat Equation: From One to Higher Dimensions

The heat equation is given by

$$\partial_t u = \partial_x^2 u$$

and

$$u(x,0) = g(x).$$

The former equation is often refered to by (H). We will consider periodic boundary conditions, thus functions defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1$. In that sense, we further understand L^2 as $L^2(\mathbb{T})$.

We may solve (H) by the separation of variables method. We find solutions of the form

$$u(x,t) = Ae^{-k^2 t} e^{ikx}, k \in \mathbb{Z}.$$

By superposition,

$$u(x,t) = \sum_{finite} A_k e^{-k^2 t} e^{ikx}$$

also solves (H). With these solutions, we understand that we can find solutions with initial conditions of the form

$$g(x) = \sum_{finite} A_k e^{ikx}.$$

We may then ask the following classical question: when does $\sum_{k\in\mathbb{N}} A_k e^{-k^2 t} e^{ikx}$ solve (H)? We will seek an answer by concidering the following three smaller problems:

1. When does the Fourier series $\sum_{k\in\mathbb{N}} a_k e^{ikx}$ converge?
2. When it does, when is its derivative term by term? That is, when do we have
$$\frac{\partial}{\partial x}\left(\sum_{k\in\mathbb{Z}} a_k e^{ikx}\right) = \sum_{k\in\mathbb{N}} ika_k e^{ikx} \ ?$$
3. What are the required conditions to have
$$\frac{\partial}{\partial t}\left(\sum_{k\in\mathbb{Z}} a_k(t) e^{ikx}\right) = \sum_{k\in\mathbb{N}} a_k'(t) e^{ikx} \ ?$$

2.1.1 L^2-Theory of Fourier Series

In this section, we will adress question 1. Let $e_k(x) := e^{ikx}$ be defined. Then, we have

$$\langle e_k, e_m\rangle_{L^2} = \int_0^{2\pi} e_k(x)\overline{e_m(x)}dx = 2\pi\delta_{km}.$$

It follows that for $f_m = \sum_{|k|\le m} a_k e_k$, we obtain

$$\|f_m\|^2 = \langle \sum_{|k|\le m} a_k e_k, \sum_{|j|\le m} a_j e_j\rangle = 2\pi \sum_{|k|\le m} |a_i|^2.$$

Hence, if $a = (a_k)_{k\in\mathbb{Z}} \in \ell^2$, that means we have $\|f_m\| \le \sqrt{2\pi}\|a\|_{\ell^2}$. Thus w.l.o.g., assuming $k \le m$, we conclude from the fact that

$$\|f_m - f_k\|^2 = 2\pi \sum_{k<|j|\le m} |a_i|^2 \le 2\pi \sum_{k<|j|} |a_i|^2 \longrightarrow 0 \ \text{ as } \ k, m \to \infty,$$

that $(f_m)_{m\in\mathbb{Z}}$ is Cauchy in L^2, and so that $\exists f \in L^2$ s.t. $f_m \underset{L^2}{\longrightarrow} f$ as $m \to \infty$.

Proposition 1. *If $a = (a_k)_{k\in\mathbb{Z}} \in \ell^2$, then $\sum_{k\in\mathbb{Z}} a_k e^{ikx}$ converges in $L^2(\mathbb{T})$ to some $f \in L^2(\mathbb{T})$ and we have $\|f\| = \sqrt{2\pi}\|a\|_{\ell^2}$.*

Conversly, let $f \in L^2$ and suppose that $f_m \underset{L^2}{\longrightarrow} f$ as $m \to \infty$, i.e. $f = \sum_{k\in\mathbb{Z}} a_k e_k$. Then $\langle f - f_m, e_j\rangle \longrightarrow 0$ as $m \to \infty$, but $\langle f - f_m, e_j\rangle = \langle f, e_j\rangle - 2\pi a_j$ does not depend on $m \in \mathbb{Z}$, and so we must have $\langle f, e_j\rangle - 2\pi a_j = 0$, $\forall j \in \mathbb{Z}$. Hence,

$$a_k = \frac{1}{2\pi}\langle f, e_k\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{e_k}dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx.$$

Moreover, we observe that $\langle f - f_m, e_j\rangle = 0$ also implies that

$$f - f_m \perp \Sigma_m := \operatorname{span}\{e_k : |k| \le m\},$$

i.e. $f - f_m$ is orthogonal to Σ_m. The pythagorean identity hence yields

$$\|f\|^2 = \|f - fm\|^2 + \|f_m\|^2,$$

which further leads to the conclusion that

$$\|f - f_m\| = \inf_{g\in\sum_m} \|f - g\|. \tag{2.1.1}$$

So in fact, if $f \in L^2$ and f_m is defined as above, $f_m \longrightarrow f$ as $m \to \infty$. Indeed, recall the two following facts:

- ⋄ $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$;
- ⋄ $\cup_{m\in\mathbb{Z}}\Sigma_m$ are dense in $C(\mathbb{T})$ with respest to the L^∞-norm;

and let $\epsilon > 0$. Choose $g \in C$ such that $\|f - g\| < \epsilon$ and m large enough so that we may find $h \in \Sigma_m$ such that $\|g - h\|_\infty < \epsilon$. Then,

$$\begin{aligned} \|f - f_m\|_{L^2} &\le \|f - h\|_{L^2} \\ &< \epsilon + \sqrt{2\pi}\|g - h\|_\infty \\ &= (1 + \sqrt{2\pi})\epsilon, \end{aligned} \tag{2.1.2}$$

where (2) holds from (1).

Proposition 2. *If $f \in L^2(\mathbb{T})$, then $f = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\langle f, e_k\rangle e_k$ and $\|f\|^2 = \frac{1}{2\pi}\sum_{k\in\mathbb{N}} |\langle f, e_k\rangle|^2$, i.e. any $L^2(\mathbb{T})$ function can be written as a Fourier series converging in L^2 and its coefficients may be given explicitely.*

2.1.2 Weak Derivatives and Sobolev Spaces

We will now be concerned with question 2. Consider a function $f \in L^2(\mathbb{T}) \cap C^1(\mathbb{T})$ and define $\hat{f}(k) := \frac{1}{2\pi}\langle f, e_k\rangle$. If we compute the coefficients of $g = f'$ using integration by parts, we obtain

$$\hat{g}(k) = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}dx = \frac{ik}{2\pi}\int_0^{2\pi} f(x)\overline{e_k}dx = ik\hat{f}(k).$$

Hence,

$$f' = g = \sum_{k\in\mathbb{Z}} \hat{g}(k)e_k = \sum_{k\in\mathbb{Z}} ik\hat{f}(k)e_k,$$

which is the term by term derivative of $f = \sum_{k\in\mathbb{Z}} \hat{f}(k)e_k$, i.e. $f'_m = \sum_{|k|\le m} ik\hat{f}(k) \longrightarrow f'$ in L^2 as $m \to \infty$.

Now, for an arbitrary $f \in L^2(\mathbb{T})$, proposition 1 implies that if $(k\hat{f}(k))_{k\in\mathbb{Z}} \in \ell^2$, then $\sum_{k\in\mathbb{Z}} ik\hat{f}(k)e_k$ converges to some function $h \in L^2(\mathbb{T})$. The above computations further show that if $h \in C^1$, then h could be viewed as the classical derivative. In general though, it is not the case that for any $f \in L^2$, the sequence of finite sums of the term by term derivatives converges in the analogous way. We therefore identify h differently if it exists and call it the strong derivative (or the Weyl derivative) of f.

Definition (**Strong Derivative**). Let $f \in L^2$ and $\{f_1, f_2, ...\} \subset C^1$ s.t. $f'_k \longrightarrow h$ and $f_k \to f$ in L^2 as $k \to \infty$, then we call h the strong derivative of f.

Exercise. The strong derivative doesn't depend on the approximating sequence of C^1 function.

Definition (**Weak Derivative**). If $f, h \in L^2(\mathbb{T})$ satisfy $\langle h, v\rangle = -\langle f, v'\rangle$ for any test function v, i.e. $\forall v \in C^1(\mathbb{T})$, then we say that h is a weak derivative of f.

Proposition 3. *A function $h \in L^2(\mathbb{T})$ is a weak derivative of f if and only if it is a strong derivative of f.*

Proof. Let h be the strong derivative of f and $v \in C^1(\mathbb{T})$. Integration by parts yield

$$\begin{aligned}\langle h, v\rangle = \langle\sum_{k\in\mathbb{Z}} ik\hat{f}(k)e_k, v\rangle = \sum_{k\in\mathbb{Z}} \hat{f}(k)\langle ike^{ikx}, v\rangle \\ = \sum_{k\in\mathbb{Z}} \hat{f}(k)\int_0^{2\pi}(e^{ikx})'\bar{v}dx = -\sum_{k\in\mathbb{Z}} \hat{f}(k)\int_0^{2\pi} e^{ikx}\bar{v}'dx = -\langle f, v'\rangle.\end{aligned}$$

Conversely, if $h \in L^2(\mathbb{T})$ is a weak derivative of f, then

$$\hat{h}(k) = \frac{1}{2\pi}\langle h, e_k\rangle = \frac{1}{2\pi}\langle f, e'_k\rangle = \frac{ik}{2\pi}\langle f, e_k\rangle = ik\hat{f}(k).$$

$\square$

Definition (**Sobolev Space**). For $s \ge 0$ in $\mathbb{N}$, $H^s(\mathbb{T}) = \left\{f \in L^2(\mathbb{T}) : \hat{f} \in \ell_s^2\right\}$, where

$$\ell_s^2 = \left\{a \in \ell^2 : (|k|^s a_k)_{k\in\mathbb{Z}} \in \ell^2\right\},$$

is called a Sobolev Space.

Exercise. $H^s(\mathbb{T})$ is a Hilbert Space with respect to the inner product

$$\langle f, g\rangle_s = \sum_{k\in\mathbb{Z}} \left(1 + |k|^{2s}\right)\hat{f}(k)\overline{\hat{g}(k)}.$$

We adopt the following notation:

$$|f|_s = \sqrt{\sum |k|^{2s}|\hat{f}(k)|^2}$$

is a semi-norm and

$$\|f\|_s = \sqrt{\|f\|_{L^2}^2 + |f|_s^2}$$

is the induced norm on $H^s(\mathbb{T})$. Furthermore, for a sequence $(a_k)_{k\in\mathbb{Z}} \in \ell_s^2$, we define

$$\|a_k\|_{\ell_s^2} = \sum_{k\in\mathbb{Z}} \left(1 + |k|^{2s}\right) |a_k|^2.$$

Notice that

$$\langle f, g\rangle = \langle \sum_{k\in\mathbb{Z}} \hat{f}(k)e_k, \sum_{k\in\mathbb{Z}} \hat{g}(k)e_k\rangle = 2\pi \sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)}$$

and

$$\langle f^{(s)}, g^{(s)}\rangle = 2\pi \sum_{k\in\mathbb{Z}} \hat{f^{(s)}}(k)\overline{\hat{g^{(s)}}(k)} = 2\pi \sum_{k\in\mathbb{Z}} |k|^{2s}\hat{f}(k)\overline{\hat{g}(k)}.$$

Hence, we find that the inner products associated to $L^2(\mathbb{T})$ and $H^s(\mathbb{T})$ are related by

$$\langle f, g\rangle_s = \frac{1}{2\pi}\left(\langle f, g\rangle_{L^2} + \langle f^{(s)}, g^{(s)}\rangle_{L^2}\right).$$

Other interesting facts are that for $f \in H^s(\mathbb{T})$ and $s \geq 0$, if $f_m \to f$ in L^2, then $f_m \longrightarrow f$ in $|\cdot|_s$ and

$$\|f - f_m\|_{L_2}^2 \leq m^{-s}|f|_s.$$

Indeed,

$$\|f - f_m\|^2 = 2\pi \sum_{|k|>m} |\hat{f}(k)|^2 \leq m^{-2s} \sum_{|k|\geq m} |k|^{2s}|\hat{f}(k)|^2 = m^{-2s}|f|_s^2,$$

and $|f - f_m|_s^2 = \sum_{|k|>m} |k|^{2s}|\hat{f}(k)|^2 \longrightarrow 0$ as $m \to \infty$, since $f \in H^s(\mathbb{T}) \implies \hat{f} \in \ell_s^2$ by definition.

2.1.3 Differentiation in Banach Spaces

We will now answer the third and last question. Let $u(x,t) = \sum_{k\in\mathbb{Z}} a_k e^{-k^2 t} e^{ikx}$ and recall that we are investigating the equation $\partial_t u = \partial_x^2 u$ for solutions. Observe that in the weak sense of the derivative,

$$\partial_x^2 u = \sum_{k\in\mathbb{Z}} -k^2 a_k e^{-k^2 t} e^{ikx},$$

and if we were to differentiate w.r.t. t term by term,

$$\partial_t u = \sum_{k\in\mathbb{Z}} -k^2 a_k e^{-k^2 t} e^{ikx}.$$

2.1.3.1 The Generalized Derivative

In this short section, we define what we mean by "derivative" when discussing vector-valued functions. Otherwise specified, the generalized derivative will be loosely called the "derivative", because it is the natural extension of the latter to Banach spaces.

Definition (Generalized Derivative). Let X be a Banach space and $I \subset \mathbb{R}$ be an interval. A function $f : I \to X$ is said to be differentiable at a point $a \in I$ if

$$f(t) - f(a) = \lambda(t-a) + o\left(|t-a|\right)$$

as $t \to a$ for $\lambda \in X$. We write $f'(a) = \frac{df}{dt}(a) = \lambda$ and call $f'(a)$ the derivative of f at a.

Proposition 4. *Let $U : I \to H^s(\mathbb{T})$ be differentiable at $b \in I$, then $U'(b) = \sum_{k\in\mathbb{Z}} a_k'(b) e_k$, where*

$$a_k(t) = U\hat{(t)}(k) = \frac{1}{2\pi}\langle U(t), e_k\rangle$$

are the Fourier coefficients of $U(t)$.

Proof.

$$U'\hat{(b)} = \frac{1}{2\pi}\langle U'(b), e_k\rangle = \frac{1}{2\pi}\langle \lim_{t\to 0}\frac{U(b+t)-U(b)}{t}, e_k\rangle = \frac{1}{2\pi}\lim_{t\to 0}\frac{\langle U(b+t), e_k\rangle - \langle U(b), e_k\rangle}{t}$$
$$= \lim_{t\to 0}\frac{a_k(b+t)-a_k(t)}{t} = a_k'(b)$$

□

Remark. Lemma 4 also proves that the a_k coefficients are differentiable, as it shows that their derivatives exist. Moreover, observe that if $U : I \to H^s$, $s \geq 0$, is differentiable, then it is differentiable as a function in $U : I \to L^2$. It is convinient to succintly remember the result in Lemma 4 as the commutativity statement that $\widehat{\frac{d}{dt}U(t)} = \frac{d}{dt}\widehat{U(t)}$ if U is assumed differentiable.

Proposition 5. *Let $u \in C^1(I \times \mathbb{T})$, where $I \subset \mathbb{R}$ is an open interval and define both $U(t) = u(\cdot, t)$ and $V = \partial_t u(\cdot, t)$. Then, in $L^2(\mathbb{T})$,*

$$U'(t)(x) = V(t)(x) \;\; \forall t \in I.$$

Proof. Let

$$w_h(x) = \frac{u(x, t+h) - u(x,t)}{h} - \partial_t u(x,t),$$

with $w_0(x) = 0$. To complete the proof, we are required to show that, for $t \in I$ fixed, $\|w_h\|_{L^2} \longrightarrow 0$ as $h \to 0$, but as $(x, h) \mapsto w_h(x)$ is continuous in $\mathbb{T} \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$, it is sufficient to observe that

$$\|w_h\|_{L^2} \leq \sqrt{2\pi}\|w_h\|_\infty,$$

because $\sup_{x\in\mathbb{T}} |w_h(x)| \longrightarrow 0$ as $h \to 0$. □

Remark. Proposition 5 shows that if u is differentiable in the classical sense, then in L^2, its generalized derivative, considered from the point of view of $U(t)$, corresponds to its classical derivative $\partial_t u(\cdot, t)$ as one would expect.

2.1.4 Solutions of the Heat Equations in L^2

We are now ready to prove the main results of the L^2-theory of the heat equation in one dimension.

2.1.4.1 Solving the Heat Equation

Theorem (Existence of Strong H^s-solutions). *Assume $t \geq 0$. Then, for any $s \geq 0$ and $g \in L^2(\mathbb{T})$,*

$$U(t) = \sum_{k\in\mathbb{Z}} \hat{g}(k) e^{-k^2 t} e_k$$

satisfies

$$U'(t) = \partial_x^2 U(t), \tag{2.1.3}$$

in $H^s(\mathbb{T})$, where $\partial_x^2 U(t)$ is the strong derivative, and

$$\|U(t) - g\|_{L^2} \longrightarrow 0 \ \text{as}\ t \to 0^+.$$

Proof. Proposition 2 allows us to work with $g = \sum_{|k|\in\mathbb{Z}} \hat{g}(k) e_k$. Let

$$a_k(t) = \widehat{U(t) - g}(k) = \frac{1}{2\pi}\langle U(t) - g, e_k\rangle = \frac{1}{2\pi}\langle \sum_{k\in\mathbb{Z}} \hat{g}(k) e^{-k^2 t} e_k, e_k\rangle - \hat{g}(k) = \hat{g}(k)\left(e^{-k^2 t} - 1\right).$$

We will first prove the last claim of the theorem. To do so, we want to show that for any $t \geq 0$ fixed, $\|(a_k(t))_{k\in\mathbb{Z}}\|_{\ell^2} \longrightarrow 0$ as $t \to 0^+$. Now, it is immediate that we have both $|a_k(t)| \leq |\hat{g}(k)|$ and $(a_k(t))_{k\in\mathbb{Z}} \in \ell^2$. Moreover, since $e^y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we get that if $0 < |y| < 1$, then

$$|e^y - 1| \leq \sum_{n=1}^{\infty} |y|^n = \frac{|y|}{1 - |y|},$$

which further implies that for $|y| \leq \frac{1}{2}$, $|e^y - 1| \leq 2|y|$. Thus, for $t \geq 0$ small enough, $|k^2 t| \leq \frac{1}{2}$ and we also have that

$$|a_k(t)| \leq 2|\hat{g}(k)||k^2 t|.$$

Hence to acheive our goal, it is sufficient to show that for $\epsilon > 0$ given, we may find $t > 0$ small and $N \in \mathbb{N}$ such that

$$\|a_k(t)\|_{\ell^2}^2 = \sum_{k\in\mathbb{Z}} |a_k(t)|^2 \leq \underbrace{\sum_{|k|>N} |\hat{g}(k)|^2}_{A} + 4t^2 \underbrace{\sum_{|k|\leq N} |\hat{g}(k)|^2 k^4}_{B} < \epsilon.$$

But since $(\hat{g}(k))_k \in \ell^2$ and $B < \infty$, it is given that we can take N large so that $A < \frac{\epsilon}{2}$ and choose $0 < t$ small to obtain $\|a_k(t)\|_{l^2}^2 \leq A + B < \epsilon$. By proposition 1,

$$\|U(t) - g\|_{L^2} = \sqrt{2\pi}\|a_k(t)\|_{\ell^2} \text{ for any } t > 0,$$

and we conclude that $\|U(t) - g\|_{L^2} \longrightarrow 0$ as $t \to 0^+$.

It is clear that $U(t) \in L^2$, since $\forall k \in \mathbb{Z}$, $t \geq 0$, $e^{-k^2 t} \leq 1$; but we also observe, for $t > 0$, that $U(t) \in H^s(\mathbb{T})$ for any $s \geq 0$. This follows from the fact that

$$|U(t)|_s^2 = \sum_{|k|\in\mathbb{Z}} |k|^{2s} |\hat{U}(k)|^2 = \sum_{|k|\in\mathbb{Z}} |k|^{2s} e^{-2k^2 t} |\hat{g}(k)|^2 < \infty.$$

Indeed, assume, w.l.o.g., that $k > 0$ and let $q(k) = k^{2s} e^{-2k^2 t}$. Then, by solving

$$0 = \frac{d}{dk} q(k) = 2sk^{2s-1} e^{-2k^2 t} + -4tk^{2s+1} e^{-2k^2 t} = e^{-2k^2}\left(2sk^{2s-1} - 4tk^{2s+1}\right)$$

in k, we find by maximality that $q(k) \leq q\left(\sqrt{\frac{s}{2t}}\right) = \left(\frac{s}{2t}\right)^s e^{-s} \leq C \cdot t^{-s}$. Hence,

$$|U(t)|_s \leq \sqrt{C \cdot t^{-s}} \|g\| < \infty, \tag{2.1.4}$$

which shows that $\|U(t)\|_s < \infty$.

We would now like to show that the $\widehat{U'(t)} = -k^2 \hat{g}(k) e^{-k^2 t}$. Let

$$\eta_k(h) = \frac{\hat{g}(k)e^{-k^2(t+h)} - \hat{g}(k)e^{-k^2 t}}{h} + k^2 \hat{g}(k) e^{-k^2 t} = \hat{g}(k) e^{-k^2 t} \left(\frac{e^{-k^2 h} - 1}{h} + k^2 \right).$$

We want to show that $\|n_k(h)\|_{\ell_s^2} \longrightarrow 0$ as $h \to 0$. Using the properties of the exponential function again, we easily find that both $|e^y - 1 - y| \leq 2|y|^2$, $|y| \leq \frac{1}{2}$, and $\left|\frac{e^y - 1}{|y|}\right| \leq e^{|y|}$ holds. Hence,

$$\left| \frac{e^{-k^2 h} - 1}{h} + k^2 \right| \leq k^2 \left(e^{k^2 |h|} + 1 \right),$$

and

$$\left| \frac{e^{-k^2 h} - 1}{h} + k^2 \right| = \left| \frac{e^{-k^2 h} - 1 + hk^2}{h} \right| \leq 2k^4 |h| \text{ if } k^2|h| \text{ is small.} \tag{2.1.5}$$

We thus have

$$\|\eta_k(h)\|^2_{\ell_s^2} \leq \underbrace{\sum_{|k|>N} (1 + |k|^{2s}) |\hat{g}(k)|^2 e^{-2k^2 t} k^4 (e^{k^2|h|} + 1)^2}_{C} + \underbrace{\sum_{|k| \leq N} 2(1 + |k|^{2s}) |\hat{g}(k)|^2 e^{-2k^2 t} k^8 |h|^2}_{D}.$$

Since we are differentiating with respect to fixed t, it follows from (4) that as $N \to \infty$, $C \longrightarrow 0$ uniformly in h as long as $|h| < \frac{t}{2}$. Hence $\epsilon > 0$ given, choose N large enough for $C < \epsilon$ to hold for all $|h| \leq \frac{t}{2}$ and h small enough so that $|h| \leq \frac{t}{2}$, $k^2|h|$ is small enough for (5) to hold and $D < \epsilon$. □

Theorem (Uniqueness). *The only solution to $U'(t) = \partial_x^2 U(t)$ with $U(0) = 0$ in L^2 is $U \equiv 0$.*

Proof. The conclusion follows from the derivation

$$\begin{aligned}
\frac{d}{dt}\|U(t)\|^2 &= \frac{d}{dt}\langle U(t), U(t)\rangle \\
&= \lim_{h\to 0}\langle \frac{U(t+h) - U(t)}{h}, U(t)\rangle + \lim_{h\to 0}\langle U(t), \frac{U(t+h) - U(t)}{h}\rangle \\
&= \langle U'(t), U(t)\rangle + \langle U(t), U'(t)\rangle \\
&= 2Re\left(\langle U'(t), U(t)\rangle\right) \\
&= 2Re\left(\langle \partial_x^2 U(t), U(t)\rangle\right) \\
&= -2Re\left(\langle \partial_x U(t), \partial_x U(t)\rangle\right) \\
&= -2\|\partial_x U(t)\|^2 \leq 0.
\end{aligned}$$

□

Remark. Uniqueness comes from the fact that this proof can be applied to $u(0) - v(0) = 0$ if u and v both solves the heat equation with same initial data. This theorem should not be a suprise, as the heat diffusion decays in time. It is intuitively clear that if the initial data have modes with vanishing frequencies on the whole domain, then the related solution must stay unchanged when evolving in time. That is to say, what the above proof really shows is that the rate of change of the norm of U is negative with respect to time.

Theorem (Regularity). *For $t > 0$, $U(t)$ agrees with a smooth function almost everywhere in L_2. In fact, if*

$$u(x,t) := U(t)(x) \text{ for } (x,t) \in \mathbb{T} \times (0,\infty),$$

then in L^2, $u \in C^\infty(\mathbb{T} \times (0,\infty))$ up to a null set.

Proof. We will first prove the regularity statement in space. If $f \in H^s(\mathbb{T})$, then f is C^m if $s > m + \frac{1}{2}$. Indeed,

$$\begin{aligned}
|\partial^m f(x)| &= |\sum_{k\in\mathbb{Z}} (ik)^m \hat{f}(k) e^{ikx}| \\
&\leq \sum_{k\in\mathbb{Z}} |k|^m |\hat{f}(k)| \quad := \|(\hat{f}(k))_k\|_{\ell^1_m} \\
&\leq \sum_{k\in\mathbb{Z}} \left(|k|^s |\hat{f}(k)|\right) |k|^{m-s} \\
&\leq \left(\sum_{k\in\mathbb{Z}} \left(|k|^{2s} |\hat{f}(k)|^2\right)\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} |k|^{2(m-s)}\right)^{\frac{1}{2}} \qquad (2.1.6) \\
&\leq C\|f\|_s, \qquad (2.1.7)
\end{aligned}$$

where (6) was obtained from the Cauchy-Schwartz inequality and (2) by the convergence of the geometric series under the hypothesis $s - m > \frac{1}{2}$. C is a real constant.

We conclude from the above that $\|f\|_{C^m} \leq C\|f\|_s$ whenever $s - m > \frac{1}{2}$. Using now the density of the C^∞ functions in $H^s(\mathbb{T})$, let $f_k \in C^\infty(\mathbb{T})$ be such that $f_k \longrightarrow f$ in $H^s(\mathbb{T})$. The convergence

$$\|f_k - f_{k'}\|_{C^m} \leq C\|f_k - f_{k'}\|_s \longrightarrow 0 \text{ as } k \to \infty$$

implies that $\{f_k\}_k$ is Cauchy in $C^m(\mathbb{T})$, and thus converges to some function $g \in C^m(\mathbb{T})$. On the one hand, it follows from this convergence in $C^m(\mathbb{T})$ that $f_k \longrightarrow g$ in $C^0(\mathbb{T})$, and thus in L^2. On the other hand, convergence to f in $H^s(\mathbb{T})$ also means that $f_k \longrightarrow f$ in $L^2(\mathbb{T})$. So we must have $g = f$ in L^2, i.e. they agree almost everywhere. Since we've shown in the existence theorem that $U(t)$ is in $H^s(\mathbb{T})$ for any $s \geq 0$, we conclude that it agress a.e. with a function $f \in C^\infty(\mathbb{T})$.

Let $t \in (0,\infty)$. If we define $U(t)(x) = u(x,t)$, then we have

$$\frac{u(x,t+h) - u(x,t)}{h} - U'(t)(x) = \frac{U(t+h)(x) - U(t)(x)}{h} - U'(t)(x) \underset{h\to 0}{\longrightarrow} 0 \text{ in } H^s(\mathbb{T}),$$

and here again, it follows that we have convergence in $C^0(\mathbb{T})$ as $h \to 0$ (as $s > \frac{1}{2}$), i.e. the above converges uniformly has a function of $x \in \mathbb{T}$, thus we have that $\partial_t u(x,t)$ exists everywhere on $\mathbb{T}$. So we have shown yet that for a given $t \in (0,\infty)$,

$$\underbrace{\partial_t u(x,t)}_{\text{Classical}} = \underbrace{U'(t)(x)}_{\text{Generalized}} = \underbrace{\partial_x^2 U(t)(x)}_{\text{Strong}} = \underbrace{\partial_x^2 u(x,t)}_{\text{Classical}}.$$

We know from the previous existence theorem that

$$U'(t) = -\sum_{k\in\mathbb{Z}} k^2 e^{-k^2 t} e_k,$$

and in fact, it follows from an analogous proof (we use the boundedness of the heat propagator with respect to any H^s-norm) that

$$\begin{aligned}U''(t) &= \sum_{k\in\mathbb{Z}} k^4 e^{-k^2 t} e_k \\ &\vdots \\ U^{(n)}(t) &= (-1)^n \sum_{k\in\mathbb{Z}} k^{2n} e^{-k^2 t} e_k,\end{aligned}$$

because the mentionned argument reduces to observing that an appropriate exponential growth overrule a polynomial growth of any order. Hence,

$$U \in C^\infty\left((0,\infty), H^s(\mathbb{T})\right).$$

Repeating the argument we have used on the first derivative for the derivatives of higher orders, we conclude that $\underbrace{\partial_t^m u(x,t)}_{\text{Classical}} = \underbrace{U^{(m)}(t)(x)}_{\text{Generalized}}$ for any $m \in \mathbb{N}$.

Now, the inequality

$$\begin{aligned}&|\partial_t^n \partial_x^m u(x+\epsilon, t+h) - \partial_t^n \partial_x^m u(x,t)| \\ &\qquad \le |\partial_t^n \partial_x^m u(x+\epsilon, t+h) - \partial_t^n \partial_x^m u(x+\epsilon, t)| + |\partial_t^n \partial_x^m u(x+\epsilon, t) - \partial_t^n \partial_x^m u(x,t)|,\end{aligned}$$

illustrates that $\partial_t^n \partial_x^m u \in C^1(\mathbb{T}\times(0,\infty))$ for any $m, n \in \mathbb{N}$, hence u has partial derivatives of all order, i.e. u is smooth: $u \in C^\infty(\mathbb{T}\times(0,\infty))$. □

Remark. The above regularity theorem shows that the heat equation is classicaly satisfied. Moreover, we know from the existence theorem that $U(t) \longrightarrow g$ in H^s as $t \to 0^+$, hence the above argument shows that if $s > \frac{1}{2}$, then $U(t) \longrightarrow g$ uniformly on $\mathbb{T}^n$.

2.1.4.2 Instantaneous Smoothing and Long Term Decay

More can be said about the nature of the solutions of the heat equation. Suppose we understand the existence theorem from section 2.1.4 as a proceedure to build solutions for the heat equation out of an L^2 function, say g. The initial data g is decomposed into modes $\hat{g}(k)e_k, k \in \mathbb{Z}$, that is, into terms of different frequencies, and each k^{th} mode is damped with the rate $e^{-k^2 t}$. In this way, the exponentials act on each frequencies and high frequencies decay fast. This results into a smoothing, and that is what we have shown in section 2.4. In other words, by acting on the frequencies, the exponentials push g into any Sobolev space, causing a very strong decay.

Another interesting fact is that the exponentials actually make $g \in L^2$ decay into a constant as $t \to \infty$. Consider, for $C = \hat{g}(0)$,

$$\begin{aligned}\|\sum_{k\in\mathbb{Z}}^{\infty} \hat{g}(k) e^{-k^2 t} e_k - C\|_\infty &= \|\sum_{k\neq 0}^{\infty} \hat{g}(k) e^{-k^2 t} e_k\|_\infty \\ &\le \sum_{k\neq 0}^{\infty} |\hat{g}(k)| e^{-k^2 t} \\ &\le Ce^{-t} \longrightarrow 0 \text{ as } t \to 0.\end{aligned}$$

2.1.5 Extension of the Theory to n-dimensions

The above theory extends naturally to higher finite dimensions.

2.1.5.1 Generalized L^2-definitions for Finite Dimensional Spaces

The statements and the proofs of many theorems in this section are omitted. Some of them are easy extensions of arguments seen in previous sections, - we refer the reader to the question 1 of the first assignment (appendix A), and the majority of the past basic results have a straight foward generalization to higher dimensions.

We introduce the following multi-index notation. We write $x = (x_1, ..., x_n) \in \mathbb{T}^n$ and we define $\partial_j := \frac{\partial}{\partial x_j}$, $j = 1, ..., n$. For example, we can encounter expressions such as $\partial_1^3 \partial_2^2 \partial_4^3$. A multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ is a vector in $\mathbb{N}_0^n = \mathbb{N}^n \backslash \{0\}$ with an index order defined as $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. We may call the index order the length of the multi-index. We define

$$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} \;\; \& \;\; y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} ... y_n^{\alpha_n},$$

where $y \in \mathbb{R}^n$. D^α is often defined in the same way in the litterature, and for a non-negative integer $k \in \mathbb{N}$, we read

$$D^k = \{D^\alpha : |\alpha| = k\} = \{\partial^\alpha : |\alpha| = k\}.$$

The Fourier series in n-dimensions may now be expressed as

$$\begin{aligned} f(x) &= \sum_{k_1,...,k_n \in \mathbb{Z}^n} a_k e^{ik_1 x_1} e^{ik_2 x_2} ... e^{ik_n x_n} \\ &= \sum_{k \in \mathbb{Z}^n} a_k e^{ik \cdot x} \\ &= \sum_{k \in \mathbb{Z}^n} a_k e_k(x), \end{aligned}$$

where $e_k(x) := e^{ik \cdot x}$ and $\cdot : \mathbb{N}_0^n \times \mathbb{R}^n \to \mathbb{R}$ is the dot product of the two n-dimensional vectors k, x, i.e. $k \cdot x = k_1 x_1 + .. + k_n x_n$.

For finite sums, we can use $Q_m = \{-m, ...m\}^n$, so that an anolog of $\sum_{|k|<m} a_k e_k$ is $\sum_{z \in Q_m} a_k e_k$ in higher dimensions. We note though, that Q_m is understood as an n-dimensional square with edges of length m, but that similar constructions such as finite sums over $|k| \leq m$, where $m \in \mathbb{N}$ and $|k| = \sqrt{k_1^2 + ... + k_n^2}$, that is, sums over the multi-dimensional integer lattice points contained in an n-sphere, yield the same results.

Definition. Let $g \in L^2(\mathbb{T}^n)$ and $(k^\alpha \hat{g}(k))_{k \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$, then we call

$$h = \partial^\alpha g = \sum_{k \in \mathbb{Z}^n} (ik)^\alpha \hat{g}(k) e_k$$

the strong derivative of g. If we let $\alpha \in \mathbb{N}_0^n$ be a multi-index, $g, h \in L_1(\mathbb{T}^n)$ and

$$\langle h, v \rangle = (-1)^\alpha \langle g, \partial^\alpha v \rangle \;\; \forall v \in C^\infty(\mathbb{T}^n),$$

then we say that $h = \partial^\alpha g$ is the weak sense, i.e. g is the mixed weak partial derivative of h of order α.

Theorem (Friedrichs, 1944). *In the above setting, h is a strong derivative of g if and only if it is a weak derivative of g.*

Definition. For $s \geq 0$, we define the Sobolev space

$$H^s(\mathbb{T}^n) = \left\{ f \in L^2(\mathbb{T}^n) : \hat{f} \in \ell_s^2 \right\},$$

where $\ell_s^2 = \left\{ a \in \ell^2(\mathbb{Z}^n) : (|k|^s a_k)_{k \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n) \right\}$.

We understand that if $f \in H^s(\mathbb{T}^n)$, then $\partial^\alpha f \in L^2(\mathbb{T}^n)$ for any $|\alpha| \leq s$, because then

$$k_1^{\alpha_1} k_2^{\alpha_2} ... k_n^{\alpha_n} \leq |k|^{\alpha_1} ... |k|^{\alpha_n} = |k|^{|\alpha|} \leq |k|^s.$$

Conversly, if $\partial^\alpha f \in L^2(\mathbb{T}^n)$ for all $\alpha \leq s$, then it holds that $k_i^s \hat{f}(k) \in \ell^2$ for any $i = 1, ..., n$, as we can always choose $\alpha = (0, ..., s, ...0)$ s.t. $\pi_i \alpha = s$ is the i^{th}-coordinate projection map. Thus, since

$$|k|^s = \left(k_1^2 + ... + k_2^2\right)^{\frac{s}{2}} \leq C_s \left(|k_1|^s + ... + |k_n|^s\right),$$

where C_s is a constant depending on s, then $\sum_{z \in \mathbb{Z}^k} |k|^s \hat{f}(k) e_k < \infty$, i.e. $f \in H^s(\mathbb{T}^n)$. Hence, the Sobolev space can be seen as the space of functions for which all mixed weak partial derivatives of order less than a given degree exist in $L^2(\mathbb{T}^n)$.

This is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_s = \sum_{k \in \mathbb{Z}^n} (1 + |k|^{2s}) \hat{f}(k) \bar{\hat{g}}(k).$$

If $s \in \mathbb{N}$, this inner product is equivalent to

$$\sum_{|\alpha| \leq s} \int_{\mathbb{T}^n} \partial^\alpha f(x) \partial^\alpha \overline{g(x)} dx.$$

Here again, the semi-norm and the norm associated to this space are given by

$$|f|_s^2 = \sum_{k \in \mathbb{Z}^n} |k|^{2s} |\hat{f}(k)|^2$$

and

$$\|f\|_s = \sqrt{\|f\|_{L^2}^2 + |f|_s^2}.$$

Theorem (Bernstein). *Let $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$ and $s > m + \frac{n}{2}$. Then, $H^s(\mathbb{T}^n) \hookrightarrow C^m(\mathbb{T}^n)$, i.e. each $f \in H^s(\mathbb{T}^n)$ is equal to a C^m function a.e. on $\mathbb{T}^n$ (has a C^m representative) and*

$$\|f\|_{C^m} \leq C_{s,m} \|f\|_s, \quad C_{s,m} \in \mathbb{R}.$$

Proof. We have

$$|\partial^\alpha f(x)| = |\sum_{k \in \mathbb{Z}^n} (ik)^\alpha \hat{f}(k) e^{ik \cdot x}| \leq \sum_{k \in \mathbb{Z}^n} |k|^{|\alpha|} |\hat{f}(k)| \leq \left(\sum_{k \in \mathbb{Z}^n} |k|^{2s} |\hat{f}(k)|^2\right)^{\frac{1}{2}} \underbrace{\left(\sum_{k \in \mathbb{Z}^n} |k|^{2(|\alpha| - s)}\right)^{\frac{1}{2}}}_{A}.$$

We have $A < \infty$ if $2(|\alpha| - s) < n$ and if this is the case, then the same argument we have used in section 2.4 to prove the regularity theorem still applies here and it completes the proof. □

2.1.5.2 Multiplication in Sobolev Spaces

We will prove an important inequality regarding multiplications in Sobolev spaces that will be especially useful when we will investigate the maximal time of existence of solutions of some well-known PDEs in section 3 of the next chapter (also see assignement 2). In the following, we will make a change in our usual notation by replacing our beloved $k \in \mathbb{Z}^n$ for the Fourrier variables $\xi, \eta, \zeta \in \mathbb{Z}^n$.

Lemma (Young's Inequality). *For a, b sequences,*

$$\|a * b\|_{\ell^2} \leq \|a\|_{\ell^1} \|b\|_{\ell^2}.$$

Proof. For any sequence $c \in \ell^2$,

$$\begin{aligned}
\langle a * b, c\rangle &= \sum_{\xi}\sum_{\eta} a(\eta) b(\xi-\eta)\overline{c(\xi)} \\
&\leq \sum_{\eta} a(\eta) \sum_{\xi} b(\xi-\eta)\overline{c(\xi)} \\
&\leq \sum_{\eta} |a(\eta)| \max_{\eta} |\sum_{\xi} b(\xi-\eta)\overline{c(\xi)}|.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, this implies that

$$\sum_{\eta} |a(\eta)| \max_{\eta} |\sum_{\xi} b(\xi-\eta)\overline{c(\xi)}| \leq \|a\|_{\ell^1}\|b\|_{\ell^2}\|\bar{c}\|_{\ell^2} = \|a\|_{\ell^1}\|b\|_{\ell^2}\|c\|_{\ell^2},$$

and thus choosing $c = a * b$ concludes the proof.

$\square$

Theorem. *Let $u, v \in H^s(\mathbb{T}^n)$, where $s > \frac{n}{2}$. Then $uv \in H^s(\mathbb{T}^n)$ and*

$$\|uv\|_s \leq C_{s,n}\|u\|_s\|v\|_s,$$

where $C_{s,n}$ depends on s and n.

Proof. We have

$$\begin{aligned}
\widehat{uv}(\xi) &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} u(x)v(x)e^{-ix\cdot\xi}dx \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \sum_{\eta,\zeta} \hat{u}(\eta)e^{ix\cdot\eta}\hat{v}(\zeta)e^{ix\cdot\zeta}e^{-ix\cdot\xi}dx \\
&= \frac{1}{(2\pi)^n}\sum_{\eta,\zeta}\hat{u}(\eta)\hat{v}(\zeta)\int_{\mathbb{T}^n} e^{ix\cdot(\eta+\zeta-\xi)}dx \\
&= \sum_{\eta,\zeta}\hat{u}(\eta)\hat{v}(\zeta)\delta_{\eta+\zeta,\xi} \\
&= \sum_{\eta+\zeta=\xi}\hat{u}(\eta)\hat{v}(\zeta) \\
&= \sum_{\eta}\hat{u}(\eta)\hat{v}(\xi-n) \\
&= (\hat{u} * \hat{v})(\xi),
\end{aligned}$$

i.e. $\widehat{uv}(\xi)$ is equal to the discrete convolution of $\hat{u}$ and $\hat{v}$ at ξ. Hence,

$$\begin{aligned}
|\xi|^s|\widehat{uv}(\xi)| &\leq \sum_{\eta}|\xi|^s|\hat{u}(\xi-\eta)||\hat{v}(\eta)| \\
&\leq C_s\left[\sum_{\eta}\underbrace{|\xi-\eta|^s|\hat{u}(\xi-\eta)|}_{a(\xi-\eta)}|\hat{v}(\eta)| + \sum_{\eta}|\hat{u}(\xi-\eta)|\underbrace{|\eta|^s|\hat{v}(\eta)|}_{b(\eta)}\right] \\
&\leq C_s\left(a * \hat{v}\right)(\xi) + C_s\left(\hat{u} * b\right)(\xi),
\end{aligned}$$

where we have use the fact that for $(a+b)^p \leq \max\{1, 2^{1-p}\}\,(a^p + b^p)$ for $a, b, p > 0$. Using Young's inequality,

$$\begin{aligned}
|\widehat{uv}|_s &= \left(\sum_{\xi} |\xi|^{2s} |\widehat{uv}(\xi)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{\xi} \left(|\xi|^s |\widehat{uv}(\xi)|\right)^2\right)^{\frac{1}{2}} \\
&= \|\left(|\xi|^s |\widehat{uv}(\xi)|\right)_{\xi}\|_{\ell^2} \\
&\leq |C_s| \|\left(a * |\hat{v}|\right)\|_{\ell^2} + |C_s| \|\left(|\hat{u}| * b\right)\|_{\ell^2} \\
&\leq |C_s| C_n |u|_s \|\hat{v}\|_{\ell^1} + |C_s| C_n |v|_s \|\hat{u}\|_{\ell^1} \\
&\leq |C_s| C_n |u|_s \|\hat{v}\|_{\ell^1} + |C_s| C_n |v|_s \|\hat{u}\|_{\ell^1}
\end{aligned}$$

for any $s \geq 0$. Since, in particular, we have $s > \frac{n}{2}$ by hypothesis, its follows that $|uv|_s \leq C_{s,n} \|u\|_s \|v\|_s$, which concludes the proof.

□

2.1.5.3 The n-dimensional Heat Equation and its Solutions

The n-dimensional heat equation is the equality

$$u'(t) = \Delta u(t)$$

in $L^2(\mathbb{T}^n)$, where $\Delta = \partial_1^2 + \partial_2^2 + ... + \partial_n^2$ is the Laplace operator understood in the week sense. We want to solve this equation by finding solutions $u : [0, \infty) \longrightarrow L^2(\mathbb{T}^n)$ with $u(0) = g \in L^2(\mathbb{T})$. The corresponding main results of section 2.1.4 extend naturally and can be derived similarly. Please read the proofs related the finite dimensional homogeneous hyperdissipative heat equation in assignement 1 (appendix A). They generalize the theorems and extend the result related to the convergence of the solutions to their initial data. In this case, the solutions take the form

$$u(t) = e^{t\Delta} g := \sum_{k \in \mathbb{Z}^n} e^{-|k|^2 t} \hat{g}(k) e_k, \;\; g \in L^2(\mathbb{T}), t \geq 0,$$

where $e^{t\Delta} : L^2(\mathbb{T}) \longrightarrow H^s(\mathbb{T}^n)$ is a bounded linear operator with $|e^{t\Delta} g|_2 \leq \sqrt{C_s t^{-s}} \|g\|$ for $t > 0$ and $s \geq 0$.

We call the $e^{t\Delta}$ the heat propagator. It is interesting to note that we can view $\left\{e^{t\Delta} : t \in \mathbb{R}^+\right\}$ as a C_0 semigroup. Indeed, we have

1. $e^{0\Delta} = Id.$,
2. $e^{(t+s)\Delta} = e^{t\Delta} e^{s\Delta}$, and
3. $\|e^{t\Delta} g - g\|_s \longrightarrow 0$ as $t \rightarrow 0^+$.

2.2 The Nonhomogeneous Heat Equation: In the Presence of External Energy

2.2.1 Riemann Integration in Banach Spaces

We will define the integral of a function $f : [a, b] \longrightarrow X$, where X is a Banach space.

A partition of $[a,b]$ is a sequence of points $P = \{t_0, ..., t_n\}$ satisfying $a = t_o < t_1 < ... < t_n = b$. We say that this partition is tagged by $\xi_P = \{\xi_1, ..., \xi_n\}$ if $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, ..., n$. We define, for $f \in B([a,b], X)$, the Riemann sum with respect to P and ξ to be

$$S_{P,\xi}(f) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

We say f is Riemann integrable over $[a,b]$ if $\lim_{|p|\to 0} S_{p,\xi_P}(f) < \infty$ in X independent of the choice of P, where $|P| = \max_j |t_j - t_{j-1}|$, and if it is, we define its Riemann integral as

$$\int_a^b f(t)dt = \lim_{|p|\to 0} S_{P,\xi_P}(f).$$

We adopt the convention $\int_a^b f(t)dt = -\int_a^b f(t)dt$.

Definition. We define oscillation on an interval and with respect to a partition as

$$\operatorname{osc}(f, [c,d]) = \sup_{t,s\in[c,d]} \|f(s) - f(t)\|_X$$

and $\operatorname{osc}(f, P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \operatorname{osc}(f, [t_{i-1}, t_i])$.

Lemma. *$f \in B([a,b], X)$ is Riemann integrable if $\forall \epsilon > 0$, $\exists$ a tagged partition P of $[a,b]$ with $osc(f,p) < \epsilon$.*

Proof. If $P' \supset P$, i.e. if P' is refinement of P, then it is clear from the above definitions that

$$\operatorname{osc}(f, P') \leq \operatorname{osc}(f, P), \tag{2.2.1}$$

and

$$\|S_{P,\xi_P}(f) - S_{P',\xi_{P'}}(f)\|_X \leq \operatorname{osc}(f, P). \tag{2.2.2}$$

Using refinement and (1), we can choose a nested sequence of partition $P_1 \subset P_2 \subset ... \subset P_n \subset ...$ with an associated sequence of tagging partitions $(\xi_{P_k})_{k=1}^{\infty}$ such that $\operatorname{osc}(f, P_k) \longrightarrow 0$ as $k \to \infty$. This implies, from (2), that $\left(S_{p_k,\xi_{P_k}}\right)_{k=1}^{\infty}$ is Cauchy in X. Since X is Banach, $\exists x \in X$ s.t. $\lim_{k\to\infty} S_{p_k,\xi_{P_k}}(f) = x$ in X.

This convergence is independent of the tags, since if $\xi_{P_k'}$ is another tagging of P_k, then

$$\begin{aligned}
\|x - S_{P_k',\xi_{P_k'}}(f)\|_X &\leq \|x - S_{p_k,\xi_{P_k}}(f)\| + \|S_{p_k,\xi_{P_k}}(f) - S_{P_k',\xi_{P_k'}}(f)\|_X \\
&\leq \|x - S_{p_k,\xi_{P_k}}(f)\| + \operatorname{osc}(f, P_k) \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

To conclude, we need to show that for any tagged partition Q of $[a,b]$, $\lim_{|Q|\to 0} S_{Q,\xi_Q}(f) = x$ in X, i.e. x doesn't depend on the choice of partition. Let $\epsilon > 0$ be given. We have

$$\begin{aligned}
\|S_{Q,\xi_Q}(f) - x\| &\leq \underbrace{\|S_{Q,\xi_Q}(f) - S_{Q',\xi_{Q'}}(f)\|}_{A} + \|S_{Q',\xi_{Q'}}(f) - S_{P_k,\xi_{P_k}}(f)\| + \underbrace{\|S_{P_k,\xi_{P_k}}(f) - x\|}_{B} \\
&\leq A + \operatorname{osc}(f, P_k) + B,
\end{aligned}$$

where $Q' = P_k \cup Q$. We can take k large enough so that $\operatorname{osc}(f, P_k) + B < \frac{\epsilon}{2}$. Now, it also follows from (9) and (8) that $A \leq \operatorname{osc}(f, Q') \leq \operatorname{osc}(f, P_k)$, hence our choice of k ensures that $\|S_{Q,\xi_Q}(f) - x\| < \epsilon$ and we're done. $\square$

Corollary. *Functions in $C([a,b], X)$ are Riemann integrable.*

Proof. Choose a uniform partition $P = \{t_0, ..., t_n\}$ with $|P| = h$. Then,

$$\operatorname{osc}(f, P) = \sum_{i=1}^{n} h \cdot \operatorname{osc}(f, [t_{i-1}, t_i]) \leq \sum_{i=1}^{n} h \cdot \max_{t \in [a,b]} \operatorname{osc}(f, [t, t+h]) \leq (b-a) \max_{t \in [a,b]} \operatorname{osc}(f, [t, t+h]).$$

Since f continuous on $[a, b]$ implies that $f \in B([a, b], X)$ and that the last term of the right hand side of the above equation can be made as small as possible by using the continuity of f, the conclusion follows from last lemma. □

Lemma. *If f is Riemann integrable on $[a, b]$, then its Riemann integral is linear, i.e.*

$$\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx,$$

additive, i.e.

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

for $c \in [a, b]$ and bounded from

$$\| \int_a^b f(x) dx \|_X \leq \int_a^b \|f(x)\|_X dx \leq (b-a) \|f\|_\infty.$$

Theorem (Fundamental Theorem of Calculus). *The FTC has two parts.*

1. *If $u \in C^1((a, b), X)$, then*

$$u(b) - u(a) = \int_a^b u'(t) dt.$$

2. *If $f \in C([a, b], X)$, then the function*

$$F(x) = \int_a^x f(t) dt, \ x \in [a, b]$$

satisfies $F \in C^1((a, b), X)$ and $F'(x) = f(x)$ on $[a, b]$.

Proof. To prove the first part, let $P = \{t_0, ..., t_n\}$ be a uniform partition with stepsize h tagged by ξ_P at the left endpoints. Then,

$$\begin{aligned}
\|S_{P,\xi_P}(u') - (u(b) - u(a))\| &= \| \sum_{i=1}^{n} h u'(t_{i-1}) - (u(b) - u(a)) \| \\
&= \| \sum_{i=1}^{n} h u'(t_{i-1}) - \sum_{i=1}^{n} (u(t_i) - u(t_{i-1})) \| \\
&\leq \sum_{i=1}^{n} h \| u'(t_{i-1}) - \frac{u(t_{i-1} + h) - u(t_{i-1})}{h} \| \underset{h \to 0^+}{\longrightarrow} 0
\end{aligned}$$

shows that $\int_a^b u'(x) dx = \lim_{|P| \to 0} S_{P,\xi_P}(u') = u(b) - u(a)$.

To prove the second part, suppose $a < x < y < b$. Then for $P' = P|_{[x,y]} \cup \{x, y\}$,

$$F(y) - F(x) = \int_a^y f(t) dt - \int_a^x f(t) dt = \int_x^y f(t) dt = \lim_{|P'| \to 0} S_{P',\xi_P}(f). \tag{2.2.3}$$

Observe that

$$\begin{aligned}
\left\|\frac{F(y)-F(x)}{y-x}-f(x)\right\| &= \|\underbrace{\frac{F(y)-F(x)-S_{P',\xi_{P'}}(f)}{y-x}}_{A}\|+\|\frac{S_{P,\xi_P}(f)-(y-x)f(x)}{y-x}\| \\
&\leq A+\|\frac{\sum_{i=1}^n h\left(f(t_{i-1})-f(x)\right)}{y-x}\| \\
&\leq A+\underbrace{\|\operatorname{osc}(f,[x,y])\|}_{B}
\end{aligned}$$

By continuity, we have that $B \longrightarrow 0$ as $y \searrow x$. So choose y s.t. $B < \frac{\epsilon}{2}$. From (3), we now choose $|P'|$ small enough so that $\|F(y)-F(x)-S_{P',\xi_{P'}}(f)\| < \frac{\epsilon|y-x|}{2}$. It follows that $A < \frac{\epsilon}{2}$ and this completes the proof. $\square$

2.2.2 Duhamel's Principle

Duhamel's principle is a general method for expressing solutions of inhomogeneous linear evolution equations.

Consider $u'(t) = \Delta u(t) + f(t)$, $0 < t < T$ in $H^s(\mathbb{T}^n)$, where $f(t) \in C\left((0,T), H^\sigma(\mathbb{T}^n)\right)$, $\sigma \geq 0$, $0 < T \leq \infty$. Further impose that $u(t) \longrightarrow g$ in $L^2(\mathbb{T})$ as $t \to 0^+$. We are looking for strong H^s-solutions, which means that we understand u' as the derivative of $u : (0,T) \longrightarrow H^s(\mathbb{T}^n)$ and $\Delta u(t)$, $t > 0$, in the weak sense.

The heuristic behind Duhamel's principle is to let $u(t) = \sum_{z\in\mathbb{Z}} a_k(t)e_k$, where

$$a_k'(t) = -k^2 a_k(t) + b_k(t) \tag{2.2.4}$$

with $b_k(t) = \hat{f}(k)$, and solve (4) using variation of parameters. For $a_k(t) = \zeta_k(t)e^{-k^2t}$, we have $a_k'(t) = \zeta_k'(t)e^{-k^2t} + k^2\zeta_k(t)e^{-k^2t}$, and thus

$$\zeta_k'(t)e^{-k^2t} + k^2\zeta_k(t)e^{-k^2t} = -k^2 a_k(t) + b_k(t).$$

So we find that $\zeta_k'(t) = b_k(t)e^{k^2t}$, and using the FTC $\zeta_k(t) = \zeta_k(0) + \int_0^t e^{k^2\tau} b_k(\tau)d\tau$. Finally, since $a_k(0) = \zeta_k(0)e^{-k^2\cdot 0} = \zeta_k(0)$, those computations lead to

$$a_k(t) = a_k(0)e^{-k^2t} + \int_0^t e^{-k^2(t-\tau)} b_k(\tau)d\tau.$$

This reveals that if we could commute the sum with this integral in expressing the inhomogeneous heat equation, then we would retreive the heat propagator and the initial data as

$$u(t) = e^{t\Delta}g + \int_0^t e^{(t-\tau)\Delta} f(\tau)d\tau.$$

We can view this equation as the FTC "twisted" by the heat semi-group.

2.2.3 Solutions of the Inhomogeneous Heat Equation in L^2

We can use the above principle to specify strong H^s-solutions to the inhomogeneous heat equation.

Theorem (**Uniqueness**). *Let $u \in C\left([0,T), L^2(\mathbb{T}^n)\right)$ and $f \in C\left((0,T), L^2(\mathbb{T}^n)\right)$ satisfiy*

$$u'(t) - \Delta u(t) = f(t)$$

in $L^2(\mathbb{T}^n)$, $0 < t < T$, and $u(0) = g$. Further assume that $\lim_{\epsilon \to 0^+} \int_\epsilon^a \|f(t)\| dt < \infty$ for $0 < a < T$. Then,

$$u(t) = e^{t\Delta} g + \lim_{\epsilon \to 0^+} \int_\epsilon^t e^{(t-\tau)\Delta} f(\tau) d\tau$$

for $0 < t < T$, where the limit $\epsilon \to 0^+$ can be replaced by the evaluation at $\epsilon = 0$ if $f \in C\left([0,T), L^2(\mathbb{T}^n)\right)$. In particular, we have uniqueness in the considered class of function.

Proof. Let $v(\tau) = e^{(t-\tau)\Delta} u(\tau)$, $o < \tau < t$, where $t \in (\tau, T)$ is fixed. For $h \in \mathbb{R}$ appropriately small,

$$\begin{aligned} v(\tau+h) - v(\tau) &= e^{(t-\tau-h)} u(\tau+h) - e^{(t-\tau)\Delta} u(\tau) \\ &= e^{(t-\tau-h)\Delta} \left(u(\tau) + hu'(t) + o(h)\right) - e^{(t-\tau)\Delta} u(\tau) \qquad (2.2.5) \\ &= e^{(t-\tau-h)\Delta} u(\tau) - e^{(t-\tau)\Delta} u(\tau) + he^{(t-\tau-h)\Delta} u'(\tau) + o(h) \\ &= -h \left(e^{(t-\tau)\Delta} u(\tau)\right)'_t + he^{(t-\tau-h)\Delta} u'(\tau) + 2o(h), \qquad (2.2.6) \end{aligned}$$

where (5) and (6) were obtained from differentiation in L^2. Dividing through by h and taking the limit $h \to 0$ yield

$$v'(\tau) = -\Delta e^{(t-\tau)\Delta} u(\tau) + e^{(t-\tau)\Delta} u'(\tau).$$

The first term was found from $e^{(t-\tau)\Delta} u(\tau)$ being a solution of the homogeneous heat equation (see section 2.5.2) and the second term comes from the continuity of the middle term divided by h with respect to h in (13). Using the hypothesis and the commutativity of the laplacian applied in the weak sense with the heat propagator,

$$v'(\tau) = -e^{(t-\tau)\Delta} \Delta u(\tau) + e^{(t-\tau)\Delta} \left(\Delta u(\tau) + f(\tau)\right) = e^{(t-\tau)\Delta} f(\tau).$$

Since $f \in C\left((0,T), L^2(\mathbb{T}^n)\right)$, $v \in C^1\left((0,t), L^2(\mathbb{T}^n)\right)$ and we have

$$v(b) - v(\epsilon) = \int_\epsilon^b e^{(t-\tau)\Delta} f(\tau) \qquad (2.2.7)$$

by the FTC. As $\epsilon \to 0$, $v(\epsilon) \longrightarrow e^{t\Delta} u(0) = e^{t\Delta} g$ in L^2, and as $b \to t$, $v(b) \longrightarrow u(t)$ in L^2. Moreover, $\tau \mapsto e^{(t-\tau)\Delta} f(\tau)$ is continuous on $(t-\epsilon, t]$ and

$$\lim_{\epsilon \to 0^+} \| \int_\epsilon^a e^{(t-\tau)\Delta} f(\tau) d\tau \| \leq \lim_{\epsilon \to 0^+} \int_\epsilon^a \| e^{(t-\tau)\Delta} f(\tau) \| d\tau \leq \lim_{\epsilon \to 0^+} \int_\epsilon^a \|f(\tau)\| d\tau \leq M,$$

where M is an upper bound independent of ϵ, so

$$\lim_{\epsilon \to 0, b \to t} \int_\epsilon^b e^{(t-\tau)\Delta} f(\tau) d\tau = \lim_{\epsilon \to 0} \int_\epsilon^t e^{(t-\tau)\Delta} f(\tau) d\tau.$$

Taking those limits on (7) thus completes the proof. □

Theorem (**Existence**). *Let $g \in H^\alpha(\mathbb{T}^n)$, $f \in C\left((0,T), H^\sigma(\mathbb{T}^n)\right)$, where $0 \leq \alpha \leq \sigma$, and*

$$\lim_{\epsilon \searrow 0} \int_\epsilon^a \|f(t)\|_\sigma dt < \infty$$

for some $0 < a < T$. Then, the function

$$u(t) = e^{t\Delta} g + \lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon} e^{(t-\tau)\Delta} f(\tau) d\tau$$

defined for $0 < t < T$ is in $C\left((0,T), H^s(\mathbb{T}^n)\right)$ for all $s < \sigma + 2$. Furthermore, $\lim_{t \to 0^+} u(t) = g$ in $H^\alpha(\mathbb{T}^n)$.

Proof. This theorem was proven in the more general case of the hyperdissipative equation in assignement 1. If we let $\theta = 2$ in the third problem of the latter assignement, then both statements are identical. $\square$

Remark. We often, by abuse of notation, write "$\int_0^t$" in instead of writting "$\lim_{\epsilon \to 0^+} \int_0^t$", as it lightens the computations. We want to stretch the fact thought, that taking the limit is needed, because viewing $e^{t\Delta}$ as a map from sobolev spaces, it is true that for $s_1 \geq 0$, $e^{t\Delta} : H^{s_1} \longrightarrow H^{s_2}$ for any s_2 when $t > 0$, but in the case where $t = 0$, $e^{0\Delta}$ acts as the identity $H^{s_1} \longrightarrow H^{s_1}$, and may be undefined (it may blow up) in H^{s_2} for $s_2 > s_1$. This is undesired, because we wish to consider the integral of f for general $s_2 \geq 0$ first, and then consider the properties of both the initial data and the external force together to further determine which are the conditions for u to satisfy the requirements of a given type of solution. In the previous theorem, we have found that $u(t)$ was in $C\left((0,T), H^s(\mathbb{T}^n)\right)$ for all $s < \sigma + 2$, but observe that for $f \in H^\sigma$, $e^{0\Delta} f \notin H^{\sigma+i}$, $i \in \{1, 2\}$, which explains why the integral must be evaluated under the limiting process.

Chapter 3

Mild Solutions of Semilinear Parabolic Equations

3.1 Local Theory

In the previous chapter, we found many interesting propreties of $u(t) = e^{t\Delta}g$, $g \in H^\alpha$. For example, we found that $u \in C\left([0,\infty), H^\alpha\right)$, $u(0) = g$, and that $e^{(t+s)\Delta}g = e^{s\Delta}e^{t\Delta}$, $t, s \geq 0$ (semi-group property). While some inequalities such as $\|u(t)\|_\alpha \leq \|g\|_\alpha$ were immediate, we have also derived suprisingly useful inequalities such as

$$|u(t)|_s \leq Ct^{-\kappa}\|g\|_\alpha,$$

and

$$|u(t) - u(t-h)|_s \leq C|h|t^{-1-\kappa}|g|_\alpha, \ \ |h| \leq \frac{t}{2},$$

for $s > \alpha$, where in both of the above equations $\kappa = \frac{s-\alpha}{2}$ (we refer the reader to the treatement of $|\eta_g(h)|^2$ in appendix A.3). In this chapter, we want to establish a theory for a generalized setting in which we could derive similar results and regard some aspects of the previous discussion as special cases.

Definition. Let X and Y be Banach spaces. We say that $f : X \longrightarrow Y$ is locally Lipschitz if $\forall r > 0$, $\exists C_r \in \mathbb{R}$ s.t.

$$\|f(x_1) - f(x_2)\|_Y \leq C_r\|x_1 - x_2\|_X$$

for all $x_1, x_2 \in X$, $\max\{\|x_1\|_X, \|x_2\|\} \leq r$.

We consider in this chapter Banach spaces X and Y such that $X \hookrightarrow Y$. We let $f : X \longrightarrow Y$ be locally Lipschitz and $\{E(t)\}_{t\geq 0}$ be a contraction semi-group in X, i.e.

- $\forall t \geq 0$, $E(t) : X \longrightarrow X$ is linear with $\|E(t)\| \leq 1$, where $\|E(t)\| = \sup_{g\in X} \frac{\|E(t)g\|_X}{\|g\|_X}$;
- For each $g \in X$, the map $t \mapsto E(t)g$ is in $C\left([0,\infty), X\right)$;
- $E(0) = id.$ and $E(t+s) = E(s)E(t)$, $\forall t, s \geq 0$.

In addition, we assume that $E(t)$, $t > 0$, can be extended to a bounded linear map $E(t) : Y \longrightarrow X$ satisfying:

- $\|E(t)y\|_X \leq C(1 + t^{-\kappa})\|y\|_Y$, and
- $\|E(t)y - E(t-h)y\|_X \leq C|h|(1 + t^{1-\kappa})\|y\|_Y$, $0 < \kappa < 1$.

We will call (E) the problem of solving equations of the form

$$u(t) = E(t)g + \underbrace{\int_0^t E(t-\tau)f(u(\tau))d\tau}_{A}, \ \ t \in I,$$

for solutions $u \in C(I, X)$, where $g \in X$, $I = [0,T)$ or $I = [0,T]$.

Remark. Here again, "$\int_0^t$" stands for "$\lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon}$". Formally, for $f : X \longrightarrow Y$, $E(0)f(u(\tau))$ is undefined, because from the above conditions, the latter only acts as the identity on X, and we assumed it can be extended to $Y \backslash X$ only for $t > 0$.

Theorem (Local Existence). *For any $r > 0$ and $R > r$, $\exists T > 0$ such that if $\|g\|_X \leq r$, (E) has a unique solution in*

$$U_{R,T} = \{u \in C([0,T], X) : \|u(t)\|_X \leq R \, \forall t \in [0,T]\}.$$

Moreover, the map $g \mapsto u : \overline{B_r} \longrightarrow U_{R,T}$, where $\overline{B_r} = \overline{B_r(0)_X} = \{g \in X : \|g\|_X \leq r\}$, is Lipschitz continuous.

Proof. Let $g \in \overline{B_r}$, $u \in U_{R,T}$ and define the map $u \mapsto \phi(u)$ by

$$\phi(u)(t) = \underbrace{E(t)g}_{u_0(t)} + \underbrace{\int_0^t E(t-\tau)\underbrace{f(u(\tau))}_{\omega(\tau)}d\tau}_{u_1(t)}.$$

It is immediate from the conditions on $\{E(t)\}_{t \geq 0}$ that this map is well-defined. We want to show that $\phi : U_{R,T} \longrightarrow C([0,T], X)$, but it immediately follows from the semi-group properties and the Lipschitz continuity of f that $u_0 \in C([0,\infty), X)$ and $\omega \in C([0,T], Y)$, so it is sufficient to derive that

$$\begin{aligned}
\|u_1(t) - u_1(t-h)\| &= \|\int_0^t E(t-\tau)\omega(\tau)d\tau - \int_0^{t-h} E(t-h-\tau)\omega(\tau)d\tau\| \\
&= \|\int_{t-h}^t E(t-\tau)\omega(\tau)d\tau + \int_0^{t-h} (E(t-\tau) - E(t-h-\tau))\,\omega(\tau)d\tau\| \\
&\leq \|\int_{t-h}^t E(t-\tau)\omega(\tau)d\tau\| + \|\int_0^{t-h} (E(t-\tau) - E(t-h-\tau))\,\omega(\tau)d\tau\| \\
&\leq |h||(1+(t-\tau)^{-\kappa})|C_1 + |h|(1+(t-\tau)^{1-\kappa})C_2 \qquad (3.1.1) \\
&\lesssim |h|^{1-\kappa} + |h|^{2-\kappa},
\end{aligned}$$

where (1) follows by definition.

We now want to show the uniqueness of the solution in $U_{R,T}$. To do so, we will use the Banach Fix Point theorem. We will first show that $\phi(U_{R,T}) \subset U_{R,T}$. We have

$$\|E(t)g\|_X \leq \|E\|_X \|g\|_X \leq \|g\|_X,$$

so

$$\begin{aligned}
\|\phi(u)(t)\|_X &\leq \|E(t)g\|_X + \|\int_0^t E(t-\tau)f(u(\tau))d\tau\|_X \\
&\leq \|g\|_X + C\int_0^t (1+(t-\tau)^{-k})\|f(u(\tau))\|_Y d\tau \\
&\leq r + \underbrace{C\int_0^t (1+(t-\tau)^{-k})\left[\|f(u(\tau)) - f(0)\|_Y + \|f(0)\|_Y\right] d\tau}_{A}.
\end{aligned}$$

But as

$$
\begin{aligned}
A = \int_0^t \|f(u(\tau)) - f(0)\|_Y d\tau + \int_0^t \|f(0)\|_Y d\tau \\
&+ \int_0^t (t-\tau)^{-\kappa} \|f(u(\tau)) - f(0)\|_Y d\tau + \int_0^t (t-\tau)^{-k} \|f(0)\|_Y d\tau \\
&\leq C_R T \|u(\tau)\|_{C([0,T],X)} + T\|f(0)\|_Y + C_R T^{1-\kappa} \|u(\tau)\|_{C([0,T],X)} + T^{1-\kappa}\|f(0)\|_Y,
\end{aligned}
$$

we can take T small such that $\|\phi(u)(t)\|_X \leq R$.

If we consider another function $v \in U_{R,T}$, then we observe that we also have

$$
\begin{aligned}
\|\phi(u)(t) - \phi(v)(t)\|_X &\leq \| \int_0^t E(t-\tau)\left(f(u(\tau)) - f(v(\tau))\right) d\tau \| \\
&\leq C \int_0^t (1 + (t-\tau)^{-\kappa}) \|f(u(\tau)) - f(v(\tau))\|_Y d\tau \\
&\leq CC_R \left[\int_0^t \|u(\tau) - v(\tau)\|_X d\tau + \int_0^t (t-\tau)^{-\kappa} \|u(\tau) - v(\tau)\|_X d\tau \right] \\
&\leq CC_R \|u(\tau) - v(\tau)\|_{C([0,T],X)} \left[t + t^{1-\kappa}\right] \\
&\leq CC_R \left(T + T^{1-\kappa}\right) \|u(\tau) - v(\tau)\|_{C([0,T],X)}
\end{aligned}
$$

Hence, choosing T small enough so that both the above and $CC_R\left(T + T^{1-\kappa}\right) < 1$ holds, it follows, since $U_{R,T}$ is closed, that $\phi : U_{R,T} \longrightarrow U_{R,T}$ is a contraction. Thus, we conclude from the fix point theorem that $\exists!\ u \in U_{R,T}$ s.t. $\phi(u) = u$. We let the Lipschitz dependence of u on g is left as an exercise. $\square$

Exercise. Prove the Lipschitz dependence of u on g stated in the last theorem (see question 2 of assignement 2 for a mathematically precise statement of the claim and its proof).

Theorem (Uniqueness). *If $u_i \in C\left([0,T_i], X\right), i = 1,2$ are two solutions of (E), then $u_1 = u_2$ on $[0,T_1] \cap [0,T_2]$.*

Proof. Let $T = \min\{T_1, T_2\}$, $t \in [0,T]$ and $R = \|u_1\|_\infty + \|u_2\|_\infty$. From

$$
\|u_1(t) - u_2(t)\|_X \leq CC_R \int_0^t (t-\tau)^{-\kappa} \|u_1(\tau) - u_2(\tau)\|_X d\tau,
$$

which we derived in the last proof, we let $y(t) = \|u_1(t) - u_2(t)\|_X$ and $b(t-\tau) = CC_R(t-\tau)^{-\kappa}$ to write

$$
y(t) \leq \int_0^t b(t-\tau) y(\tau) d\tau, \;\; y(0) = 0.
$$

Recursively, this implies, with $\tau = t_1$, that

$$
\begin{aligned}
y(t) \leq \int_0^t \int_0^{t_1} b(t_1 - t_2) b(t - t_1) y(t_2) dt_2 dt_1 \\
&\leq \int_0^t \int_0^{t_1} ... \int_0^{t_{n-1}} b(t_{n-1} - t_n) ... b(t_1 - t_2) b(t - t_1) y(t_n) dt_n dt_{n-1} ... dt_1.
\end{aligned}
$$

Using that

$$
\int_0^1 t^{x-1}(1-t)^{y-1} dt = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},
$$

we find that

$$\begin{aligned}\int_0^t \tau^a (t-\tau)^b d\tau &= t^{b+a} \int_0^t \left(\frac{\tau}{t}\right)^a \left(1-\frac{\tau}{t}\right)^b d\tau \\ &= t^{b+a} \int_0^t \left(\frac{\tau}{t}\right)^a \left(1-\frac{\tau}{t}\right)^b d\tau \\ &= t^{b+a+1} \int_0^t (u)^a (1-u)^b d\tau \\ &= t^{b+a+1} B(a+1, b+1).\end{aligned}$$

Hence, using $y(t_n) = \|u_1(t_n) - u_2(t_n)\|_X \leq \|u_1(t)\|_X + \|u_2(t)\|_X \leq R$, we integrate

$$\begin{aligned}&\int_0^t \int_0^{t_1} ... \int_0^{t_{n-1}} b(t_{n-1}-t_n)...b(t_1-t_2)b(t-t_1)y(t_n)dt_n dt_{n-1}...dt_1 \\ &\leq RC_R C \int_0^t \int_0^{t_1} ... \int_0^{t_{n-2}} b(t_{n-2}-t_{n-1})b(t_1-t_2)b(t-t_1) \int_0^{t_{n-1}} (t_{n-1}-t_n)^{-\kappa} dt_n dt_{n-1}...dt_1 \\ &= RC_R C \int_0^t \int_0^{t_1} ... \int_0^{t_{n-2}} b(t_{n-2}-t_{n-1})b(t_1-t_2)b(t-t_1)y(t_n) \int_0^1 t_{n-1}^{-\kappa+1}(1-u)^{-\kappa} du dt_{n-1}...dt_1 \\ &= RC_R^2 C^2 \frac{\Gamma(1)\Gamma(1-\kappa)}{\Gamma(2-\kappa)} \int_0^t [...] \int_0^{t_{n-2}} t_{n-1}^{-\kappa+1}(t_{n-2}-t_{n-1})^{-\kappa} dt_{n-1}...dt_1 \\ &= RC_R^3 C^3 \frac{\Gamma(1-\kappa)^2\Gamma(2-\kappa)}{\Gamma(2-\kappa)\Gamma(3-2\kappa)} \int_0^t [...] \int_0^{t_{n-3}} t_{n-2}^{2(1-\kappa)}(t_{n-3}-t_{n-2})^{-\kappa} dt_{n-2}...dt_1 \\ &= RC_R^3 C^3 \frac{\Gamma(1-\kappa)^2}{\Gamma(1+2(1-\kappa))} \int_0^t [...] \int_0^{t_{n-3}} t_{n-2}^{2(1-\kappa)}(t_{n-3}-t_{n-2})^{-\kappa} dt_{n-2}...dt_1,\end{aligned}$$

and conclude from this pattern of integration that

$$y(t) \leq 2RC_R^n C^n t^{n(1-\kappa)} \frac{\Gamma(1-\kappa)^n}{\Gamma(1+n(1-\kappa))}.$$

It follows from Sterling's approximation formula that $y(t) \longrightarrow 0$ as $n \to \infty$. Indeed,

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + O(\frac{1}{z})\right),$$

and so $\Gamma(1+n(1-\kappa)) \sim (n(1-\kappa))^{n(1-\kappa)}$, where

$$n(1-\kappa)\log(n(1-\kappa)) >> n\log(a)$$

for a constant. The conclusion thus follows from letting $a = Ct^{1-\kappa}\Gamma(1-\kappa)$ and taking $n \to \infty$. □

Definition. Let $u_i \in C(I_i, X)$, $i = 1, 2$, be two solutions of (E). Suppose, w.l.o.g., that $I_2 \subset I_1$. We call u_1 an extension of u_2 if $u_1|_{I_2} = u_2$.

Remark. For u_1 and u_2 defined as the in the previous definiton, $u \in C(I_1 \cup I_2, X)$ given by

$$u(t) = \begin{cases} u_1(t), & t \in I_1 \\ u_2(t), & t \in I_2 \end{cases}$$

is an extension of both u_1 and u_2. Indeed, it follows from the last uniqueness theorem that u is well-defined, and it is immediatly a solution of (E).

Definition. We call a solution with no proprer extension a maximal solution, and we refer to the upper boundary of its domain as the maximal time of existence.

Lemma. *Suppose that a maximal local solution of* (E) *exists. Then it is unique, and it is defined on an interval of the form* $I = [0, T)$, $0 < T \leq \infty$.

Proof. Suppose two maximal solutions u_1 and u_2 exist on I_1 and I_2 respectively. As we remarked, we can define a third solution $u \in C(I, X)$, where $I = I_1 \cup I_2$. This function u is an extension of both u_1 and u_2, and thus by maximality $I_1 = I = I_2$, and from uniqueness $u_1 = u_2$.

It is left to show that I doesn't contain u's maximal time of existence. We will proceed by contradiction. Suppose $I = [0, T]$ for some $T > 0$. Then $\|u(T)\|_X < \infty$ and we can find a solution $v \in C([0, \epsilon), X)$, for some $\epsilon > 0$, such that

$$v(t) = E(t)u(T) + \int_0^t E(t-\tau)f(v(\tau))d\tau.$$

Define

$$\omega(t) = \begin{cases} u(t), & 0 \leq t \leq T \\ v(t-T), & T \leq t < T+\epsilon \end{cases}.$$

It is readily seen that $\omega \in C([0, T+\epsilon), X)$. We will show that ω solves the same problem (E) that u solves, thus contradicting the maximality of u. For $T \leq t < T+\epsilon$,

$$\begin{aligned}
\omega(t) &= v(t-T) \\
&= E(t-T)u(T) + \int_0^{t-T} E(t-T-\tau)f(v(\tau))d\tau \\
&= E(t-T)E(T)g + E(t-T)\int_0^T E(T-\tau)f(u(\tau))d\tau + \int_0^{t-T} E(t-T-\tau)f(v(\tau))d\tau \\
&= E(t-T)E(T)g + \int_0^T E(t-\tau)f(u(\tau))d\tau + \int_T^t E(t-\tau)f(v(\tau-T))d\tau \\
&= E(t)g + \int_0^t E(t-\tau)f(\omega(t))d\tau,
\end{aligned}$$

where we have used the semi-group properties and the definition of ω. This completes the proof. $\square$

Theorem. *There exists a unique maximal solution* $u \in C([0, T), X)$ *to* (E). *Moreover, if* $T < \infty$, *then* $\|u(t)\|_X \longrightarrow \infty$ *as* $t \to T^-$.

Proof. Let $\mathcal{T}$ be the set of $T > 0$ s.t. $\exists$ a solution $u \in C([0, T], X)$ of (E). Notice that if $T \in \mathcal{T}$, then $T^* \in \mathcal{T}$ for any $T^* < T$, thus $\mathcal{T} = (0, T)$ for some $T \leq \infty$ by the existence theorem and the argument used in the above lemma, because otherwise we could find $\epsilon > 0$ s.t. a solution exists in $C([0, T+\epsilon), X)$, which would imply the contradiction $T < T + \frac{\epsilon}{2} \in \mathcal{T}$.

To each $t \in \mathcal{T}$ is associated a solution $u_t \in C([0, t], X)$. Hence, we can define a function $u \in C([0, T), X)$ by

$$u(t) = \begin{cases} u_t(t), & 0 < t < T \\ g, & t = 0 \end{cases},$$

where continuity comes from the uniqueness of the local solutions. We claim that this is a maximal solution. It is straight foward to verify that

$$u(t) = u_t(t) = E(t)g + \int_0^t E(t-\tau)f(u_t(\tau))d\tau = E(t)g + \int_0^t E(t-\tau)f(u(\tau))d\tau,$$

since the last equality holds from the fact that $u_t(t^*) = u_{t^*}(t^*) = u(t^*)$ for any $t^* \leq t$, again by uniqueness. Thus, u solves (E), and as it is immediate for the choice of T that it is maximal, we are now left to prove the claimed limit behavior of u.

Let $T < \infty$ and suppose $\|u(t)\|_X \leq M,\ t \in [0,T)$. Solve (E) with initial data $u(t_k)$ and take $t_k \to T^-$. For each $k \in \mathbb{N}$, u can be extended to a solution defined on $[0, t_k + \epsilon)$ with $\epsilon > 0$ in the same way we extended u to ω in the last lemma. Since from the local existence theorem ϵ depends only on the norm of X and the initial data g, $t_k + \epsilon > T$ for k large enough, $\longrightarrow\longleftarrow$. $\square$

3.2 Regularity of Mild Solutions

We will consider the equations

$$\left\{ \begin{array}{r} \partial_t u = \Delta u + \kappa u^2 \\ u|_{t=0} = g \in H^s(\mathbb{T}^n) \end{array} \right. , \tag{3.2.1}$$

where $\kappa \in \mathbb{R}$ and $s > \frac{n}{2}$. This example will exhibit a general approach to investigate the regularity of the mild solutions of similar equations (see Assignement 2). We know form section 3.1 that there exists a maximal mild solution $u \in C\left([0,T), H^s(\mathbb{T}^n)\right)$ for some $T \geq \infty$, as it is readily seen from the Banach algebra property of the Sobolev spaces that $\kappa u^2 \in H^\sigma(\mathbb{T}^n)$ for $u \in H^\sigma(\mathbb{T}^n)$ and that it is locally Lipschitz on $H^\sigma(\mathbb{T}^n)$ for any $\sigma > \frac{n}{2}$.

We will first show that u is regular in space. To do so, we will show that $\partial_t u = \Delta u + \kappa u^2$ strongly in $H^s(\mathbb{T}^n)$ for any $s \geq \frac{n}{2}$, and from the results in previous sections (recall the regularity in space argument of the regularity theorem in section 2.1.4.1), conclude in favor of the regularity of u. The idea is to observe that since $g \in H^s(\mathbb{T}^n)$ and $\omega(\tau) = \kappa[u(\tau)]^2 \in C\left([0,T), H^s(\mathbb{T}^n)\right)$, it follows from the existence and uniqueness theorems of the inhomogeneous heat equation of section 2.2.3 that $u \in C\left((0,T), H^{s+\alpha}\right)$, $\alpha < 2$. In particular, $u \in C\left((0,T), H^{s+1}\right)$, from which we have $u \in C\left([\epsilon,T), H^{s+1}\right)$. Now considering $u(\epsilon)$ as the initial datum,

$$\begin{aligned} u(t) &= e^{t\Delta}u(\epsilon) + \int_\epsilon^{t+\epsilon} e^{(t+\epsilon-\tau)\Delta}\omega(\tau)d\tau \\ &= e^{t\Delta}u(\epsilon) + \int_0^t e^{(t-\tau)\Delta}\omega(\tau+\epsilon)d\tau \end{aligned}$$

leads the problem back to an analogous setting in $H^{s+1}(\mathbb{T}^n)$. Indeed, $\omega(\cdot+\epsilon) \in C\left([0,T-\epsilon), H^{s+1}\right)$

$$\begin{aligned} &\Longrightarrow u(\cdot+\epsilon) \in C\left((0,T-\epsilon), H^{s+2}\right) \\ &\Longrightarrow u \in C\left((0,T), H^{s+2}\right), \end{aligned}$$

where we have used the continuity of ω and u in $H^{s+1}(\mathbb{T}^n)$. As we may repeat the above indefinitely, we conclude that $u \in C\left((0,T), H^\sigma\right)$ for any $\sigma \geq \frac{n}{2}$.

To establish regularity in time, we consider, using the above equality in $H^\sigma(\mathbb{T}^n)$, $\sigma \geq 0$,

$$\frac{u'(t+h) - u'(t)}{h} = \underbrace{\frac{\Delta u(t+h) - \Delta u(t)}{h}}_{A} + \kappa \underbrace{\frac{u(t+h)^2 - u(t)^2}{h}}_{B}.$$

We derive that

$$A = \underbrace{\Delta \underbrace{\frac{u(t+h) - u(t)}{h}}_{\longrightarrow u'(t) \text{ in } H^{\sigma+2}}}_{\text{converges in } H^\sigma} \quad \& \quad B = \underbrace{\frac{(u(t+h) - u(t))}{h}}_{\longrightarrow u'(t) \text{ in } H^\sigma} \underbrace{u(t+h) + u(t)}_{\longrightarrow 2u(t) \text{ in } H^\sigma},$$

thus $u''(t) = \Delta u'(t) + 2\kappa u(t)u'(t)$ in $H^\sigma(\mathbb{T}^n)$, and repeating the calculations, we find that

$$u'''(t) = \Delta 2\kappa u'(t)u'(t) + 2\kappa u(t)u''(t),$$

etc. Hence, $u \in C^\infty\left((0,T), H^\sigma(\mathbb{T}^n)\right)$ and it follows that (1) is classically satisfied by the mild solution, i.e. we have found that $u(x,t) := u(t)(x) \in C^\infty\left(\mathbb{T}^n \times [0,T)\right)$.

3.3 Limit Behavior of Mild Solutions in Time

3.3.1 A Comparision Principle

The following theorem will allow us to investigate the behavior of a given solution by comparing it to the solutions of simpler equations, or to solutions for which we have already considered the properties.

Theorem (**Maximum Principle**). *Let $u \in C\left(\mathbb{T}^n \times [0,T]\right) \cap C^2\left(\mathbb{T}^n \times [0,T]\right)$, $f : (0,T] \longrightarrow \mathbb{R}^n$ and c be a bounded function $\mathbb{T}^n \times (0,T] \longrightarrow \mathbb{R}$. If*

$$\partial_t u - \Delta u - f \cdot \nabla u - cu \geq 0 \ \text{on } \mathbb{T}^n \times (0,T] \text{ and } u \geq 0 \ \text{on } \mathbb{T}^n \times \{0\},$$

then $u \geq 0$ on $\mathbb{T}^n \times (0,T]$.

Proof. Suppose for contradiction that u has a negative minimum on $\mathbb{T}^n \times (0,T]$. The function $v(x,t) = e^{-\alpha t}u(x,t)$ also has a negative minimum on $\mathbb{T}^n \times (0,T]$. Suppose that v acheives this minimum at (x_0,t_0).

On the one hand, $\Delta v|_{(x_x,t_0)} \geq 0$ from the second derivative test and $\partial_t v|_{(x_0,t_0)} \leq 0$, because either it vanishes at (x_0,t_0), or it is a boundary point; thus

$$\partial_t v - \Delta v \leq 0. \tag{3.3.1}$$

On the other hand, we have by hypothesis that

$$\partial_t v - \Delta v = -\alpha e^{-\alpha t}u + e^{-\alpha t}\left(\partial_t u - \Delta u\right) \geq -\alpha e^{-\alpha t}u + e^{-\alpha t}\left(f \cdot \nabla u + cu\right) = -\alpha v + f \cdot \nabla u + cv.$$

Hence, as $\nabla u|_{(x_0,t_0)} = 0$ from (x_0,t_0) being an extrema, $\partial_t v - \Delta v \geq (c-\alpha)v$ at (x_0,t_0) in $\mathbb{T}^n \times (0,T]$; and since c is bounded, we can take $\alpha > \max_{(x,t)\in\mathbb{T}^n\times(0,T]} x(x,t)$, so that from $v < 0$, we have $\partial_t v - \Delta v > 0$, which contradicts (1). ☐

Corollary (**Comparision Principle**). *If v and u satisfy*

$$\begin{cases} \partial_t u - \Delta u - f \cdot \nabla u - cu \geq \partial_t v - \Delta v - f \cdot \nabla v - cv \ \text{on } \mathbb{T}^n \times (0,T] \\ u \geq v \ \text{on } \mathbb{T}^n \times \{0\} \end{cases},$$

then $u \geq v$ in $\mathbb{T}^n \times [0,T]$.

3.3.2 Finite Time Blow Up and Global in Time Solutions

We will investigate in this section the solutions to two different PDEs. One of these blows up in a finite amount of time, while the other doesn't.

We first compare the prototypical example

$$\begin{cases} \partial_t u = \Delta u + u^2 \\ u|_{t=0} = g \in H^s(\mathbb{T}^n), \ \ g > 0, \ s > \frac{n}{2} \end{cases} \tag{3.3.2}$$

with $v(t) = \frac{1}{\frac{1}{v(o)} - t}$, which solves the ODE $\partial_t v = v^2$. If we let $u \in C([0,T), H^s(\mathbb{T}^n))$ be the smooth maximal solution of (2) and we choose $v(0) = \min_{x \in \mathbb{T}^n} g(x) > 0$, then $u - v \geq 0$ on $\mathbb{T}^n \times \{0\}$ and

$$\partial_t(u - v) - \Delta(u - v) = u^2 - v^2 = (u + v)(u - v). \tag{3.3.3}$$

Hence, applying the Maximum Principle with $f = 0$, $c = u + v$ and viewing the equality in (3) as an equality, we find that $u \geq v = \frac{1}{(1/\min g) - t}$.

Theorem. *The solution to the above PDE with the stated boundary conditions blows up in a finite amount of time. This behavior at T can be estimated as $1/\max g \leq T \leq 1/\min g$.*

Proof. Let $u \in C([0,T), H^s)$ be a maximal solution. We already know that $u \in C^\infty(\mathbb{T}^n \times (0,T)) \cap C(\mathbb{T}^n \times [0,T))$ and that $\partial_t u = \Delta u + u^2$ classically in $\mathbb{T}^n \times (0,T)$. Suppose that $T > T_0 = \frac{1}{\min g}$. From the comparision principle, $u(x,t) \geq 1/(T_0 - t)$ for any $t < T_0$. This implies $\|u(\cdot,t)\|_{L^\infty} \longrightarrow \infty$, which is a contradiction, because $\|u(\cdot,t)\|_{L^\infty} \lesssim \|u(t)\|_s < \infty$. Hence, we conclude that $T \leq T_0$.

We now want to prove the claimed lower bound. We proceed by contradiction again and assume $T < T_1 = \frac{1}{\max g}$. We must have $\|u(t)\|_s \longrightarrow \infty$ as $t \to T$. This will be a contradiction, as we will show that neither $\|u(t)\|_{L^\infty}$ nor $\|u\|_s$ is blowing up on $[0,T]$ when we assume $T < 1/\max g$. In order to to so, we observe that since for $t < T$, $|u(x,t)| \leq 1/((1/\|g\|_\infty) - t)$, then

$$|u(x,t)| \leq \frac{1}{\frac{1}{\|g\|_\infty} - T} \in C([0,T), \mathbb{R}).$$

This yeilds the basic estimate $\sup_{0 \leq t < T} \|u(t)\|_\infty < \infty$. We conclude the proof by proceeding as in Annex 2. □

Now consider the equation $\partial_t u = \Delta u - u^2$, $u(\cdot,0) = g > 0$, which can be compared to the ODE $\partial_t v = -v^2$, for which we have the solution $v(t) = 1/((1/v(0)) + t)$.

Theorem. *The solution to the above equation $\partial_t u = \Delta u - u^2$, $u(\cdot,0) = g > 0$, with $g \in H^s(\mathbb{T}^n)$, $s > n/2$, is global in time. i.e. $T = \infty$, and*

$$\frac{1}{\frac{1}{\max g} + t} \leq u(t) \leq \frac{1}{\frac{1}{\min g} + t}$$

for all $t \in (0,\infty)$.

Proof. Again, we use the basic estimate $\|u(t)\|_\infty \leq \|g\|_\infty$ $\forall t < T$ and proceed as in Annex 2 (see question 3). □

3.3.2.1 Controllability

Definition. Suppose u is a regular solution of a fixed initial value problem on $(0,T)$. Then a quantity

$$q(t) = q(t, \{u(s), 0 < s \leq t\}$$

is said to be controlled for time T_* if $\exists$ a function $f : [\epsilon, T_*) \longrightarrow \mathbb{R}$ independent of T such that $q(t) \leq f(t)$ whenever $0 < t < T < T_*$.

Remark. Controllability may depend on other data in the problem.

Chapter 4

Mild Solutions of the Navier-Stokes Equations

After briefly introducing a new notation, we will be ready to investigate the Navier-Stokes equations given by

$$\text{NSE}\begin{cases} \partial_t u = \Delta u - u\cdot\nabla u - \nabla p \\ \operatorname{div}(u) = 0 \\ u|_{t=0} = g \quad (\operatorname{div} g = 0) \end{cases},$$

where $u(x,t) \in \mathbb{R}^n$, $p(x,t) \in \mathbb{R}$.

Einstein's notation will be used throughout the chapter; and given $u, v \in \mathbb{R}^n$, we define

$$(u\otimes v)_{ij} := u_i v_j,$$
$$(\nabla\otimes u)_{ij} := \partial_i u_j,$$
$$\nabla^k \otimes u := \{\partial^\alpha u_j : |\alpha| = k, j = 1, ..., n\},$$

and

$$\|u\|_k^2 := \|u\|_{L^2}^2 + |u|_k^2, \text{ where } |u|_k^2 = \|\nabla^k\otimes u\|_{L^2}^2.$$

As expected, Duhamel's principle may be applied to systems coordinatewise. If

$$f : H^s(\mathbb{T}^n, \mathbb{R}^m) \longrightarrow H^{s-1}(\mathbb{T}^n, \mathbb{R}^m)$$

is continuous, then the mild H^s-solution of $\partial_t u = \Delta u + f(u)$ is defined by

$$u_j(t) = e^{t\Delta} g_j + \int_0^t e^{(t-\tau)\Delta} f_j\left(u(\tau)\right) d\tau.$$

The solution of a such a PDE system may thus still be expressed as in previous chapters by

$$u(t) = e^{t\Delta} g + \int_0^t e^{(t-\tau)\Delta} f\left(u(\tau)\right) d\tau,$$

but it must be understood that for a function $h : \mathbb{T}^n \longrightarrow \mathbb{R}^m$, the action of the heat propagator is best viewed as a $m\times m$ matrix multiplication:

$$e^{t\Delta} h = \begin{pmatrix} e^{t\Delta} & 0 & \cdots & 0 \\ 0 & e^{t\Delta} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{t\Delta} \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix} = \begin{pmatrix} e^{t\Delta} h_1 \\ e^{t\Delta} h_2 \\ \vdots \\ e^{t\Delta} h_m \end{pmatrix}.$$

4.1 Existence of Unique Regular Mild Solutions

4.1.1 The Leray Projector

We immediatly observe that the main equation of the NSE is not in the form $\partial_t u = \Delta u + f(u)$ that we have been studying in the previous chapters. Exploring the consequences of the divergence condition imposed on u will allow us to work around this problem and express the equations of the NSE in the desired form.

Suppose $u(t)$ is a solution to NSE at some time $t > 0$, then $\text{div}(u) = 0$, and so it follows from

$$\partial_t \text{div}(u) = \Delta \text{div} u - \text{div}\,(u \cdot \nabla u) - \text{div} \nabla p,$$

assuming that $\partial_t \text{div}(u) = 0$, that $\text{div}\,(u \cdot \nabla u) = \nabla p$. In other words, the pressure term counter-acts the divergence caused by the nonlinear term $u \cdot \nabla u$ to ensure that $\partial_t \text{div}(u) = 0$. Informally, we may view the nonlinear term as

$$u \cdot \nabla u = \text{Divergence free part} \;+\; \text{Pure divergence part},$$

where Pure divergence part $= -\nabla p$. There exist results related to the divergence part of the nonlinear term which state that one can always rewrite the latter as the gradient of a scalar function. Hence, it is reasonable to expect that

$$\partial_t u = \Delta u - \mathbb{P}\,(u \cdot \nabla u)\,,$$

where $\mathbb{P}v$, $v : \mathbb{T}^n \longrightarrow \mathbb{R}$ is the divergence free part of v. How can we formally define such a function $\mathbb{P}$ such that this is indeed the case?

Let $v \in L^2(\mathbb{T}^n, \mathbb{R}^n)$ be a vector function and consider in the Fourier space, the integer lattice $\mathbb{Z}^n$ in which is suspended, at each $\xi \in \mathbb{Z}^n$, the vector coefficient $\hat{v}(\xi) = (\hat{v_1}(\xi), ..., \hat{v_n}(\xi))$. As

$$\begin{aligned} \widehat{\text{div}(v)}(\xi) &= \partial_1 v_1 + ... + \partial_n v_n \\ &= i\xi_1 \hat{v_1}(\xi) + ... + i\xi_n \hat{v_n}(\xi) \\ &= i\xi \cdot \hat{v}(\xi) \\ &= i|\xi| \underbrace{\frac{\xi \cdot \hat{v}(\xi)}{|\xi|}}_{R}, \end{aligned} \tag{4.1.1}$$

where R is the radial component of $\hat{v}(\xi)$ with respect to the the n-sphere centered at the origin with radius ξ, and that

$$\begin{aligned} \widehat{\text{grad}(p)}(\xi) &= \left(\widehat{\partial_1 p}(\xi), ..., \widehat{\partial_n p}(\xi)\right) \\ &= (i\xi_1 p(\xi), ..., i\xi_n p(\xi)) \\ &= i\xi p(\xi), \end{aligned}$$

with $p(\xi) \in \mathbb{R}$ is a vector parallel to the radius ξ, we understand that ∇p acts on the solution by removing the described radial part of its Fourier coefficients in the defined Fourier space lattice. In other word, the functions for which the ξ^{th} Fourier coefficient is tangential to the described n-sphere with radius ξ for every $\xi \in \mathbb{Z}^n$ is divergence free, and p acts on u by removing the associated non-zero radial components so that the divergence of u vanishes. The following definition for $\mathbb{P}$ is thus promising.

Definition (Leray Projector). The Leray projector

$$\mathbb{P} : L^2(\mathbb{T}^n, \mathbb{R}^n) \longrightarrow L^2(\mathbb{T}^n, \mathbb{R}^n)$$

is defined in the Fourier space by

$$\widehat{\mathbb{P}v}(\xi) = \hat{v}(\xi) - \frac{\xi}{|\xi|}\left(\frac{\xi}{|\xi|} \cdot \hat{v}(\xi)\right) = \left(\mathrm{Id} - \frac{\xi \otimes \xi}{|\xi|^2}\right)\hat{v}(\xi),$$

with $\widehat{\mathbb{P}v}(0) = \hat{v}(0)$ (or zero).

Lemma. *The Leray Projector satisfies the following properties.*
(i) $\mathbb{P}^2 = \mathbb{P}$;
(ii) $\langle v - \mathbb{P}v, \mathbb{P}g\rangle = 0$, $\forall v, g \in L^2(\mathbb{T}^n)^n$;
(iii) $div\mathbb{P}v = 0$, $v \in H^1(\mathbb{T}^n)^n$; *and*
(iv) $\mathbb{P} : H^s(\mathbb{T}^n)^n \longrightarrow H^s(\mathbb{T}^n)^n$ *with norm* 1 *for any* $s \geq 0$.

Proof. (i) and (ii) are immediate from $\mathbb{P}$ being projection. To prove (iii), it is sufficient, from the equality (1) derived in the above discussion, to show that $\xi \cdot \widehat{\mathbb{P}v}(\xi) = 0$. This is indeed the case. By definition,

$$\widehat{\mathbb{P}v}(\xi) = \hat{v}(\xi) - \frac{\xi}{|\xi|}\left(\frac{\xi}{|\xi|} \cdot \hat{v}(\xi)\right) = \hat{v}(\xi) - \left(\frac{\xi \cdot \hat{v}(\xi)}{|\xi|^2}\right)\xi,$$

and in this from, it is easy to see that

$$\xi \cdot \widehat{\mathbb{P}v}(\xi) = \xi \cdot \hat{v}(\xi) - \left(\frac{\xi \cdot \hat{v}(\xi)}{|\xi|^2}\right)\underbrace{(\xi \cdot \xi)}_{|\xi|^2} = 0.$$

Finally, $\hat{v}(\xi) \cdot \xi = 0$ by definition, hence

$$|\widehat{\mathbb{P}v}(\xi)|^2 = |\widehat{\mathbb{P}v}(\xi) \cdot \widehat{\mathbb{P}v}(\xi)| = |\widehat{\mathbb{P}v}(\xi) \cdot \hat{v}(\xi) - 0| \leq |\widehat{\mathbb{P}v}(\xi)||\hat{v}(\xi)|$$

and dividing both sides of the inequality by $|\widehat{\mathbb{P}v}(\xi)|$ proves (iv). □

Remark. If $\mathrm{div}(u) = 0$, then in general $\partial_i u_i = 0$, and thus $(\mathrm{div}\,(u \otimes u))_j = (\partial_i u_j u_i)_j = u_i \partial_i u_j + u_j \partial_i u_i = u_i \partial_i u_j = (u \cdot \nabla u)$, i.e. $\mathrm{div}\,(u \otimes u) = u \cdot \nabla u$.

Proposition. *The following proposition may be stated in two parts.*

(I) If $u \in C^2(\mathbb{T}^n \times (0,T), \mathbb{R}^n)$ *and* $p \in C^1(\mathbb{T}^n \times (0,T))$ *satify* $\partial_t u = \Delta u - u \cdot \nabla u - \nabla p$ *with* $div(u) = 0$ *in* $\mathbb{T}^n \times (0,T)$, *then we have*

$$\partial_t u = \Delta u - \mathbb{P} div\,(u \otimes u) \quad in\ \mathbb{T}^n \times (0,T).$$

(II) Conversly, if $u, w \in C\left([0,T), L^2(\mathbb{T}^n)^n\right)$ *satisfy* $div(u(0)) = divg = 0$ *and* $u' = \Delta u - \mathbb{P}w$ *on* $(0,T)$, *then* $div(u) = 0$ *and* $\exists p \in \left((0,T), H^1(\mathbb{T}^n)\right)$ *such that*

$$u' = \Delta u - w - \nabla p \quad on\ (0,T).$$

Proof. Since $\mathrm{div}(u) = 0$, it follows by continuity that $\mathbb{P}(u') = (\mathbb{P}u)' = u'$ and $\mathbb{P}\Delta u = \Delta \mathbb{P} u = \Delta u$. Moreover, it follows from the nature of the projectivity in the definition of $\mathbb{P}$ exhibited in the above discussion that $\mathbb{P}\nabla p = 0$. We conclude that (I) is a result of the last remark.

In order to prove (II), first observe that $(u - \mathbb{P}u)(0) = 0$ implies that $u = \mathbb{P}u$ and thus that $\mathrm{div}(u) = 0$. Secondly, since $(u - \mathbb{P}u)' = \Delta(u - \mathbb{P}u)$ and $(\mathbb{P}u)' = \mathbb{P}u' = \Delta\mathbb{P}u - \mathbb{P}w$, the required p is found by defining

$$\hat{p}(\xi) := \frac{i\xi}{|\xi|^2} \cdot \hat{w}(\xi)$$

($\hat{p}(0) = 0$), so that

$$\widehat{\nabla p}(\xi) = i\xi \frac{i\xi \cdot \hat{w}(\xi)}{|\xi|^2} = -\frac{\xi \otimes \xi}{|\xi|^2}\hat{w}(\xi),$$

from which we obtain

$$\widehat{w + \nabla p}(\xi) = \hat{w}(\xi) - \frac{\xi \otimes \xi}{|\xi|^2}\hat{w}(\xi) = \widehat{\mathbb{P}w}(\xi).$$

□

Corollary. *If $u, div\,(u \otimes u) \in C\left([0,T), L^2\right)$ are such that $u' = \Delta u - \mathbb{P}div\,(u \otimes u)$ with $divu(0) = 0$, then $\exists p \in \left((0,T), H^1(\mathbb{T}^n)\right)$ for which*

$$\begin{cases} u' = \Delta u - div\,(u \otimes u) - \nabla p \\ divu = 0 \end{cases}.$$

4.1.2 Maximal Mild Solutions

We conclude from the last two theorems that (N) may be reduced to a form for which we have already proved the existence and uniqueness of regular mild solutions in the case of a locally Lipschitz external heat energy. From now on, to simplify computations, we will assume w.l.o.g. the initial condition g of the NSE problem to be such that $\int_{\mathbb{T}^n} g = 0$, and thus that $\hat{u}(0) = 0$.

Theorem (Uniqueness, Regularity and Finite Time Blow Up of Mild Solutions). *The Navier-Stokes Equations with initial data $g \in H^s(\mathbb{T}^n)^n$, where $div(g) = 0$ and $s > \frac{n}{2}$, has a unique mild solution $u \in C\left([0,T), H^s(\mathbb{T}^n)^n\right)$ for some $T > 0$. The mild solution is smooth in $\mathbb{T}^n \times (0,T)$, hence classical. Moreover, if $T < \infty$, then $\|u(t)\|_s \longrightarrow \infty$ as $t \to T^-$.*

Proof. We need to solve the equation $u' = \Delta u + f(u)$, where $f(u) = \mathbb{P}\mathrm{div}\,(u \otimes u)$. From chapter 3, it is sufficient to show that $f : H^s(\mathbb{T}^n, \mathbb{R}^n) \longrightarrow H^{s-1}(\mathbb{T}^n, \mathbb{R}^n)$ is locally Lipshitz in order to prove the existence and uniqueness of a maximal mild solutions, but this follows immediately from the fact that f is a composition of $u \otimes u : H^s(\mathbb{T}^n, \mathbb{R}^n) \longrightarrow H^s(\mathbb{T}^n, \mathbb{R}^{n\times n})$, which is locally Lipschitz for $s > \frac{n}{2}$ by the Banach algebra property, and the two bounded linear function $\mathrm{div} : H^s(\mathbb{T}^n, \mathbb{R}^{n\times n}) \longrightarrow H^{s-1}(\mathbb{T}^n, \mathbb{R}^n)$ and $\mathbb{P} : H^{s-1}(\mathbb{T}^n, \mathbb{R}^n) \longrightarrow H^{s-1}(\mathbb{T}^n, \mathbb{R}^n)$. The rest of the proof goes through without problems. □

4.2 Existence of Global Solutions

We understand from the previous theorem that global control on $\|u(t)\|_2$ would be required (the Banach algebra property is needed) and sufficient to establish the global existence of the maximal solutions of the two and three dimensional NSE. In this section, we explore how one can acheive, or at least partly acheive, this control.

Proposition (Basic Energy Identity). *For $u \in C^\infty\left(\mathbb{T}^n \times [0,T)\right)$, a maximal solution of NSE, we have the following BEI (basic energy identity):*

$$\frac{1}{2}dt\|u\|^2 + \|\nabla \otimes u\|^2 = 0.$$

Proof. Integrating over $\mathbb{T}^n$,

$$\begin{aligned}
\frac{1}{2}dt\|u\|^2 &= \frac{1}{2}dt\int |u|^2 \\
&= \frac{1}{2}dt\int u_1^2+u_2^2 \\
&= \int u_1\partial_t u_1 + u_2\partial_t u_2 \\
&= \int u\cdot\partial_t u \\
&= \int u\cdot\Delta u - \int u\cdot\left(\operatorname{div}\left(u\otimes u\right)+\nabla p\right) \\
&= \underbrace{\int u\cdot\Delta u}_{A} - \underbrace{\int u\cdot\operatorname{div}\left(u\otimes u\right)}_{B} - \underbrace{\int u\cdot\nabla p}_{C}.
\end{aligned}$$

As $\operatorname{div}(u)=0$ by hypothesis, integration by parts immediately yields that

$$C=\int u_k\partial_k p = -\int p\partial_k u = -\int p\operatorname{div}(u) = 0.$$

Now, on the one hand,

$$\begin{aligned}
B &= \int u\cdot\operatorname{div}\begin{pmatrix} u_1u_1 & ... & u_1u_n \\ \vdots & \ddots & \vdots \\ u_nu_1 & ... & u_nu_n \end{pmatrix} \\
&= \int u\cdot\begin{pmatrix} \operatorname{div}\begin{pmatrix} u_1u_1 \\ \vdots \\ u_nu_1\end{pmatrix} \\ \vdots \\ \operatorname{div}\begin{pmatrix} u_1u_n \\ \vdots \\ u_nu_n\end{pmatrix}\end{pmatrix} \\
&= \int u\cdot\begin{pmatrix} \partial_1\left(u_1u_1\right)+...+\partial_n\left(u_nu_1\right) \\ \vdots \\ \partial_1\left(u_1u_n\right)+...+\partial_n\left(u_nu_n\right)\end{pmatrix} \\
&= \int\sum_{k=1}^n u_k\sum_{j=1}^n \partial_i\left(u_iu_k\right) \\
&= \int u_k\partial_i\left(u_iu_k\right),
\end{aligned}$$

and on the other hand, since $u_ku_k\partial_iu_i=\sum_{k=1}^n u_ku_k\sum_{i=1}^n\partial_iu_i = u_ku_k\operatorname{div}u=0$, integrating by parts again shows that we also have

$$B=\int u_k\partial_i\left(u_iu_k\right)=\int u_ku_k\partial_iu_i+\int u_ku_i\partial_iu_k = -\int u_k\partial_i\left(u_ku_i\right).$$

Hence $B=\int u_k\partial_i\left(u_iu_k\right)=0$, and we conclude, using integration by parts one last time, that

$$\frac{1}{2}dt\|u\|^2 = A = \sum_{k=1}^n\int u_j\partial_k^2u_j = -\int\left(\partial_ku_j\right)\left(\partial_ku_j\right) = -\int|\nabla\otimes u|^2 = -\|\nabla\otimes u\|^2.$$

□

Remark. Integrating both sides of $\frac{d}{dt}\|u\|^2 = -2\|\nabla \otimes u\|^2$ with respect to time on $[0,t]$ yields an equivalent and useful integral form of the BEI:

$$\begin{aligned}\|g\|^2 &= \|u(t)\|^2 + 2\int_0^t \|\nabla \otimes u(s)\|^2 ds \\ &= \|u(t)\|^2 + 2\int_0^t |u(s)|_1^2 ds.\end{aligned}$$

The last proposition shows that knowledge of a given initial initial datum g yields an initial basic control on the L^2H^1-norm and the $L^\infty L^2$-norm of the mild solution u. We would like to identify the additional conditions under which we could extend this control so that global existence is garanteed. The first step towards this objective is to consider the equation

$$\partial_t\partial_k u = \Delta\partial_k u - \partial_k\left(u\cdot\nabla u\right) - \partial_k\nabla p.$$

Using the same techniques that we have used in the last proposition, we derive that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_k u\|^2 &= \int \partial_k u\cdot\partial_t\partial_k u \\ &= \int \partial_k u\cdot\partial_t\left(\Delta\partial_k u - \partial_k\left(u\cdot\nabla u\right) - \partial_k\nabla p\right) \\ &= \underbrace{\int \partial_k u\cdot\left(\partial_k\Delta u - \partial_k\nabla p\right)}_{A} - \underbrace{\int \partial_k u\cdot\partial_k\left(u\cdot\nabla u\right)}_{B}.\end{aligned}$$

We further find that

$$\begin{aligned}B &= \int \partial_k u\cdot\partial_k\left(u\cdot\nabla u\right) \\ &= \int \begin{pmatrix}\partial_k u_1\\ \vdots\\ \partial_k u_n\end{pmatrix}\cdot\partial_k\left(u_1\partial_1 u + ... + u_n\partial_n u\right) \\ &= \int \begin{pmatrix}\partial_k u_1\\ \vdots\\ \partial_k u_n\end{pmatrix}\cdot\begin{pmatrix}\partial_k\left(u_1\partial_1 u_1\right) + ... + \partial_k\left(u_n\partial_n u_1\right)\\ \vdots\\ \partial_k\left(u_1\partial_1 u_n\right) + ... + \partial_k\left(u_n\partial_n u_n\right)\end{pmatrix} \\ &= \int \partial_k u_j\left(\partial_k\left(u_i\partial_i u_j\right)\right) \\ &= -\int \partial_k^2 u\cdot\left(u\cdot\nabla u\right),\end{aligned}$$

where the last step follows from integration by parts, and that

$$\begin{aligned}
A &= \int \partial_k u \cdot \partial_k \Delta u - \int \partial_k u \cdot \partial_k \nabla p \\
&= \sum_{i=1}^{n} \int \partial_k u_j \partial_k \left(\partial_i^2 u_j\right) - \int \left(\partial_k u_j\right) \partial_k \left(\partial_j p\right) \\
&= -\sum_{i=1}^{n} \int \partial_k \partial_i u_j \partial_k \left(\partial_i u_j\right) + \int \left(\partial_k^2 u_j\right) \partial_j p \\
&= -\sum_{i=1}^{n} \sum_{j=1}^{n} \int \left(\partial_k \partial_i u_j\right)^2 - \int p \partial_k^2 \underbrace{\mathrm{div} u}_{=0} \\
&= -\int |\nabla \otimes \partial_k u|^2 \\
&= -\|\nabla \otimes \partial_k u\|^2,
\end{aligned}$$

where again, integration by parts was a precious tool in the derivation of the last equalities. Hence,

$$\begin{aligned}
\frac{d}{dt}\|\partial_k u\|^2 &= -2\|\nabla \otimes \partial_k u\|^2 + 2\int \partial_k^2 u \cdot (u \cdot \nabla u) && (4.2.1) \\
&\leq -2\|\nabla \otimes \partial_k u\|^2 + 2\int |\partial_k^2 u||u \cdot \nabla u| \\
&\leq -2\|\nabla \otimes \partial_k u\|^2 + \int |\partial_k^2 u|^2 + \int |u \cdot \nabla u|^2 && (4.2.2) \\
&= -2\|\nabla \otimes \partial_k u\|^2 + \|\partial_k^2 u\|^2 + \|u \cdot \nabla u\|^2,
\end{aligned}$$

where (2) follows from the fact that $ab \leq (a^2 + b^2)/2$ for any real numbers $a, b > 0$. The above derivation allows us to find an analog inequality for the norm of $\nabla \otimes u$. Indeed,

$$\begin{aligned}
\frac{d}{dt}\|\nabla \otimes u\|^2 &= \frac{d}{dt}\int |\nabla \otimes u|^2 \\
&= \frac{d}{dt}\sum_{k,i=1}^{n} \int \left(\partial_k u_i\right)^2 \\
&= \frac{d}{dt}\sum_{k=1}^{n} \int |\partial_k u|^2 \\
&= \sum_{k=1}^{n} \frac{d}{dt}\|\partial_k u\|^2,
\end{aligned}$$

thus using (1) and going through the above arguments once again,

$$\begin{aligned}
\frac{d}{dt}\|\nabla \otimes u\|^2 &= -2\sum_{k=1}^{n} \|\nabla \otimes \partial_k u\|^2 + 2\int \sum_{k=1}^{n} \left(\partial_k^2 u \cdot (u \cdot \nabla u)\right) \\
&= -2\int \sum_{k,j,i=1}^{n} \left(\partial_i \partial_k u_j\right)^2 + 2\int \Delta u \cdot (u \cdot \nabla u) \\
&\leq -2\int |\nabla^2 \otimes u|^2 + 2\int |\Delta u||u \cdot \nabla u| \\
&\leq -2\|\nabla^2 \otimes u\|^2 + \int |\Delta u|^2 + \int |u \cdot \nabla u|^2 \\
&= -2\|\nabla^2 \otimes u\|^2 + \|\Delta u\|^2 + \|u \cdot \nabla u\|^2.
\end{aligned}$$

This is an enlightening inequality, because since $\|\Delta u\|^2 \leq \|\nabla^2 \otimes u\|^2$, we have

$$\begin{aligned} \frac{d}{dt}\|\nabla \otimes u\|^2 &\leq -2\|\nabla^2 \otimes u\|^2 + \|\Delta u\|^2 + \|u \cdot \nabla u\|^2 \\ &= -\|\nabla^2 \otimes u\|^2 + \|u \cdot \nabla u\|^2, \\ &\leq -\|\nabla^2 \otimes u\|^2 + \|u\|_\infty^2 \|\nabla u\|^2 \end{aligned} \tag{4.2.3}$$

and it follows from Berstein's Theorem that

$$\begin{aligned} \frac{d}{dt}\|\nabla \otimes u\|^2 &\leq -\|\nabla^2 \otimes u\|^2 + C_\sigma \|u\|_\sigma^2 \|\nabla u\|^2 \\ &\leq -\|\nabla^2 \otimes u\|^2 + C_\sigma \|u\|_\sigma^2 \|\nabla \otimes u\|^2 \end{aligned}$$

for any $\sigma > \frac{n}{2}$, where C_σ is a constant which depends on σ. In other words,

$$\frac{d}{dt}|u|_1^2 \leq -|u|_2^2 + C_\sigma \|u\|_\sigma^2 |u|_1^2 \lesssim \|u\|_\sigma^2 |u|_1^2,$$

and applying Gronwall's inequality to the latter yields

$$|u(t)|_1^2 \lesssim |u(0)|_1^2 \exp\left\{\int_0^t \|u(s)\|_\sigma^2 ds\right\}.$$

This suggests that we could acheive global control on $\partial_k u$ provided we have control on $\int_0^t \|u(s)\|_\sigma^2 ds$, i.e. provided the L^2H^σ-norm of u is controlled, where the only condition imposed on the degree of H^σ is that $\sigma > \frac{n}{2}$. In the two dimensional NSE, this requirement is that $\sigma - 1 > 0$, and one can show that the problem is globally solvable. In three dimensions though, the L^2H^1-norm control on u given by the BEI is not sufficient to establish control over the derivative, because the condition $\sigma - 1 > \frac{1}{2}$ requires to overcome a positive gap between our current degree of control and the critial one, a challenge for which no mathematical tools or techniques were found effective yet. We will thus need to consider an additional condition on the initial data to obtain global solvability in the 3D NSE problem.

In order to prove the two dimensional case, we need to equipe ourselves with some essential inequalities.

Lemma (One Dimensional Agmon's Type Inequality). *For $f \in C^1$ compactly supported in $\mathbb{R}$, then*

$$f(x)^2 \leq \|f\|_{L^2(\mathbb{R})} \|f'\|_{L^2(\mathbb{R})}.$$

Proof. Since

$$f(x)^2 \leq 2\int_{-\infty}^x f(s)f'(s) \leq 2\int_{-\infty}^x |f(s)||f'(s)|$$

and

$$f(x)^2 \leq -2\int_x^\infty f(s)f'(s) \leq 2\int_x^\infty |f(s)||f'(s)|,$$

we have

$$f(x)^2 \leq \int_{-\infty}^\infty |f(s)||f'(s)|,$$

and it follows immediatly from Cauchy-Schwartz inequality that

$$f(x)^2 \leq \left(\int_{\mathbb{R}} |f(s)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f'(s)|^2\right)^{\frac{1}{2}}.$$

$\square$

Corollary (Two Dimensional Ladyzhenskaya Inequality). *For a C^1 compactly supported function u over $\mathbb{R}^2$,*

$$\|u\|^2_{L^4(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})} \|\nabla u\|_{L^2(\mathbb{R})}.$$

Proof. The proof relies on the application of the 1D Agmon's type inequality to each variable of u. This yields

$$\begin{aligned} u(x,y)^4 &= u(x,y)^2 u(x,y)^2 \\ &\leq \left[\left(\int_{\mathbb{R}} |u(s_1,y)|^2 ds_1 \right) \left(\int_{\mathbb{R}} |u_x(s_1,y)|^2 ds_1 \right) \left(\int_{\mathbb{R}} |u(x,s_2)|^2 ds_s \right) \left(\int_{\mathbb{R}} |u_y(x,s_2)|^2 ds_s \right) \right]^{\frac{1}{2}}, \end{aligned}$$

from which

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} u(x,y)^4 dx dy \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(s_1,y)|^2 ds_1 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u_x(s_1,y)|^2 ds_1 \right)^{\frac{1}{2}} dy \\ \cdot \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(x,s_2)|^2 ds_2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u_y(x,s_2)|^2 ds_2 \right)^{\frac{1}{2}} dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, we find that

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} u(x,y)^4 dx dy &\leq \left[\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u(s_1,y)|^2 ds_1 dy \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x,s_2)|^2 ds_2 dx \right) \right]^{\frac{1}{2}} \\ &\quad \cdot \left[\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u_x(s_1,y)|^2 ds_1 dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u_y(x,s_2)|^2 ds_2 dx \right)^{\frac{1}{2}} \right] \\ &\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x,y)|^2 dx dy \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u|^2 dx dy \right) = \|u\|^2_{L^2} \|\nabla u\|^2_{L^2}. \end{aligned}$$

□

Remark. The above inequalities can be generalized. The stated versions are sufficient to acheive our goals in the following theorems, but it is interesting to know that the conditions imposed on u and f can be loosen to weak differentiability requirements.

Corollary. *If $u \in H^1(\mathbb{T}^2)$ then there exists a real constant $C_u > 0$ such that*

$$\|u\|^2_{L^4(\mathbb{T}^2)} \leq C_u \|u\|_{L^2(\mathbb{T}^2)} \|u\|_{H^1(\mathbb{T}^2)}.$$

Proof. We want want to use the periodicity of u to create a functional setting in which we can apply Ledyzhenskaya inequality. Let $\phi \in C^\infty(\mathbb{R}^2)$ be such that $\phi = 1$ on $\Gamma = [0, 2\pi]^2$, $\phi = 0$ on $\mathbb{R}^2 \backslash (-2\pi, 4\pi)^2$ and $0 \leq \phi(x) \leq 1$ $\forall x \in \mathbb{R}^2$. Then for $u \in C^1(\mathbb{T}^2)$, $\phi u \in C^1(\mathbb{R}^2)$ and $\operatorname{supp}(\phi u) = [-2\pi, 4\pi]^2 = \Omega$. Since Ω is compact in $\mathbb{R}^2$, it follows from Ledyzhenskaya inequality that

$$\|u\|^2_{L^4(\mathbb{T}^2)} = \|\phi u\|^2_{L^4(\Gamma)} \leq \|\phi u\|^2_{L^4(\mathbb{R})} \leq \|\phi u\|_{L^2(\mathbb{R})} \|\nabla \phi u\|_{L^2(\mathbb{R})}.$$

On the one hand,

$$\|\phi u\|_{L^2(\mathbb{R})} \leq \|\phi\|_{L^\infty(\Omega)} \|u\|_{L^2(\Gamma)} = \|\phi\|_{L^\infty(\Omega)} \|u\|_{L^2(\mathbb{T}^2)} \lesssim \|u\|_{L^2(\mathbb{T}^2)}.$$

On the other hand, since $\|\nabla u\|_{L^2(\mathbb{T}^2)} = |u|_{H^1(\mathbb{T}^2)}$, we also have

$$\begin{aligned} \|\nabla \phi u\|_{L^2(\mathbb{R})} &= \|\phi \nabla u\|_{L^2(\Omega)} \|u \nabla \phi\|_{L^2(\Omega)} \\ &\leq \|\phi\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\mathbb{T}^2)} + \|\nabla \phi\|_{L^\infty(\Omega)} \|u\|_{L^2(\mathbb{T}^2)} \\ &\lesssim \left(\|u\|_{L^2(\mathbb{T}^2)} + \|\nabla u\|_{L^2(\mathbb{T}^2)} \right) \\ &\lesssim \|u\|_{H^1(\mathbb{T}^2)}. \end{aligned}$$

Combining the two inequalities yields

$$\|u\|^2_{L^4(\mathbb{T}^2)} \lesssim \|u\|_{L^2(\mathbb{T}^2)} \|u\|_{H^1(\mathbb{T}^2)}.$$

$\square$

Remark. In the next theorems, we will in particular use the inequality $\|u\|^2_{L^4(\mathbb{T}^2)} \leq C_u \|u\|_{L^2(\mathbb{T}^2)} |u|_{H^1(\mathbb{T}^2)}$. The latter holds immediatly in the current setting because the assumption $\hat{u}(0) = 0$ implies that $\|u\|_{L^2(\mathbb{T}^2)} = 0$. In fact, this hypothesis implies in general that $\|\cdot\|_{H^{s_1}} \leq \|\cdot\|_{H^{s_2}}$ if $s_1 \leq s_2$.

Theorem (Global Solvability of the Two Dimensional NSE). *The two dimensional NSE is globally solvable for smooth initial data.*

Remark. The smoothness assumption is for convinience. The theorem holds for initial data in H^1, or more generally, in H^s for any $s \geq 1$.

Proof. Recall that before proving the above inequalities, we were considering the equation

$$\partial_t \partial_k u = \Delta \partial_k u - \partial_k (u \cdot \nabla u) - \partial_k \nabla p.$$

In trying to globally control $\|u\|_1$, we had derived equation (3):

$$\frac{d}{dt} \|\nabla \otimes u\|^2 \leq -\|\nabla^2 \otimes u\|^2 + \|u \cdot \nabla u\|^2.$$

Assuming u is a mild solution of the 2D NSE, we can use Ladyzhenskaya inequality to reach an inequality which is viable for the comparision principle to yield this desired control. Indeed, it follows from (3) and the Cauchy-Schwartz inequality that

$$\begin{aligned}
\frac{d}{dt} \|\nabla \otimes u\|^2 &\leq -|u|_2^2 + \|u \cdot \nabla u\|^2 \\
&\leq -|u|_2^2 + \int_{\mathbb{T}^n} |u \cdot \nabla u|^2 \\
&\leq -|u|_2^2 + \int_{\mathbb{T}^n} |u|^2 |\nabla u|^2 \\
&\leq -|u|_2^2 + \left(\int_{\mathbb{T}^n} |u|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} |\nabla u|^4 \right)^{\frac{1}{2}} \\
&\leq -|u|_2^2 + \|u\|^2_{L^4} \|\nabla u\|^2_{L^4}.
\end{aligned}$$

Applying Ladyzhenskaya inequality to the right and side of the last inequality yields

$$\begin{aligned}
\frac{d}{dt} |u|_1^2 &\leq -|u|_2^2 + C_{u,\nabla u} \|u\|_{L^2} |u|_1 \|\nabla u\|_{L^2} |\nabla u|_1 \\
&= -|u|_2^2 + C_{u,\nabla u} \|u\|_{L^2} |u|_1^2 |u|_2 \\
&\lesssim -|u|_2^2 + \|u\|^2_{L^2} |u|_1^4 + |u|_2^2 \\
&= \|u\|^2_{L^2} |u|_1^2 |u|_1^2
\end{aligned}$$

Hence, Gronwall's inequality implies that

$$|u(t)|_1^2 \lesssim |u(0)|_2^2 \exp\left\{ \int_0^t \|u(s)\|_{L^2} |u(s)|_1^2 ds \right\},$$

and since the BEI yields control over both the $L^\infty L^2$ and the $L^2 H^1$-norm of u, this inequality may be simplified to

$$|u(t)|_1^2 \lesssim |u(0)|_2^2 \exp\left\{ \int_0^t |u(s)|_1^2 ds \right\}.$$

We thus have control on $\|u\|_1$.

We now consider the equation

$$\partial_t \partial_i \partial_k u = \Delta \partial_i \partial_k u - \partial_i \partial_k (u \cdot \nabla u) - \nabla \partial_i \partial_k p$$

in the hope that the previous techniques that we have used to bound $\|u\|_1$ will also allow us to control $\|u\|_2$. Following the same formal arguments, we apply integration by parts to derive

$$\begin{aligned} \frac{d}{dt}\|\partial_i \partial_k u\|^2 &= -2\|\nabla \otimes \partial_i \partial_k u\|^2 - \int \partial_i \partial_k u \cdot \partial_i \partial_k (u \cdot \nabla u) \\ &= -2\|\nabla \otimes \partial_i \partial_k u\|^2 + \int \partial_i^2 \partial_k u \cdot \partial_k (u \cdot \nabla u), \end{aligned}$$

which further implies that

$$\begin{aligned} \frac{d}{dt}\|\nabla^2 \otimes u\|^2 &\leq -2\|\nabla^3 \otimes u\|^2 + \int \sum_{k,i=1}^{n} \partial_i^2 \partial_k u \cdot \partial_k (u \cdot \nabla u) \\ &= -2\|\nabla^3 \otimes u\|^2 + \int \sum_{k=1}^{n} \Delta \partial_k u \cdot \partial_k (u \cdot \nabla u) \\ &\leq -2\|\nabla^3 \otimes u\|^2 + \int \sum_{k=1}^{n} \Delta \partial_k u \cdot \partial_k (u \cdot \nabla u) \\ &\leq -2\|\nabla^3 \otimes u\|^2 + \int \sum_{k=1}^{n} |\Delta \partial_k u||\partial_k (u \cdot \nabla u)| \\ &= -2\|\nabla^3 \otimes u\|^2 + \left(\int \sum_{k,i,j}^{n} \left(\partial_i^2 \partial u_k u_j\right)^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \left(\partial_k u_i \partial_i u\right)^2\right)^{\frac{1}{2}} \\ &\leq -2\|\nabla^3 \otimes u\|^2 + \|\nabla^3 \otimes u\|\|\nabla \otimes (u \cdot \nabla u)\|. \end{aligned} \tag{4.2.4}$$

Now, observe that from the product rule and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|\nabla\otimes(u\cdot\nabla u)\|^2 &= \int|\nabla\otimes(u\cdot\nabla u)|^2\\
&= \int\sum_{i,j=1}^n\left(\partial_i\left(u_k\partial_k u_j\right)\right)^2\\
&= \int\sum_{i,j=1}^n\left(\sum_{k=1}^n(\partial_i u_k)(\partial_k u_j)+\sum_{k=1}^n u_k\partial_i\partial_k u_j\right)^2\\
&\le 2\int\sum_{i,j=1}^n\left[\left(\sum_{k=1}^n(\partial_i u_k)(\partial_k u_j)\right)^2+\left(\sum_{k=1}^n u_k\partial_i\partial_k u_j\right)^2\right]\\
&\le 2\int\sum_{i,j=1}^n\left(\sum_{k=1}^n(\partial_i u_k)^2\right)\left(\sum_{k=1}^n(\partial_k u_j)^2\right)+2\int\sum_{i,j=1}^n\left(\sum_{k=1}^n u_k\partial_i\partial_k u_j\right)^2\\
&= 2\int\sum_{i,k=1}^n(\partial_i u_k)^2\sum_{j,k=1}^n(\partial_k u_j)^2+2\int\sum_{i,j=1}^n|u\cdot\partial_i\nabla u_j|^2\\
&\le 2\int|\nabla\otimes u|^4+2\int\sum_{i,j=1}^n|u|^2|\partial_i\nabla u_j|^2\\
&= 2\|\nabla\otimes u\|_{L^4}^4+2\int|u|^2\sum_{i,j,k=1}^n(\partial_i\partial_k u_j)^2\\
&= 2\|\nabla\otimes u\|_{L^4}^4+2\int|u|^2|\nabla^2\otimes u|^2\\
&\le 2\|\nabla\otimes u\|_{L^4}^4+2\|u\|_{L^4}^2\|\nabla^2\otimes u\|_{L^4}^2.
\end{aligned}$$

Hence, it follows from Ladyzhenskaya inequality that

$$\begin{aligned}
\|\nabla\otimes(u\cdot\nabla u)\| &\le 2\|\nabla\otimes u\|_{L^2}|\nabla\otimes u|_1+2\|u\|_{L^2}^{\frac12}|u|_1^{\frac12}\|\nabla^2\otimes\|_{L^2}^{\frac12}|\nabla^2\otimes u|_1^{\frac12}\\
&\lesssim |u|_1|u|_2+\|u\|_{L^2}^{\frac12}|u|_1^{\frac12}|u|_2^{\frac12}|u|_3^{\frac12}. \qquad (4.2.5)
\end{aligned}$$

Combining the inequalities (4) and (5), we find that

$$\frac{d}{dt}|u|_2^2\lesssim -2|u|_3^2+|u|_1|u|_2|u|_3+\|u\|_{L^2}^{\frac12}|u|_1^{\frac12}|u|_2^{\frac12}|u|_3^{\frac32}.$$

Hence, as $|u|_1|u|_2|u|_3\le|u|_1^2|u|_2^2+|u|_3^2$ by the Cauchy-Schwarz inequality and

$$\|u\|_{L^2}^{\frac12}|u|_1^{\frac12}|u|_2^{\frac12}|u|_3^{\frac32}\le\frac14\left(\|u\|_{L^2}^{\frac12}|u|_1^{\frac12}|u|_2^{\frac12}\right)^4+\frac34|u|_3^2$$

by choosing $a=|u|_3^{\frac32}$, $b=\|u\|_{L^2}^{\frac12}|u|_1^{\frac12}|u|_2^{\frac12}$, $p=4$ and $\epsilon=1$ in Young's inequality which states that

$$ab\le\frac{\epsilon}{p}a^p+\frac{1}{\epsilon^{1/(p-1)}q}b^q,$$

for all $a,b>0$, $1<p<\infty$ with $q=p/(p-1)$ and $\epsilon>0$, we conclude that

$$\frac{d}{dt}|u|_2^2\lesssim|u|_1^2|u|_2^2\|u\|_{L^2}^2+|u|_1^2|u|_2^2=|u|_2^2\left(|u|_1^2\|u\|_{L^2}^2+|u|_1^2\right).$$

Since we have control on both the L^2 and the H^1 norm of u, the proof is completed by gainning global control over $\|u(t)\|_2$, either by comparing the above equation with the ODE $v'=cv$ for an appropriate $c\in\mathbb{R}$ and applying the comparision principle, or by using Gronwall's inequality again. □

Theorem. *The NSE in $\mathbb{T}^3$ has a global solution if the initial data $|g|_2$ is small enough.*

Proof. The proof is completely analogous to the one found in the question 3 of Annex 3. $\square$

Remark. This doesn't fully solve the general NSE problem, because it leaves unkown the behavior of the solutions when the H^2-norm of the initial data is big. Unfortunatly, a short investigation of scaling arguments soon illustrates that they are inefficient to carry this method further into solving the general problem. In the next chapter, we study a new approach towards solving the NSE.

Chapter 5

Weak Solutions of the Navier-Stokes Equations

The approach we have taken so far in trying to solve the NSE was primarily concerned with solving the equations directly. In this chapter however, the heuristic of the regularization method which we are going to use is quite different. The idea is to consider a family NSE(ϵ) of problems for which a global solution can be found more easily. We must define this set of problems so that there is a formal equivalence between NSE(0) and NSE. We then hope that the solvability of the altered problems will yield the desired solvability of the NSE through the limit as $\epsilon \to 0$. In other words, we will investigate the convergence of NSE(ϵ) as $\epsilon \to 0$, and if a limit exists, we will verify if it solves the original NSE. The well-known regularization procedures comprise the definition of the Hyperdissipative NSE

$$\partial_t u = -\epsilon \Delta^2 u + \Delta u - \mathbb{P}\mathrm{div}\,(u \otimes u)\,,$$

the Hopf-Galerkin method

$$\partial_t u = \Delta u - P_m \mathbb{P}\mathrm{div}\,(P_m u \otimes P_m u)\,,$$

where P_m is the Fourier truncation operator, and other Leray's regularization procedures.

5.1 Leray's Regularization

Leray's regularization is the main regularization method that we are going to investigate.

5.1.1 Leray-Navier-Stokes Equations

For a bounded linear operator

$$J : H^s(\mathbb{T}^n) \longrightarrow H^{s+\theta}(\mathbb{T}^n), \tag{5.1.1}$$

where $\theta > 0$ is constant and $s \geq 0$ is arbitrary, the Leray-Navier-Stokes equations (LNS) are defined to be

$$\begin{cases} \partial_t u = \Delta u - \mathbb{P}\mathrm{div}\,(Ju \otimes u) \\ \mathrm{div} u = 0 \end{cases}.$$

Some examples of LNS are given by the following operators.

* $J := (1 - \epsilon\Delta)^{-1}$ acting in Fourier space as

$$\widehat{Ju}(\xi) = (1 + \epsilon|\xi|^2)^{-1}\hat{u}(\xi).$$

This defines a LNS family of problems with $\theta = 2$ in (5.1.1).

* $Ju = \phi * u$, which translates in Fourier space as

$$\widehat{Ju}(\xi) = \hat{\phi}(\xi)\hat{u}(\xi).$$

Here, LNS can be defined for θ arbitrary in (5.1.1).

* $Ju = P_m u$ defined in Fourier space by

$$\widehat{P_m u}(\xi) = \chi_{Q_m}\hat{u}(\xi),$$

where $Q_m = [-m, m]^n$ and χ is the characterisitc function, will be of special interest. Observe that θ may be chosen arbitrarily, and we find

$$|P_m u|_s^2 = \sum_{\xi \in Q_m} |\xi|^{2s+2\theta}|\hat{u}(\xi)|^2 \leq \sup_{\xi \in Q_m} |\xi|^{2\theta} \sum_{\xi \in \mathbb{Z}^n} |\xi|^2 s|\hat{u}(\xi)|^2.$$

Thus $|P_m|_s \leq C_m^\theta |u|_s$, and it easily follows that $\|P_m\| \leq C_m^\theta \|u\|_s$.

5.1.2 Global Solvability of the LNS

All of the operators found in the examples of section 5.1.1 are Fourier multiplyers. They act on $u \in H^s(\mathbb{T}^n)$ by index wise multiplication of its Fourier coefficients. It is then clear from the introductory discussion of chapter 4 that for any of the above bounded linear operator J,

- ⋄ $\int_{\mathbb{T}^n} u = 0 \implies \int_{\mathbb{T}^n} Ju = 0$, and
- ⋄ $\mathrm{div}(u) = 0 \implies \mathrm{div}(Ju) = 0$.

We will thus assume those properties for a fact, along with the inequality $\|Ju\|_s \leq C\|u\|_s$, which is the general analog to the one we had derived when considering $J = P_m$ in section 5.1.1.

Under the above assumptions, the LNS problem is well-posed in $H^s(\mathbb{T}^n)$ if $s > \frac{n}{2}$, because from the multiplicative properties found in Sobolev spaces, we have

$$\|Ju \otimes u\|_s \lesssim \|Ju\|_s\|u\|_s \lesssim \|u\|_s^2.$$

Moreover, the global existence of its solutions can be shown by the same integral arguments that were used in chapter 4. Indeed, the energy method easily yields the basic energy estimate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2 &= -|u|_1^2 - \int_{\mathbb{T}^n} \mathbb{P}\mathrm{div}\,(Ju \otimes u) \cdot u \\
&= -|u|_1^2 - \int_{\mathbb{T}^n} \mathrm{div}\,(Ju \otimes u) \cdot u \\
&= -|u|_1^2,
\end{aligned}$$

which as we recall, leads to $\|u(t)\|^2 + 2\int_0^t |u(s)|_1^2 ds = \|g\|^2$. For the higher orders k with $k > \frac{n}{2}$, we go through the energy method again to find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u|_k^2 &= -|u|_{k+1}^2 - \int_{\mathbb{T}^n} \nabla^k \otimes \mathrm{div}\,(Ju \otimes u) : \nabla^k \otimes u \\
&\leq -|u|_{k+1}^2 + |Ju \otimes u|_{k+1}|u|_k \qquad (5.1.2)\\
&\leq -|u|_{k+1}^2 + C_s|Ju|_{k+1}|u|_{k+1}|u|_k \qquad (5.1.3)\\
&\leq C_s^2|Ju|_{k+1}^2|u|_k^2, \qquad (5.1.4)
\end{aligned}$$

where (5.1.2) and (5.1.4) were derived by using Cauchy-Schwarz and (5.1.3) by the Banach algebra property of the Sobolev spaces under the hypothesis $k > \frac{n}{2}$. From this inequality, we have

$$\frac{1}{2}\frac{d}{dt}|u|_k^2 \leq C|u|_{k+1-\theta}^2|u|_k^2,$$

and choosing $k = \theta$ in the definition of J yields

$$\frac{1}{2}\frac{d}{dt}|u|_k^2 \leq C|u|_1^2|u|_k^2,$$

which shows that an H^k-norm control over the solution u implies its global existence, because control of its H^1-norm is ensured by the basic energy estimate.

Theorem. *An H^k-norm control a of solution u of the LNS implies the global existence in time of that solution with respect to the H^k-norm.*

5.2 Function Spaces Theory of the Weak Solutions

Two questions that one should certainly adress when discussing the existence and properties of the LNS solutions and whether they converge to solutions to the classical NS problem is are:

1. In which space can we find those solutions?
2. What type of convergence should be investigated, and if convergence is to be found, do the limits solve the original Navier-Stokes equations?

The aim of this section is to answer the above questions by formalizing the proceedure discussed in the beginning of chapter 5.

The basics of the function spaces theory of the weak solution will be derived, and its particular application to the NSE will be realized as an ungoing example. From now on, let $\langle \cdot, \cdot \rangle$ denote the duality pairing for the action of a functional and $\langle \cdot, \cdot \rangle_X$ denote the inner product of the space X.

5.2.1 Negative Order Sobolev Spaces

It is natural to formally extend the characterization of the H^s-norm, $s \geq 0$, in Fourier space given by

$$\|u\|_s^2 = \sum \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 = \|\hat{u}\|_{\ell_s^2}^2 \tag{5.2.1}$$

to $s \in \mathbb{R}$. For the case where $s > 0$, no work is needed: we use simply use (5.2.1) and it naturally corresponds to H^s. The negative case leads to an interesting theory.

Definition. For $s < 0$, we define $H^s(\mathbb{T}^n)$ to be the completion of $L^2(\mathbb{T}^n)$ with respect to the norm $\|u\|_s := \|\hat{u}\|_{\ell_s^2}^2$ defined in the Fourier space.

If $\{u_m\}_m \subset L^2$ is a Cauchy sequence with respect to $\|\cdot\|_s$, it is a member of the equivalence class of some $u \in H^s$; that is, by definition, $\{\widehat{u_m}\}_m$ is Cauchy with respect to $\|\cdot\|_{\ell_s^2}$. That $\left\{\left((1+|\xi|)^s\, \widehat{u_m}(\xi)\right)_\xi\right\}_m$ is Cauchy is ℓ_s^2 implies that there exists a sequence $a \in \ell_s^2$ such that $\widehat{u_m} \longrightarrow a$ in ℓ_s^2. Now, $\hat{u} := \mathscr{F}u := a$, where $\mathscr{F}u$ is the Fourier series of u, is well-defined, because if any other Cauchy sequence $\{v_k\}_k$ is a representative for u, then $u_1, v_1, u_2, v_2, ...$ being Cauchy will lead to $\widehat{\{v_k\}}_k$ converging to a with respect to $\|\cdot\|_{\ell_s^2}$ as $k \to \infty$.

Proposition. *$C^\infty(\mathbb{T}^n)$ is dense in $H^s(\mathbb{T}^n)$.*

Proof. The fact that the continuous extension of the norm $\|\cdot\|_s$ from L^2 to H^s is given by definition through $\|u\|_s = \|\widehat{u}\|_{\ell^2_s}$ implies that the partial sums

$$\sum_{\xi \in Q_m} \widehat{u}(\xi) e^{i\xi\cdot x} \longrightarrow u$$

in H^s as $m \to \infty$. As these are trigonometric polynomials, the above shows in particular that $C^\infty(\mathbb{T}^n)$ is dense in $H^s(\mathbb{T}^n)$. $\square$

Remark. For any $s \in \mathbb{R}$, it is clear from viewing ℓ^2 has the image in Fourier space of Bessel potentials that H^s can be given an Hilbert structure, because $J^s : \ell^2_s \longrightarrow \ell^2$ defined by $(J^s a)(\xi) = \left(1 + |\xi|^2\right)^{\frac{s}{2}} a(\xi)$ for $\{a(\xi)\}_\xi \in \ell^2_s$ is easily seen to be an isometry.

We now want to characterize H^s, $s < 0$, in such a way that we can make sense of $u \in H^s$ in a more pratical fashion than through a limiting process. We will show that for $s \in \mathbb{R}$, $H^{-s} \cong (H^s)^* \cong {}^*(H^s)$, where the last two terms of the equality are respectively the dual and the anti-dual dual of H^s. Hence, instead of understanding $u \in H^s$ as a limit point of L^2 functions, we will consider it as an object acting on the later.

Definition. Let X and Y be complex linear spaces. A map $A : X \longrightarrow Y$ is said to be linear if for any $x_1, x_2 \in X$ and $\forall \lambda, \mu \in \mathbb{C}$,

$$A(\lambda x_1 + \mu x_2) = \lambda A(x_1) + \mu A(x_2).$$

It is called anti-linear if

$$A(\lambda x_1 + \mu x_2) = \bar{\lambda} A(x_1) + \bar{\mu} A(x_2).$$

When $Y = \mathbb{C}$, the space of all bounded linear map $A : X \longrightarrow \mathbb{C}$ is called the topological dual of X and is denoted X^*, while the space of all such anti-linear map is called the anti-dual of X and is denoted ${}^*X$.

Example. Let $u \in L^2$ and define, for $k = 1, 2, 3, 4$, $j_k(u) : L^2 \longrightarrow \mathbb{C}$ by

$$\begin{aligned}
(j_1(u))(v) &= \int uv = \langle u, \bar{v} \rangle \\
(j_2(u))(v) &= \int u\bar{v} = \langle u, v \rangle \\
(j_3(u))(v) &= \int \bar{u}v = \langle v, u \rangle = \overline{\langle u, v \rangle} \\
(j_4(u))(v) &= \int \bar{u}\bar{v}.
\end{aligned}$$

Then $j_1(u), j_3(u) \in (L^2)^*$ while $j_2(u), j_4(u) \in {}^*(L^2)$. We also have that

$$\begin{aligned}
&j_1 : L^2 \longrightarrow (L^2)^* \text{ is linear,} \\
&j_2 : L^2 \longrightarrow {}^*(L^2) \text{ is linear,} \\
&j_3 : L^2 \longrightarrow (L^2)^* \text{ is anti-linear, and} \\
&j_4 : L^2 \longrightarrow {}^*(L^2) \text{ is anti-linear.}
\end{aligned}$$

In some sense, the best choices in practice are j_1 and j_3, as they allow us to work in standard duals.

Lemma. *For $s \in \mathbb{R}$, H^{-s} can be identified with $(H^s)^*$.*

Proof. Assume $s > 0$ and let $u \in H^{-s}$ (the case where $s < 0$ is similar). Define $T : H^s \longrightarrow (H^s)^*$ by

$$\langle Tu, v \rangle = \sum \overline{\hat{u}(\xi)} \hat{v}(\xi), \;\; v \in H^s.$$

T is well-defined, because from the Cauchy-Schwarz inequality,

$$|\langle Tu, v\rangle| \leq \sum \left(1+|\xi|^2\right)^{-\frac{s}{2}} |\overline{\hat{u}(\xi)}| \left(1+|\xi|^2\right)^{\frac{s}{2}} |\hat{v}(\xi)| \leq \|\hat{u}\|_{\ell^2_{-s}} \|\hat{v}\|_{\ell^2_s} < \infty, \tag{5.2.2}$$

i.e. $Tu \in (H^s)^*$.

We want to show that the functional T is in fact an isometry. From the inequality (5.2.2), $\|Tu\|_{(H^s)^*} \leq \|u\|_{\ell^2_{-s}}$. It is thus sufficient to show that the supremum in the definition of $\|Tu\|_{(H^s)^*}$ is acheived by some $v \neq 0$ to complete the equality. We find the supremum to be acheived at $v \in H^s$ such that $\hat{v}(\xi) = \hat{u}(\xi)\left(1+|\xi|^2\right)^{-s}$. Indeed, for such a choice of v,

$$\langle Tu, v\rangle = \sum (1+|\xi|^2)^{-s} |\hat{u}(\xi)|^2 = \|u\|^2_{\ell^2_{-s}} = \|\hat{u}\|^2_{\ell^2_{-s}} \|\hat{v}\|_{\ell^2_s},$$

where the last equality on the right hand side holds from

$$\|\hat{v}\|^2_{\ell^2_s} = \sum \left(1+|\xi|^2\right)^s |\hat{v}(\xi)|^2 = \sum \left(1+|\xi|^2\right)^s |\hat{u}(\xi)\left(1+|\xi|^2\right)^{-s}|^2 = \|\hat{u}\|^2_{\ell^2_s}.$$

We now want to prove that T is surjective. Letting $f \in (H^s)^*$, we will show that $\exists u \in H^{-s}$ such that $Tu = f$. By linearity, f acts on $v \in H^s$ in Fourier space by

$$f(v) = f\left(\sum \hat{v}(\xi) e^{i\xi\cdot x}\right) = \sum \hat{v}(\xi) f(e^{i\xi\cdot x}) = \sum \overline{\overline{f(e^{-i\xi\cdot x})}}\hat{v}(\xi).$$

We thus suspect $u = \sum f(e^{-i\xi\cdot x})e^{i\xi\cdot x}$, i.e. $\hat{u}(\xi) = f(e^{-i\xi\cdot x})$, to be the element in H^{-s} we are looking for, since its action would be, by definition of T, the same as the action of f on any element $v \in H^s$. It is sufficient to show that $\left(f(e^{-i\xi\cdot x})\right)_\xi \in \ell^2_{-s}$ to confirm that u is indeed the required element. By definition, we have, for $v \neq 0$, $v \in H^s$,

$$|f(v)| = \frac{|f(v)|\|\hat{v}\|_{\ell^2_s}}{\|\hat{v}\|_{\ell^2_s}} \leq \sup_{v\neq 0} \frac{|f(v)|}{\|\hat{v}\|_{\ell^2_s}} \|\hat{v}\|_{\ell^2_s} = \|f\|_{(H^s)^*} \|\hat{v}\|_{\ell^2_s}.$$

Dividing through by $\|\hat{v}\|_{\ell^2_s}$ and rewritting some terms more explicitely yield

$$\frac{|\sum \hat{v}(\xi)\left(1+|\xi|^2\right)^{\frac{s}{2}} f(e^{i\xi\cdot x})\left(1+|\xi|^2\right)^{-\frac{s}{2}}|}{\sqrt{\sum |\hat{v}(\xi)|^2 \left(1+|\xi|^2\right)^s}} \leq \|f\|_{(H^s)^*},$$

from which, using that $a, b > 0$, $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ and $\left(1+|\xi|^2\right)^{-s} \leq \left(1+|\xi|^2\right)^{-\frac{s}{2}}$, we find that

$$|\sum f(e^{i\xi\cdot x})\left(1+|\xi|^2\right)^{-s}| \leq \|f\|_{(H^s)^*}. \tag{5.2.3}$$

As $|\sum a(\xi) b_\xi| = |\sum \overline{a(\xi)} b_\xi|$ for $b_\xi \in \mathbb{R}$, we derive from multiplying both sides by $|\sum f(e^{-i\xi\cdot x})\left(1+|\xi|^2\right)^s|$ and applying Cauchy-Schwarz inequality in the form $\left(\sum a^{\frac{1}{2}} a^{\frac{1}{2}}\right)^2 \leq \left(\sum a^2\right)\left(\sum a^2\right)$ that

$$|\sum f(e^{i\xi\cdot x})\overline{f(e^{i\xi\cdot x})}\left(1+|\xi|^2\right)^{-s}| \leq \|f\|^2_{(H^s)^*}.$$

Thus $\|\hat{u}\|_{\ell^2_{-s}} < \infty$, which shows that $u \in H^{-s}$.

Finally, it is readily seen from the definition that if $(Tu)(v) = 0$ $\forall v \in H^s$, then $\hat{u} = 0$. Hence, we conclude that T is also injective and the proof is complete. $\square$

5.2.2 Distributions

In this section, we generalize the notion of strong derivative to more abstract objects called distributions, which are continuous functionals on a given space of test functions.

Definition. $H^{-\infty}(\mathbb{T}^n) := \bigcup_{s\in\mathbb{R}} H^s(\mathbb{T}^n)$.

Remark. $H^s(\mathbb{T}^n)$ is defined as in section 5.2.1 when $s < 0$.

Definition (Continuity of Linear Functionals). A linear map $f : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ is said to be continuous if $\exists c \in \mathbb{R}, m \in \mathbb{N}$ such that $|\langle f, u\rangle| \le c\|u\|_{C^m(\mathbb{T}^n)}$ $\forall u \in C^\infty(\mathbb{T}^n)$.

Remark. Recall that $\langle f, u\rangle := f(u)$ denotes the duality pairing of f and u.

Definition (Distributions). The space of all continuous linear map from $C^\infty(\mathbb{T}^n)$ to $\mathbb{C}$ is denoted $\mathscr{D}^{'}(\mathbb{T}^n)$. An element $u \in \mathscr{D}'(\mathbb{T}^n)$ is called a distribution.

Theorem. $H^{-\infty}(\mathbb{T}^n) = \mathscr{D}'(\mathbb{T}^n)$.

Proof. If $f \in H^{-\infty}$, then by the lemma of section 5.2.1, $f \in (H^s)^*$ for some $s \in \mathbb{R}$ and $\exists c \in \mathbb{R}$ such that $|\langle f, u\rangle| \le c\|u\|_s$ $\forall u \in H^s$. On $C^s \subset H^s$, this inequality leads to $|\langle f, u\rangle| \le c_2\|u\|_{C^s}$, which shows that $f \in (C^m)^*$. Conversely, if $f \in (C^m)^*$, then by Bernstein's theorem,

$$|\langle f, u\rangle| \le c_1\|u\|_{C^m} \le c_2\|u\|_s$$

for any $s > \frac{n}{2} + m$, hence $f \in H^{-\infty}$. $\square$

Definition. For $u \in \mathscr{D}'$, we define $\partial^\alpha u \in \mathscr{D}'$ by $\widehat{\partial^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$.

Remark. This is a generalization of the strong derivative. We could alternatively generalized differentiation through the weak derivative and require that $\partial^\alpha u = \omega \in \mathscr{D}'$ if and only if

$$\langle \omega, v\rangle_{L^2} = (-1)^{|\alpha|}\langle u, \partial^\alpha v\rangle_{L^2} \;\; \forall v \in C^\infty.$$

Indeed, if $\partial^\alpha u$ satisfy the above definition, then

$$\begin{aligned}
\langle \partial^\alpha u, v\rangle_{L^2} &= (2\pi)^n \sum (i\xi)^\alpha \hat{u}(\xi)\overline{\hat{v}(\xi)} \\
&= (2\pi)^n \sum (-1)^{|\alpha|}\hat{u}(\xi)\overline{(i\xi)^\alpha \hat{v}(\xi)} \\
&= (-1)^{|\alpha|}\langle u, \partial^\alpha v\rangle_{L^2}
\end{aligned}$$

for all $v \in C^\infty$. Conversely, if $\omega \in \mathscr{D}'$ is such that $\langle \omega, v\rangle_{L^2} = (-1)^{|\alpha|}\langle u, \partial^\alpha v\rangle_{L^2}$ $\forall v \in C^\infty$, then

$$\hat{\omega}(\xi) = \frac{1}{(2\pi)^n}\langle \omega, e^{i\xi\cdot x}\rangle = \frac{1}{(2\pi)^n}\langle u, (i\xi)^\alpha e^{i\xi\cdot x}\rangle = (i\xi)^\alpha \hat{u}(\xi).$$

Lemma. *If $s > \frac{n}{2}$, then $L^1(\mathbb{T}^n) \hookrightarrow H^{-s}(\mathbb{T}^n)$.*

Proof. Suppose $f \in L^1(\mathbb{T}^n)$ and define $\langle Tf, u\rangle := \int_{\mathbb{T}^n} fu$. By Bernstein's theorem,

$$|\langle Tf, u\rangle| \le \|u\|_{L^\infty(\mathbb{T}^n)}\|f\|_{L^1(\mathbb{T}^n)} \le \|f\|_{L^1(\mathbb{T}^n)}\|u\|_{H^s},$$

hence $Tu \in (H^s)^*$. Now, $T : L^1 \longrightarrow (H^s)^*$ is linear by definition and easily seen to be injective, because if $\int_{\mathbb{T}^n} fu = 0$ $\forall u \in C^\infty$, then $f = 0$ almost everywhere by the du Bois-Reymond lemma, i.e. $f = 0$ in L^1. $\square$

5.2.3 Pettis Theorem

Let X be a Hilbert space and $I \subset \mathbb{R}$ be an open, possibly unbounded, interval. We will denote by $C_c(I,X)$ the space of all compactly supported X-valued continuous functions on I.

Definition ($L^p(I,X)$**-norm).** For $1 \le p \le \infty$, we define the norm $\|\cdot\|_{L^p(I,X)}$ on $C_c(I,X)$ by

$$\|u\|_{L^p(I,X)} = \|\|u(t)\|_X\|_{L^p(I)}.$$

Remark. More explicitly,

$$\|u\|_{L^p(I,X)} = \left(\int_I \|u(t)\|_X^p dt\right)^p$$

when $1 \le p < 1$, and $\|u\|_{L^p(I,X)} = \inf\{\lambda \ge 0 : |\{t \in I : \|u(t)\|_X > \lambda\}| = 0\}$ when $p = \infty$.

Tentatively, we will define the space $L^p(I,X)$ as the completion of $C_c(I,X)$ with respect to the $L^p(I,X)$-norm. This definition is temporary. More rigorous grounds for $L^p(I,X)$ will be established in the next section with the aid of strong measurability (yet to be defined). The current definition will appear as a consequence of this upcoming formalization.

Lemma. *Let $\{u_k\} \in C_c(I,X)$ be a Cauchy sequence with respect to $\|\cdot\|_{L^p(I,X)}$. There exists a subsequence $\{u_{k_n}\}$ and a function $u : I \longrightarrow X$ such that $u_{k_n} \longrightarrow u$ as $n \to \infty$ almost everywhere in I. Moreover, we have both that*

$$\int_I \|u - u_k\|_X^p dt \longrightarrow 0$$

as $k \to \infty$ for the full sequence and that$\int_I \|u(t)\|_X^p dt < \infty$.

Proof. Since $\{u_k\}$ is Cauchy, we can choose a subsequence $\{u_{k_n}\}_n$ such that $\|u_k - u_{k_n}\|_{L^p(I,X)} \le 2^{-n}$ $\forall k \ge k_n$. For this subsequence,

$$\|u_{k_1}\|_{L^p(I,X)} + \sum_{n=1}^{\infty} \|u_{k_{n+1}} - u_{k_n}\|_{L^p(I,X)} \le \|u_{k_1}\|_{L^p(I,X)} + \sum_{n=1}^{\infty} 2^{-n} = \|u_{k_1}\|_{L^p(I,X)} + 1 < \infty.$$

Hence, for $g_k(t) := \|u_{k_1}(t)\|_X + \sum_{n=1}^{k} \|u_{k_{n+1}}(t) - u_{k_n}(t)\|_X$, it follows from Minkowski's inequality that

$$\|g_k(t)\|_{L^p(I)} \le \|u_{k_1}(t)\|_{L^p(I)} + 1 < \infty.$$

We conclude that $\{g_k\}$ is a bounded increasing sequence, and from the Monotone convergence theorem, this implies $\exists g \in L^p(I)$ such that $g_k \longrightarrow g$ almost everywhere in I with respect to $L^p(I,X)$, i.e. $\exists$ a set $J \subset I$ with $|I \backslash J| = 0$ such that $g_k(t) \longrightarrow g(t)$ $\forall t \in J$ in $L^p(I,X)$.

Now observe that $u_{k_1}(t) + \sum_{n=1}^{k} \left(u_{k_{n+1}}(t) - u_{k_n}(t)\right) = u_{k_{n+1}}(t)$ is Cauchy in X, because

$$\|u_{k_{n+m}}(t) - u_{k_n}(t)\|_X \le \sum_{j=n}^{\infty} \|u_{k_{j+1}}(t) - u_{k_j}(t)\|_X$$

and the left hand side converges to 0 as $n \to \infty$. So $u_{k_{n+1}} \longrightarrow u(t)$ for $t \in J$. Finally, since $\|u_{k_{n+1}}(t)\|_X^p \le |g(t)|^p$ for $t \in J$, we have

$$\begin{aligned}
\int \|u(t)\|_X^p dt &\le \int |g(t)|^p dt \\
&\le \int \lim_{k\to\infty} |g_k(t)|^p dt \\
&\le \lim_{k\to\infty} \int |g_k(t)|^p dt \\
&\le \lim \left(\|u_{k_1}\|_{L^p} + \sum_{n=1}^{k} \|u_{k_{n+1}}(t) - u_{k_n}(t)\|_{L^p}\right) \le M,
\end{aligned}$$

which concludes the proof. □

Remark. In general, the above argument holds for any sequence $\{u_k\}$ such that $\|u_1\|_{L^p(I,X)}+\sum_{n=1}^{\infty}\|u_{k+1}-u_k\|_{L^p(I,X)} < \infty$. Moreover, if $\{u_k\} \subset C(I,X)$ with $u_k \longrightarrow u$ a.e. on I and $x^* \in X^*$, then $f_k(t) = \langle u_k(t), x^*\rangle$ and $f(t) = \langle u(t), x^*\rangle$ satisfy $f_k \in C(I,\mathbb{R})$, where $f_k \longrightarrow f$ a.e. in I with f measurable. Indeed, the f_k's are measurable by continuity and the pointwise limit of measurable functions is measurable.

Definition (Weak Measurability). A function $u : I \longrightarrow X$ is called weakly measurable (WM) if the map $t \mapsto \langle u(t), x^*\rangle$ is measurable for each $x^* \in X^*$.

Remark. While $\langle\cdot,\cdot\rangle$ denotes the duality pairing, we could require $t \mapsto \langle u(t), x\rangle_X$ to be measurable, because since X is an Hilbert space, it is its own dual by the Riesz representation theorem.

Definition (Simple Functions). A function $s : I \longrightarrow X$ is called simple if it is of the form $s = \sum_{k=1}^{n} a_k \chi_{A_k}$, where $a_k \in X$ and $A_k \subset I$ is measurable for $k = 1, 2..., n$.

Remark. In the above definition, χ_B is the characteristic function on the set B.

Definition (Strong Measurability). A function $u : I \longrightarrow X$ is called strongly measurable (SM) if $\exists$ a sequence $\{u_k\}$ of simple functions such that $u_k \longrightarrow u$ pointwise almost everywhere on I.

Remark. We will notice in the following that the simple functions u_k could be replaced by functions in $C_c(I,X)$ in the above definition.

Theorem (Pettis Theorem). *If X is separable, then strong measurability is equivalent to weak measurability.*

Proof. Suppose that u is strongly measurable, i.e. that $\exists$ a sequence of simple functions $\{u_k\}$ such that $u_k \longrightarrow u$ pointwise a.e. on I. Then, for $x^* \in X^*$, $f_k(t) = \langle u_k(t), x^*\rangle$ and $f(t) = \langle u(t), x^*\rangle$ are such that $f_k \longrightarrow f$ a.e. $t \in I$. Since the f_k are simple scalar valued functions, f is measurable. This shows that u is weakly measurable.

Conversly, suppose that u is weakly measurable. Let $\{x_k\}$ be a dense sequence in X and define, for $x \in X$, $s_n(x) = x_k$, where $\|x - x_k\|_X = \min_{1\le j\le n}\|x - x_j\|_X$ with k minimal amongst the x_j's satisfying the latter equality. From the densitiy of $\{x_k\}$ in X, it is obvious that $s_n(x) \longrightarrow x$ as $n \to \infty$. Hence, we may further define $u_k(t) = s_k(u(t))$ for $t \in I$, which has finitely many values and converges to $u(t)$ as $k \to \infty$. If we define the measurable sets

$$A_{n,k} = \{t \in I : u_k(t) = x_n\},$$

for $n \le k$, then writting $u_k(t) = \sum_{m=1}^{n} x_m \chi_{A_{n,k}}$ shows that $\{u_k\}$ is a sequence of simple functions converging to u pointwise almost everywhere in I. □

Corollary. *If X is separable, almost everywhere limit of strongly measurable functions are strongly measurable.*

Proof. Suppose $u_k \longrightarrow u$ a.e. with u_k SM. Let $x^* \in X^*$. We have $f_k(t) = \langle u_k(t), x^*\rangle$ is measurable by Pettis Theorem and $f_k \longrightarrow f$ a.e. with $f(t) = \langle u(t), x^*\rangle$ measurable, because the pointwise limit of measurable functions is measurable. Thus u is WM and from Pettis Theorem again, u is SM. □

Corollary. *Let $u : I \longrightarrow X$ be SM and let Y be a Hilbert space. If $\phi : X \longrightarrow Y$ is continuous, then $\phi \circ u : I \longrightarrow Y$ is SM.*

Proof. There exists simple functions $u_k \longrightarrow u$ a.e. on I. By continuity, this implies that $\phi\circ u_k \longrightarrow \phi\circ u$ a.e. on I. Since $\phi \circ u_k$ is simple for all k, $\phi \circ u$ is SM. □

Remark. If $u : I \longrightarrow X$ is Borel measurable, i.e. $u^{-1}(\mathcal{O}) = \{t \in I : u(t) \in \mathcal{O}\} \subset I$ is measurable for any open set $\mathcal{O} \subset X$, then $f(t) = \langle u(t), x^*\rangle = x^*(u(t)) = (x^* \circ u)(t)$ is measurable for any $x^* \in X^*$. It follows that u is WM, and thus SM. The converse is also true.

5.2.4 The Bochner Integral

The hypothesis of section 5.2.3 are carried in this section.

Definition. For a simple function $s = \sum_{k=1}^{n} a_k \chi_{A_k}$, we define

$$\int_I s = \sum_{k=1}^{n} a_k |A_k| \in X.$$

Remark. This integral does not depend of the particular representation of s.

Definition. A function $u : I \longrightarrow X$ is called Bochner integrable (B-integrable) if $\exists$ a sequence $\{u_k\}$ of simple functions such that $u_k \longrightarrow u$ pointwise a.e. on I with $\int_I \|u_k(t) - u(t)\|_X dt \longrightarrow 0$ as $k \to \infty$. For such a function u, we define

$$\int_I u = \lim_{k\to\infty} \int u_k.$$

Lemma. *The following statements are immediate.*
i. A Bochner integrable function is SM.
ii. The function $f(t) = \|u_k(t) - u(t)\|_X$ is measurable (by the last corollary of section 5.2.3).
iii. The Bochner integral is well-defined ($\| \int u_n - \int u_m\|_X \le \int \|u_n - u_m\|_X$).
iv. The integral does not depend on the sequence $\{u_k\}$ ($\| \int u_k - \int v_n\|_X \le \int \|u_k - u\|_X + \int \|u - v_n\|_X$).

Remark. We have mentionned that the u_k's could be replaced by $C_c(I, X)$-functions in the definition of SM. If we do so in the definition of the Bochner integral, the latter appears as the continuous extension of the Riemann integral from $C_c(I, X)$ to $L^1(I, X)$-functions.

Theorem (Properties of the Bochner Integral). *The following holds.*
(a) $\int(u + v) = \int u + \int v$
(b) $\| \int u\|_X \le \int \|u(t)\|_X dt$
(c) If $A : X \longrightarrow Y$ is a bounded linear map and $u : I \longrightarrow X$ is B-integrable, then Au is B-integrable and $\int Au = A \int u$.
(d) In particular, if $x^* \in X^*$, then $\int\langle u, x^*\rangle = \langle \int u, x^*\rangle$.

Proof. The proof of (a) is obvious. In order to prove (b), let $\epsilon > 0$ and observe that $\exists N \in \mathbb{N}$ s.t. if $n \ge N$, then

$$\begin{aligned}
\| \int u\|_X &\le \| \int u - \int u_n\|_X + \| \int u_n\|_X \\
&\le \epsilon + \int \|u_n - u + u\|_X \\
&\le \epsilon + \int \|u_n - u\|_X + \int \|u\|_X \\
&\le 2\epsilon + \int \|u\|_X.
\end{aligned}$$

In (c), the continuity of A implies that $Au_n \longrightarrow Au$ a.e. in I. Hence, since $\int u_n$ is a sum, $A \int u_n = \int Au_n$, and we derive

$$\begin{aligned}
\| \int Au - A \int u\|_X &\le \| \int Au - \int Au_n + A \int u_n - A \int u\|_X \\
&\le \int \|Au - Au_n\|_X + \|A\| \| \int u_n - \int u\|_X \qquad (5.2.4) \\
&\le \epsilon(1 + \|A\|),
\end{aligned}$$

where (b) was used in (5.2.4). ☐

5.2.5 Bochner-Lebesgue Spaces

In this section, X will always be understood as an Hilbert space, and $I \subset \mathbb{R}$ as an open interval. These two objects will help us establish the theoretical grounds on which formalism will meet the intuition behind the tentative definition of the Bochner-Lebesgue space $L^p(I,X)$ stated in the previous section.

Observe that if $u = 0$ a.e., then $\|u\|_{L^p(I,X)} = 0$ and if $\|u\|_{L^p(I,X)} = 0$, then $u = 0$ a.e.Hence, it is natural to define the following equivalence relation. For u and v with finite $L^p(I,X)$-norm, we will write $u \sim v$ if $u = v$ a.e., i.e. $u \sim v$ if and only if $\|u - v\|_{L^p(I,X)} = 0$.

Definition. We define $L^p(I,X) = \left\{u : I \longrightarrow X \text{ SM} : \|u\|_{L^p(I,X)} < \infty\right\} / \sim$.

Theorem. *$L^p(I,X)$ is a Banach space. In particular, $L^2(I,X)$ is a Hilbert space with inner product $\langle u, v\rangle_{L^2(I,X)} = \int_I \langle u(t), v(t)\rangle dt$.*

The following theorem shows that the tentative definition of $L^p(I,X)$ that we have stated in section 5.2.3 was justified.

Theorem. *The following theorem is stated in parts.*
(a) A function $u : I \longrightarrow X$ is B-integrable if and only if $[u] \in L^1(I,X)$.
(b) Simple functions are dense in $L^p(I,X)$ for $1 \leq p \leq \infty$.
(c) $C_c(I,X)$ is dense in $L^p(I,X)$ for $1 \leq p < \infty$.

Proof. In order to prove (b), we observe that for $u \in L^p(I,X)$, we can find simple function $\{v_n\}$ such that $v_n \longrightarrow u$ pointwise a.e. If we define

$$u_n(t) = \begin{cases} v_n(t) & \text{if } \|v_n(t)\|_X \leq 2\|u(t)\|_X \\ 0 & \text{otherwise} \end{cases},$$

then u_n are simple funtions such that $u_n \longrightarrow u$ a.e. Moreover, $\|u_n\|_{L^p(I,X)} \leq 2\|u\|_{L^p(I,X)}$, i.e. $u_n \in L^p(I,X)$. Hence, since $\|u(t) - u_n(t)\|_{L^p} \longrightarrow 0$ pointwise for a.e. t and that $\|u(t) - u_n(t)\|_{L^p} \leq C_p\|u(t)\|_X^p$, part ($b$) now follows from the Lebesgue dominated convergence theorem.

We will now prove (a). Let u be B-integrable. Then

$$\int \|u\|_X \leq \int \|u - u_n\|_X + \int \|u_n\|_X < \infty,$$

where $\{u_n\}$ is a sequence as in the definition of Bochner integrability. It is thus readily seen that $u \in L^1(I,X)$. Conversely, if $u \in L^1(I,X)$, then $\|u - u_n\|_{L^1(I,X)}$ by (b).

Now approximate $u \in L^p(I,X)$ by simple functions. Choose those simple functions to be compactly supported. Proving that we can approximate χ_A by continuous functions for any A of bounded measure will complete the proof of (c). To do so, we define the continuous function $\phi : I \longrightarrow \mathbb{R}$ by

$$\phi(x) = \frac{\text{dist}\,(x, \mathbb{R}\backslash\Omega)}{\text{dist}\,(x, \mathbb{R}\backslash\Omega) + \text{dist}\,(x, K)},$$

where $K \subset A \subset \Omega$ were found such that $|\Omega\backslash K| < \epsilon$. □

Corollary. *$L^p(I,X)$ is the completion of $C_c(I,X)$ with respect to $\|\cdot\|_{L^p(I,X)}$.*

Theorem. *If $1 \leq p, q \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then $u \in L^q(I,X)$ induces a bounded linear functional $Tu \in L^p(I,X)^*$ by*

$$\langle Tu, v\rangle = \int_I \langle u(t), v(t)\rangle_X dt. \tag{5.2.5}$$

Moreover, we have $\|Tu\|_{L^p(I,X)^*} = \|u\|_{L^q(I,X)}$.

Proof. We first want to show that $\|Tu\|_{L^p(I,X)^*} \leq \|u\|_{L^q(I,X)}$. In order to do so, we use Cauchy-Schwarz inequality to derive that

$$|\langle Tu, v\rangle| = |\int_I \langle u(t), v(t)\rangle_X dt| \leq \int_I |\langle u(t), v(t)\rangle_X| dt \leq \int_I \|u(t)\|_X \|v(t)\|_X dt.$$

Let $v \in L^p(I,X)$ be arbitrary. If $p = 1$, then $u \in L^\infty(I,X)$ and

$$|\langle Tu, v\rangle| \leq \sup_{t\in I} |\|u(t)\|_X| \int_I \|v(t)\|_X dt = \|u\|_{L^\infty(I,X)} \|v\|_{L^1(I,X)}.$$

We similarly find that $|\langle Tu, v\rangle| \leq \|u\|_{L^1(I,X)} \|v\|_{L^\infty(I,X)}$ in the case where $p = \infty$. If $1 < p < \infty$, then we may also find a similar inequality using Hölder's inequality:

$$|\langle Tu, v\rangle| \leq \|u\|_{L^p(I,X)} \|v\|_{L^q(I,X)}.$$

Dividing through the three above inequalities by $\|v\|_{L^p(I,X)}$ and taking the supremum over the elements $v \in L^p(I,X)$, $v \neq 0$, yields the more general inequality

$$\|Tu\|_{L^p(I,X)^*} = \sup_{\substack{v\in L^p(I,X)\\ v\neq 0}} \frac{|\langle Tu, v\rangle|}{\|v\|_{L^p(I,X)}} \leq \|u\|_{L^q(I,X)}, \tag{5.2.6}$$

which holds for $1 \leq p, q \leq \infty$. This proves the claim. Hence, since the inner product $\langle\cdot,\cdot\rangle_X$ on X is linear in its second argument and that $u \in L^q(I,X)$ implies $\|u\|_{L^\infty(I,X)} < \infty$, (5.2.1) indeed defines a bounded linear functional.

We will conclude the proof by showing that in fact, $\|Tu\|_{L^p(I,X)^*} = \|u\|_{L^q(I,X)}$; in other words, that $\|u\|_{L^q(I,X)}$ is the supremum of $|\langle Tu, v\rangle|/\|u\|_{L^q(I<X)}$ in (5.2.2) when taken over $v \in L^p(I,X)$. We proceed by letting u_n, $n = 1, 2, 3, ...$ be a sequence of simple functions converging to u in $L^q(I,X)$. This is possible, because simple functions are dense in $L^q(I,X)$ for any $1 \leq q \leq \infty$. The elements of this sequence are of the form $u_n = \sum_{k=1}^{m_n} a_{n,k}(t)\cdot\chi_{A_{n,k}}$, where w.l.o.g., the $A_{n,k}$ can be assumed disjoint, and where χ_B is the usual characterisitc evaluating to the identity over $B \subset \mathbb{R}$ and vanishing elsewhere.

We will prove the case where $1 < q < \infty$, we can thus define, $\forall n \in \mathbb{N}$, an associate $v_n = \sum_{k=1}^{m_n} a_{n,k}(t)\|a_{n,k}(t)\|_X^{q-2}\cdot \chi_{A_{n,k}}$, and observe that

$$\begin{aligned}
\langle u_n, v_n\rangle &= \int_I \langle u_n(t), v_n(t)\rangle_X dt \\
&= \int_I \sum_{k=1}^{m_n} \|a_{n,k}(t)\cdot\chi_{A_{n,k}}\|_X^q dt \\
&= \int_I \|u_n(t)\|_X^q dt.
\end{aligned}$$

Since $pq - p = pq(1 - \frac{1}{q}) = q$ and $\frac{q}{p} = q(1 - \frac{1}{q}) = q - 1$ by hypothesis, this definition also implies that

$$\begin{aligned}
\int_I \|v_n\|^p &= \int_I \|\sum_{k=1}^{m_n} a_{n,k}(t)\|a_{n,k}(t)\|_X^{q-2}\cdot\chi_{A_{n,k}}\|_X^p dt \\
&= \int_I \|u_n\|_X^{pq-2p}\|u_n\|_X^p dt \\
&= \int_I \|u_n\|_X^q,
\end{aligned}$$

or equivalently, that $\|v_n\|_{L^p(I,X)} = \|u_n\|_{L^q(I,X)}^{\frac{q}{p}} = \|u_n\|_{L^qX}^{q-1}$. We then find that the heart of the argument now lies in the following inequality:

$$\begin{aligned}
\langle Tu, v_n\rangle &= \int_I \langle u(t) - u_n(t), v_n(t)\rangle_X dt + \int_I \langle u_n(t), v_n(t)\rangle_X dt \\
&= \langle u_n, v_n\rangle + \int_I \langle u(t) - u_n(t), v_n(t)\rangle_X \\
&= \|u_n\|_{L^qX}^{q-1+1} + \int_I \langle u(t) - u_n(t), v_n(t)\rangle_X \\
&= \|u_n\|_{L^qX}\|v_n\|_{L^pX} + \int_I \langle u(t) - u_n(t), v_n(t)\rangle_X \\
&\geq \|u_n\|_{L^qX}\|v_n\|_{L^pX} - \|u - u_n\|_{L^q(I,X)}\|v_n\|_{L^p(I,X)} \\
&\geq \left(\|u\|_{L^q(I,X)} - 2\|u - u_n\|_{L^q(I,X)}\right)\|v_n\|_{L^p(I,X)},
\end{aligned}$$

because upon dividing both sides by $\|v_n\|_{L^P(I,X)}$, taking $n \to \infty$ shows that for any $\epsilon > 0$ given, one can find an element $v \in L^p(I,X)$ s.t. $\left(\|Tu\|_{L^p(I,X)^*}\right) - \epsilon \leq \frac{|\langle Tu,v\rangle|}{\|v\|_{L^p(I,X)}} \leq \|u\|_{L^q(I,X)}$. □

Corollary. *If X is separable, $1 \leq p < \infty$ and $T : L^q(I,X) \longrightarrow L^p(I,X)^*$ acts on $L^p(I,X)$ as defined as in the previous theorem, then T is invertible.*

Lemma. *If X is separable and $1 \leq p < \infty$, then $L^p(I,X)$ is separable.*

5.2.6 Banach–Alaoglu Theorem

From now on, we will adopt the notation $L^p_TX := L^p((0,T),X)$ and similarly for H^k.

Definition (Strong Convergence in X^*). If $\{x_n^*\} \in X^*$ is a sequence such that

$$\sup_{\|x\|_X < 1} x_n^*(x) \longrightarrow 0$$

as $n \to \infty$ in X^*, then we say that x_n^* converges strongly in X^*.

Remark. This definition is applicable to the whole space X^*, since from the linearity of x_n^*,

$$\sup_{\|x\|_X < 1} x_n^*(Rx) = R\left(\sup_{\|x\|_X < 1} x_n^*(x)\right) \longrightarrow 0$$

as $n \to \infty$ for any scalar $R > 0$. Moreover, recall that $\|u\|_{X^*} = \sup_{\|x\|<1} u(x)$ is a norm under which X^* is a Banach space. We called its induced topology the strong dual topology.

Definition (Weak * Convergence). The pointwise convergence of a sequence $\{x_n^*\} \subset X^*$, i.e. $x_n^*(x) \longrightarrow 0$ for each $x \in X$, is called weak * convergence (or weak dual convergence) and it induces a topology called the weak dual topology. We write $u_n^* * \longrightarrow 0$.

Example. Let $x = (x_1, ..., x_n, ...) \in \ell^2$, and define $f_n \in (\ell^2)^*$ by $f_n(x) = x_n$. On the one hand, since $x_n \longrightarrow 0$ as $n \to \infty$, then $f_n(x) \longrightarrow 0$ as $n \to \infty$ pointwise, so shows $f_n * \longrightarrow 0$. On the other hand, $\|f_n\|_{(\ell^2)^*}$.

Theorem (Banach-Alaoglu Theorem, Sequential Version). *Let X be a separable normed space and suppose $\{u_n^*\} \subset X^*$ is bounded. Then $\{u_n^*\}$ has a converging subsequence in the weak dual topology.*

Proof. Since X is separable, we can find a sequence $x = \{x_n\}$ such that every finite subset of its elements is linearly independent, i.e.

$$\text{span}\,\{x_1\} \not\supseteq \text{span}\,\{x_1, x_2\} \not\supseteq \text{span}\,\{x_1, x_2, x_3\} \dots,$$

and such that span$\{x\}$ is dense in X. Moreover, we can extract subsequences $\{u^*_{1k}\} \supset \{u^*_{2k}\} \supset \dots$ such that

$$\begin{aligned}
\langle u^*_{1k}, x_1\rangle &\longrightarrow \alpha_1 \\
\langle u^*_{2k}, x_2\rangle &\longrightarrow \alpha_2 \\
\langle u^*_{3k}, x_3\rangle &\longrightarrow \alpha_3 \\
&\vdots
\end{aligned}$$

and consider the diagonal sequence $\langle u^*_{nn}, x_k\rangle \longrightarrow \alpha_k$ as $n \to \infty$, where convergence holds for any $k \in \mathbb{N}$. Now, we define the functional $\tilde{u}^* : \text{span}\{x\} \longrightarrow \mathbb{C}$ by

$$\langle \tilde{u}^*, y\rangle = \sum a_k \alpha_k$$

for any $y = \sum a_k x_k \in \text{span}\{x\}$. We have

$$\begin{aligned}
\langle \tilde{u}^*, y\rangle &= \sum a_k \alpha_k \\
&= \sum a_k \langle u^*_{nn}, x_k\rangle - \sum \left(a_k \langle u^*_{nn}, x_k\rangle - a_k \alpha_k\right) \\
&= \langle u^*_{nn}, y\rangle + \sum a_k \left(\langle u^*_{nn}, x_k\rangle - \alpha_k\right),
\end{aligned}$$

hence $\langle \tilde{u}^*, y\rangle \longrightarrow \langle u^*_{nn}, y\rangle$ as $n \to \infty$ and we have found $\tilde{u}^* \in (\text{span}\{x\})^*$ such that a subsequence of u^*_n converges in $\tilde{u}^*$ on span$\{x\}$. By showing that $\tilde{u}^*$ extends to $u^* \in X^*$ with $u_{nn} \longrightarrow u^*$ weakly, we will complete the proof.

Notive that given $\epsilon > 0$ and any $y \in \text{span}\{x\}$, there must always exists $n \geq N$ such that by the reverse triangle inequality, we have

$$||\langle \tilde{u}^*, y\rangle| - |\langle u^*_{nn}, y\rangle|| \leq |\langle \tilde{u}^*, y\rangle - \langle u^*_{nn}, y\rangle| < \epsilon.$$

We deduce that there exists a scalar M such that

$$|\langle \tilde{u}^*, y\rangle| \leq |\langle u^*_{nn}, y\rangle| + \epsilon \leq M\|y\|_X + \epsilon, \tag{5.2.7}$$

This is a key fact, because since span$\{x\}$ is dense in X, we can find, for any $\omega \in X$, $\omega_k \in \text{span}\,\{x_1, \dots, x_k\}$ with $\omega_k \longrightarrow \omega$ in X, and thus a well-defined unique extension $u^* \in X^*$ of $\tilde{u}^*$ by letting $\langle u^*, \omega\rangle := \lim_{k\to\infty}\langle \tilde{u}^*, \omega_k\rangle$. Indeed, (5.2.7) shows that $|\langle u^*, \omega\rangle| \leq M\|\omega\|_X < \infty$ too and that if ω'_k is any other sequence in span$\{x\}$ such that $\omega'_k \longrightarrow \omega$ as $k \to \infty$, then

$$\begin{aligned}
\lim_{k\to\infty} |\langle u^*, \omega\rangle| &= \lim_{k\to\infty} |\langle \tilde{u}^*, \omega_k - \omega'_k\rangle + \langle \tilde{u}^*, \omega'_k\rangle| \\
&\leq \lim_{k\to\infty} |\langle \tilde{u}^*, \omega_k - \omega'_k\rangle| + \lim_{k\to\infty} |\langle \tilde{u}^*, \omega'_k\rangle| \\
&\leq \lim_{k\to\infty} M\|\omega_k - \omega'_k\|_X + \lim_{k\to\infty} |\langle \tilde{u}^*, \omega'_k\rangle| \\
&= \lim_{k\to\infty} |\langle \tilde{u}^*, \omega'_k\rangle|.
\end{aligned}$$

Finally, from a similar derivation,

$$\begin{aligned}
\langle u^*, \omega\rangle &= \langle u^* - u^*_{nn}, \omega\rangle + \langle u^*_{nn}, \omega\rangle \\
&= \langle u^* - u^*_{nn}, \omega - \omega_k\rangle + \langle u^* - u^*_{nn}, \omega_k\rangle + \langle u^*_{nn}, \omega\rangle
\end{aligned}$$

and substracting the farmost right term of the right and side on both sides of the above equation leads to

$$\begin{aligned}|\langle u^* - u^*_{nn}, \omega\rangle| &= |\langle u^* - u^*_{nn}, \omega - \omega_k\rangle + \langle u^* - u^*_{nn}, \omega_k\rangle| \\ &\leq |\langle u^*, \omega - \omega_k\rangle| + |\langle u^*_{nn}, \omega - \omega_k\rangle| + |\langle \tilde{u}^* - u^*_{nn}, \omega_k\rangle| \\ &\leq 2M\|\omega_k - \omega\|_X + |\langle \tilde{u}^* - u^*_{nn}, \omega_k\rangle|.\end{aligned}$$

The proof is completed by choosing n and k large enough. □

Corollary. *The closed unit ball in X^* of a separable normed space X is weakly sequentially compact.*

Lemma. *Let X be a normed space. If $\{y^*_k\} \subset X^*$ is a sequence converging weakly to $y^* \in X^*$, then*

$$\|y^*\|_{X^*} \leq \liminf \|y^*_k\|_{X^*}.$$

Proof. On the one hand, the supremum definition $\|y^*\|_{X^*} = \sup_{\|x\|_X=1}\langle y^*, x\rangle$ implies that for any $\epsilon > 0$, $\exists x_\epsilon \in X$ with $\|x_\epsilon\|_X = 1$ such that

$$\|y^*\|_{X^*} - \epsilon \leq \langle y^*, x_\epsilon\rangle. \tag{5.2.8}$$

On the other hand, there exists by hypothesis $K \in \mathbb{N}$ such that for $k \geq K$,

$$\langle y^*, x_\epsilon\rangle - \langle y^*_k, x_\epsilon\rangle \leq \epsilon. \tag{5.2.9}$$

Combining (5.2.8) and (5.2.9), we obtain

$$\|y^*\|_{X^*} \leq \langle y^*, x_\epsilon\rangle + \epsilon \leq \langle y^*_k, x_\epsilon\rangle + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary and y^*_k is continuous, we conclude that $\|y^*\|_{X^*} \leq \langle y^*_k, \lim_{\epsilon\to 0} x_\epsilon\rangle \leq \|y^*_k\|_{X^*}$, from which conclusion follows. □

Lemma. *Let X be a Hilbert space. If y^*_n converges weakly to y^* in X^* and that $\|y^*_n\|_{X^*} \longrightarrow \|y^*\|_{X^*}$ as scalars, then $y_n \longrightarrow y$ in X^*.*

Proof. By the Riesz representation theorem, X may be identified with X^*, and so assume y_n and y are elements of X representing y^*_n and y^* respectively. It is immediatly seen that weak * convergence becomes equivalent to convergence in X^*, because then the duality pairing becomes the inner product

$$\langle y_n - y, x\rangle_X \longrightarrow 0$$

as $n \to \infty$ for all $x \in X$, which implies that $\lim_{n\to\infty} y_n = y$, equivalently that $y^*_n \longrightarrow y^*$ as $n \to \infty$ in X^*. □

Remark. We now want to consider weak convergence of a sequence $\psi_n \longrightarrow 0$ in X^{**}, i.e. $\psi_n(u) \longrightarrow 0$ as $n \to \infty$ for all $u \in X^*$. Define $j : X \longrightarrow X^{**}$ by $j(x)(u) = u(x)$ for all $u \in X^*$. The function j inherites is linear, because it inherites the linearity of the elements in X^*. It is also immediatly continuous, and easily found to be injective. Indeed, if $j(x) = 0$, then $u(x) = 0$ for all $u \in X^*$, and thus $x = 0$, i.e. j is injective.

It follows from the Hahn–Banach theorem that j is an isometry, and thus that the weak * topology of X^{**} induces a subspace topology in $X \subset X^{**}$. We call the later the weak topology of X. More precisely, in this topology, $x_n \longrightarrow 0$ weakly, or $x_n \longrightarrow x$ in X, if and only if $u(x_n) \longrightarrow 0$ $\forall u \in X^*$, or in other words, if and only if $j(x_n) \longrightarrow 0$ weakly in X^{**}. To use a notation which enlighten the symmetry between these spaces, we can rewrite this last condition as $\langle u, x_n\rangle \longrightarrow 0$ $\forall u \in X^*$. Note that if $x_n \longrightarrow x$ and $x_n \longrightarrow y$, then $x = y$.

For X a reflexive and separable Banach space, i.e. $j : X \longrightarrow X^{**}$ is not only an injective isometry, but an isometric isomorphism. Then if a sequence $\{x_n\} \subset X$ is bounded, $\{j(x_n)\}$ is bounded in X^{**}. Passing to a subsequence, $j(x_n)$ converges weakly in X^{**} to some $j(x)$, and thus by surjectivity, it holds that $x_n \longrightarrow x$ in X.

In the case of an Hilbert space H. The function $j : H \longrightarrow H^*$ defined by $j(x)(y) = \langle x, y\rangle$ is already an isomorphism. Hence $x_n \longrightarrow 0$ in H if and only if $\langle x_n, y\rangle$ $\forall y \in H$, i.e. if and only if $u(x_n) \longrightarrow 0$ $\forall u \in H^*$.

Example. We need to establish a key basic result about the convergence of the solutions of the following LNS which we will use in the example of section 5.2.7. Let $u_m \in C^\infty([0,\infty), H^s(\mathbb{T}^n))$ satisfy

$$\partial_t u_m = \triangle u_m - \mathbb{P}(P_m u_m \otimes u_m)$$

with $\widehat{u_m}(0) = 0$ and $u_m(0) = g$. Further assume a priori the energy identity

$$\|u_m(T)\|^2_{L^2} + \|u_m\|^2_{L^2_T H^1} = \|g\|^2_{L^2}$$

Under these hypotheses, u_m is bounded in $L^2_T H^1$ and in $L^\infty_T L^2 = \left(L^1_T L^2\right)^*$ for each $T \in (0,\infty)$. Hence, u_m converge to some u in $L^2_T H^1$ and by the Banach-Alaoglu theorem, we can also pass to a subsequence u_{m_k} converging weakly to some v in $L^\infty_T L^2$. The question that arises naturally is: does $v = u$? It turns out it does, i.e. one can show $v = u \in L^\infty_T L^2 \cap L^2_T H^1$.

5.2.7 Bochner-Sobolev Spaces

Let X be an Hilbert space and $I \subset \mathbb{R}$ an interval.

Definition (**Spaces** $\widetilde{C^1}(I,X)$ **and** $\widetilde{H^1}(I,X)$)**.** For $u \in C^1(I,X)$, let

$$\|u\|^2_{H^1(I,X)} := \int_I \|u(t)\|^2_X + \|u'(t)\|^2_X dt$$

and

$$\widetilde{C^1}(I,X) := \left\{u \in C^1(I,X) : \|u\|_{H^1(I,X)} < \infty\right\}.$$

Further define $\widetilde{H^1}(I,X)$ to be the completion of $\widetilde{C^1}(I,X)$ with respect to $\|\cdot\|_{H^1(I,X)}$.

Remark. If $\{u_k\} \in \widetilde{C^1}$ is a Cauchy sequence in H^1, then $\{u_k\}$ and $\{u'_k\}$ are Cauchy in L^2. Hence, we have an injection $j = (j_0, j_1) : H^1 \longrightarrow L^2 \times L^2$ defined by $j_0\{u_k\} = \lim_{k\to\infty} u_k$ and $j_1\{u_k\} = \lim_{k\to\infty} u'_k$ in L^2. It is easy to see that it is well-defined, because if $\lim_{k\to\infty} v_k = \lim_{k\to\infty} u_k$, then $u_1, v_1, u_2, v_2, \ldots$ is Cauchy, and $\{u_k\} \sim \{v_k\}$, i.e. they belong to the same equivalence class of Cauchy sequences, which shows that they are the same element in the defined space.

Definition (**Strong Derivative of a Locally Integrable** $L^2(I,X)$**-Function**)**.** For $u, v \in L^2_{loc}(I,X)$, we say that $u' = v$ in the strong L^2-sense if $\forall K \subset\subset I$, $\exists\{\phi_n\} \subset \widetilde{C^1}(K,X)$ such that $\phi_k \longrightarrow u$ and $\phi'_k \longrightarrow v$ in $L^2(K,X)$.

Lemma. *If they exist, the strong derivatives defined in the above definition are unique. Moreover, if* $u \in L^2_{loc}(I,X)$ *is strongly differentiable, we have the integration by parts formula*

$$\int_I \langle \phi, u'\rangle_X = -\int \langle u, \phi'\rangle_X \;\forall \phi \in C^1_c(I,X).$$

Proof. Suppose that we have both $u' = v$ and $u' = w$ in the strong L^2-sense and let $\phi \in C_c^1(I, X)$. By definition, $\exists \{\phi_k\} \subset \widetilde{C^1}(I, X)$ such that $\psi_k \longrightarrow u$ and $\psi_k' \longrightarrow v$ in $L^2(I, X)$. Now,

$$\begin{aligned}\int \langle v, \phi\rangle_X &= \int \langle v - \psi_k', \phi\rangle + \int \langle \psi_k', \phi\rangle \\ &= \int \langle v - \psi_k', \phi\rangle - \int \langle \psi_k, \phi'\rangle \\ &= \int \langle v - \psi_k', \phi\rangle + \int \langle u - \psi_k, \phi'\rangle - \int \langle u, \phi'\rangle,\end{aligned} \tag{5.2.10}$$

where (5.2.10) follows from the classical integration by parts, and using the Cauchy-Schwarz inequality twice, this implies that

$$\begin{aligned}|\int \langle v, \phi\rangle + \int \langle u, \phi'\rangle| &\leq \int \|v - \psi_k'\|_X \|\phi\|_X + \int \|u - \psi_k\|_X \|\phi'\|_X \\ &\leq \|v - \psi_k'\|_{L^2(I,X)} \|\phi\|_{L^2(I,X)} + \|u - \psi_k\|_{L^2(I,X)} \|\phi'\|_{L^2(I,X)}.\end{aligned}$$

Hence, we conclude $\int \langle v, \phi\rangle = -\int \langle u, \phi'\rangle$. This proves the stated integration by parts formula. Now, the above proof implies similarly that $\int \langle w, \phi\rangle = -\int \langle u, \phi'\rangle$, and so we must have $\int \langle v - w, \phi\rangle = 0$. Since this holds $\forall \phi \in \widetilde{C^1}(I, X)$, it follows from du Bois-Reymond lemma that $v = w$. $\square$

Corollary. *The map $j_0 : H^1(I, X) \longrightarrow L^2(I, X)$ defined in the last remark by $j_0\{u_k\} = \lim_{k\to\infty} u_k$ in L^2 is a continuous linear projection.*

Remark. A priori, it seem that there could exist $\{u_k\}, \{v_k\} \subset \widetilde{C^1}(I, X)$ such that $u_k \longrightarrow u$, $v_k \longrightarrow u$ with $u_k' \longrightarrow u'$ and $v_k' \longrightarrow v'$ in L^2. By the last lemma, this cannot be the case and we must have $u' = v'$. Morevover, we see that the range of j_0 is a subset of $\{u \in L^2(I, X) : u' \in L^2(I, X)\}$. Finally, remark that j_1 is not injective, because v and $v + 1$ have the same derivative.

Lemma. $H_T^1 X \hookrightarrow C([0, T], X)$ *and for* $0 \leq s \leq t \leq T$,

$$u(t) - u(s) = \int_s^t u'(\tau) d\tau.$$

Proof. Let $u \in \widetilde{C^1}((0, T), X)$, $0 < s < t \leq T$. By the Fundamental theorem of calculus and the Cauchy-Schwarz inequality,

$$\|u(t) - u(s)\|_X \leq \|\int_s^t u'(\tau) d\tau\|_X \leq \int_s^t \|u'(\tau)\|_X d\tau \leq |t - s|^{\frac{1}{2}} \|u'(\tau)\|_{L_T^2 X},$$

and thus u is uniformly continuous on $(0, T)$ and can be extended to a function $u \in C([0, T], X)$. The above argument also shows that in fact,

$$\|u(t)\|_X \leq \|u(s)\|_X + T^{\frac{1}{2}} \|u'(\tau)\|_{L_T^2 X}$$

for all $s, t \in [0, T]$. Taking powers on both sides of the above equation further yields

$$\begin{aligned}\|u(t)\|_X^2 &\leq \|u(s)\|_X^2 + T^{\frac{1}{2}} \|u(s)\|_X \|u'(\tau)\|_{L_T^2 X} + T\|u'(\tau)\|_{L_T^2 X}^2 \\ &\leq 2\|u(s)\|_X^2 + 2T\|u'(\tau)\|_{L_T^2 X}^2,\end{aligned}$$

and since this holds for all $s, t \in [0, T]$, we may use the continuity of u on compact $[0, T]$ to find $s \in [0, T]$ such that $\|u(s)\|_X$ is minimal. From this choice, it is now clear that multiplying both sides by T leads to

$$\|u\|_{L_T^\infty X}^2 \leq C_T \|u\|_{H_T^1 X}^2.$$

This shows that if u_k is Cauchy in $H^1(I,X)$, then it is Cauchy with respect to the L^∞-norm, and thus $u_k \longrightarrow v \in C([0,T],X)$. Since by completion $u_k \longrightarrow u$ in $L^2_T X$, we have $u = v$ a.e. on $[0,T]$ and this completes the proof. □

Remark. The above lemma establishes the adequate formulation of the Fundamental theorem of calculus in this setting.

Example. In the previous example, we had $u_m \in C([0,\infty],H^s)$ satisfying

$$\partial_t u_m = \triangle u_m - \mathbb{P}(P_m u_m \otimes u_m)) \tag{5.2.11}$$

with $u_m(0) = g$ and

$$\|g\|^2_{L^2} = \|u_m(T)\|^2_{L^2} + \|u_m\|^2_{L^2_T H^1} = \|u_m(T)\|^2_{L^2} + \int_0^T \|\nabla u_m(\tau)\|^2_{L^2} d\tau,$$

were the last equality on the left hand side holds because we assumed $\widehat{u_m}(0) = 0$. We had found $u_m \longrightarrow u$ in $L^2_T H^1$ and $u_m * \longrightarrow u$ in $L^\infty_T L^2$.

Now, observe that we can conclude that $\{\Delta u_m\}$ bounded in $L^2_T H^{-1}$ and moreover, from the inequality

$$\|P_m u_m \otimes u_m\|_{L^1} \leq \|P_m u_m\|_{L^2}\|u_m\|_{L^2} \leq \|u_m\|^2_{L^2},$$

that $\{P_m u_m \otimes u_m\}$ is bounded in $L^\infty_T L^1$, hence in $L^\infty_T H^{-s}$ for $s > \frac{n}{2}$, because $L^1(\mathbb{T}^n) \hookrightarrow H^{-s}(\mathbb{T}^n)$ by the first lemma in section 5.2.2. This further implies that $\{\mathbb{P}\mathrm{div}\,(P_m u_m \otimes u_m)\}$ is bounded in $L^2_T H^{-s-1}$, and so finally, form (5.2.11), that $\{\partial_t u_m\}$ is bounded in $L^2_T H^{-s-1}$ for $0 < T < \infty$. This is a fact of great importance, because to passing to a subsequence, we have $u_m \longrightarrow u$ in $H^1_T H^{-s-1}$, which shows that $u \in H^1_T H^{-s-1}$, and thus by the last lemma, u is a.e. equal to a $C\left([0,T],H^{-s-1}\right)$ function, which can be evaluated at 0.

So we have $u(0) \in H^{-s-1}$. For $\psi \in C^\infty(\mathbb{T}^n)$, it follows from the results of section 5.2.6 that $f \in \left(H^1_T H^{-s-1}\right)^*$ defined by $f(v) = \langle v(0),\psi\rangle_{L^2}$ for $v \in H^1_T H^{-s-1}$, will be such that $f(u_m) \longrightarrow f(u)$ as $n \to \infty$ in this case. Since by hypothesis $u_m(0) = g$ for all $m \in \mathbb{N}$, we have

$$\langle g,\psi\rangle = \lim_{m\to\infty}\langle u_m(0),\psi\rangle = \langle u(0),\psi\rangle,$$

i.e. $\langle g - u(0),\psi\rangle = 0$. As ψ was arbitrary, we conclude from du Bois-Reymond lemma that $u(0) = g$. This means that after solving these LNS equations with initial value g, the inital value is preserved in the limit function u with which we hope to solve NSE.

5.2.8 Weak Solutions of the NSE

Example. We have seen in the previous example in section 5.2.7 that it is possible to transfer information from the solutions u_m to their limit u. An explicit and enlightnening example of this phenomenon is given by the heat equation. Suppose that the u_m's satisfy both $\partial_t u_m = \Delta u_m$ and the previous convergence. Integrating by parts, we see that $\langle\partial_t u_m,\phi\rangle_{L^2} = -\langle u_m,\partial_t\phi\rangle_{L^2}$ and $\langle\Delta u_m,\phi\rangle_{L^2} = \langle u_m,\Delta\phi\rangle_{L^2}$. It follows that $\forall\phi \in C^1_c\left((0,T),H^2\right)$,

$$0 = \int_0^T \langle u_m,\partial_t\phi + \Delta\phi\rangle_{L^2}.$$

Now, observe that

$$F(v) = \int_0^T \langle v,\partial_t\phi + \Delta\phi\rangle_{L^2}$$

defines a linear function $F \in \left(L^2_T L^2\right)^*$. Hence, $0 = \lim_{m\to\infty} F(u_m) = F(u) = \int_0^T\langle u,\partial_t\phi + \Delta\phi\rangle_{L^2}$ for all $\phi \in C^1_c\left((0,T),H^2\right)$, which shows that this particular property also holds for the limit u.

Definition. A function $u \in L^2_{loc}\left((0,T), L^2\right)$ is called a weak solution to the NSE problem if

$$\int_0^T \left(\langle u, \partial_t \phi + \triangle \phi\rangle - \langle u \otimes u, \nabla \otimes u\rangle\right) = 0$$

for all $\phi \in C^1_c\left((0,T), C^2\right)$ such that $\mathrm{div}\phi = 0$ and

$$\int_0^T \langle u, \nabla \psi\rangle = 0$$

for all $\psi \in C^1_c\left((0,T), C^1\right)$. Moreover, if

$$\|u\|_{L^\infty_T L^2} + \|\nabla u\|^2_{L^2_T L^2} \leq \|g\|_{L^2},$$

then we call u a Leray-Hopf weak solution.

The idea is now that we want to obtain a minimal regularity from a natural weak convergence argument. Furthermore, we want the solutions to be locally integrable functions, in opposition to distributions. If a solution u was to be smooth, then we could solve NSE classically.

Remark. In the following theorem, we require Rellich's theorem proved in Annex 4.

Theorem. *If g is a divergence free smooth vector field, then there exists a Leray-Hopf weak solution $u \in L^2\left((0,\infty), H^1\right) \cap L^\infty\left((0,\infty), L^2\right)$ to the NSE satisfying $u \in C\left([0,\infty), H^{-s}\right)$ and $u(0) = g$ for $s > \frac{n}{2} + 1$.*

Proof. From

$$\int_0^T \langle u_m, \partial_t \phi + \Delta \phi\rangle - \langle P_m u_m \otimes u_m, \nabla \otimes \phi\rangle = 0 \ \ \forall \phi \in C^1_c C^2,$$

we find that

$$\int_0^T \langle u, \partial_t \phi + \Delta\phi\rangle - \langle u \otimes u, \nabla\rangle = \underbrace{\int_0^T \langle u - u_m, \partial_t \phi + \nabla \phi\rangle}_{A_m} - \underbrace{\int_0^T \langle u \otimes u - P_m u_m \otimes u_m, \nabla \phi\rangle}_{B_m} \ \ \forall \phi \in C^1_c C^2.$$

Since $A_m \longrightarrow 0$ as $n \to \infty$, we conclude that $\{u_m\}$ is bounded in $L^2_T H^1 \cap H^1_T H^{-s}$. By Rellich's theorem, $L^2_T H^1 \cap H^1_T H^{-s} \hookrightarrow L^2_T L^2$. We may thus, by passing to a subsequence, conclude to a strong convergence of $u_m \longrightarrow u$ in $L^2_T L^2$.

Now,

$$\begin{aligned} u \otimes u - P_m u_m \otimes u_m &= u \otimes u - P_m u \otimes u + + P_m u \otimes u - P_m u_m \otimes u_m \\ &= (u - P_m u) \otimes u + P_m\left(u - u_m\right) \otimes u + P_m u_m \otimes \left(u - u_m\right), \end{aligned}$$

and thus by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|u \otimes u - P_m u_m \otimes u_m\|_{L^1} &\leq \|u - P_m u\|_{L^2}\|u\|_{L^2} + \|P_m\left(u - u_m\right)\|_{L^2}\|u\|_{L^2} + \|P_m u_m\|_{L^2}\|u - u_m\|_{L^2} \\ &\leq \underbrace{\|u - P_m u\|_{L^2}\|u\|_{L^2} + \|u - u_m\|_{L^2}\|u\|_{L^2} + \|u_m\|_{L^2}\|u - u_m\|_{L^2}}_{D_m}. \end{aligned}$$

This imples, using the Cauchy-Schwarz inequality again, that

$$|B_m| \lesssim \int D_m \leq \|u - P_m u\|_{L^2_T L^2}\|u\|_{L^2_T L^2} + \|u - u_m\|_{L^2_T L^2}\|u\|_{L^2_T L^2} + \|u_m\|_{L^2_T L^2}\|u - u_m\|_{L^2_T L^2}.$$

Since we had found $\{u_m\}$ bounded and strongly converging in $L^2_T L^2$,

$$\|u - u_m\|_{L^2_T L^2} \|u\|_{L^2_T L^2} \lesssim \|u - u_m\|_{L^2_T L^2} \longrightarrow 0$$

and

$$\|u_m\|_{L^2_T L^2} \|u - u_m\|_{L^2_T L^2} \lesssim \|u - u_m\|_{L^2_T L^2} \longrightarrow 0$$

as $m \to \infty$. Finally, $\|u(t) - P_m u(t)\|_{L^2} \longrightarrow 0$ a.e. in $(0, T)$, thus since $\|u - P_m u\|_{L^2} \leq 2\|u\|_{L^2}$, the Lebesgue dominated convergence theorem yields that $\|u - P_m u\|_{L^2_T L^2} \longrightarrow 0$ as $m \to \infty$. $\square$

Remark. We have thus found global solutions to the NSE. However, it is not known if the solutions are smooth.

Appendix A

Assignment 1

Question 1

Prove the following.

(a) If $a \in \ell^2(\mathbb{T}^n)$, then there exists $g \in L^2(\mathbb{T}^n)$ such that

$$g = \lim_{m\to\infty} \sum_{k\in Q_m} a_k e_k, \tag{A.0.1}$$

with convergence in $L^2(\mathbb{T}^n)$, where $Q_m = \{-m, ..., m\}^n$ and $e_k(x) = e^{ik\cdot x}$ for $k \in \mathbb{Z}^n$.

(b) Conversely, if $g \in L^2(\mathbb{T}^n)$ and

$$\hat{g}(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(x) e^{ik\cdot x} dx = \frac{1}{(2\pi)^n} \langle g, e_k \rangle, \ \ k \in \mathbb{Z}^n,$$

then we have $\hat{g} \in \ell^2$ and (1) holds with $a_k = \hat{g}(k)$.

Solution.

(a) For $f_m(x) = \sum_{k\in Q_m} a_k e_k(x)$, we have

$$\|f_m\|_{L^2}^2 = \langle \sum_{k\in Q_m} a_k e_k, \sum_{j\in Q_m} a_j e_j \rangle = \sum_{k\in Q_m} a_k \left(\sum_{j\in Q_m} \overline{a_j} \langle e_k, e_j \rangle \right) = (2\pi)^n \sum_{k\in Q_m} |a_k|^2 = (2\pi)^n \|a\|_{\ell^2},$$

because from Fubini's theorem,

$$\begin{aligned}
\langle e_k, e_j \rangle &= \int_{\mathbb{T}^n} e^{ik\cdot x} e^{-ij\cdot x} dx \\
&= \int_{\mathbb{T}} e^{ik_1x_1} e^{-ij_1x_1} \int_{\mathbb{T}} e^{k_2x_2} e^{-ij_2x_2} ... \int_{\mathbb{T}} e^{k_nx_n} e^{-ij_nx_n} dx_n ... dx_1 \\
&= (2\pi)^n \delta_{k_1j_1} ... \delta_{k_nj_n} \\
&= (2\pi)^n \delta_{kj}.
\end{aligned}$$

Thus, if w.l.o.g. we assume $n \le m$, then since $a \in \ell^2$, it follows from

$$\|f_m - f_n\|_{L^2}^2 = (2\pi)^n \sum_{k\in Q_m\setminus Q_n} |a_k|^2 \le (2\pi)^n \sum_{k\in \mathbb{Z}^n\setminus Q_n} |a_k|^2 \underset{m,n\to\infty}{\longrightarrow} 0$$

that f_m is Cauchy in $L^2(\mathbb{T}^n)$, and thus that $\exists g \in L^2(\mathbb{T}^n)$ s.t. $f_m \longrightarrow g$ in $L^2(\mathbb{T}^n)$ as $m \to \infty$.

(b) We first assume that $g_m = \sum_{k\in Q_m} a_k e_k \longrightarrow g$ in $L^2(\mathbb{T}^n)$ as $m \to \infty$. We must have

$$\langle g - g_m, e_k\rangle \longrightarrow 0$$

as $m \to \infty$, but since

$$\langle g - g_m, e_k\rangle = \langle g, e_k\rangle - \langle g_m, e_k\rangle = \langle g, e_k\rangle - (2\pi)^n a_k$$

does not depend on m, then it follows that $a_k = \frac{1}{(2\pi)^n}\langle g, e_k\rangle$ for all $k \in Q_m$.

In order to prove convergence, i.e. that $\hat{g} \in \ell^2$, we observe that $\langle g - g_m, e_k\rangle = 0$ also implies that

$$g - g_m \perp \Sigma_m := \operatorname{span}\{e_k : k \in Q_m\},$$

so the pythagorian identity holds and $\|g - g_m\|_{L^2} = \inf_{f\in\Sigma_m}\|g - f\|$. Hence, if $\epsilon > 0$ is given, we use the density of $C(\mathbb{T})$ in $L^2(\mathbb{T})$ to find $h_1 \in C(\mathbb{T})$ s.t. $\|h_1 - g\| < \epsilon$ and the density of $\cup_{n\in\mathbb{N}}\Sigma_n$ in $C(\mathbb{T}^n)$ with respect to the L^∞ norm to find $h_2 \in \Sigma_n$ for some n large enough s.t. $\|h_1 - h_2\|_\infty < \epsilon$, and we have

$$\|g - g_n\| = \inf_{f\in\sum_n}\|g - f\| \le \|g - h_2\| \le \|g - h_1\| + \|h_1 - h_2\| < \epsilon + \sqrt{2\pi}\|h_1 - h_2\|_\infty < (1 + \sqrt{2\pi})\epsilon.$$

Question 2

Consider the hyperdissipative heat equation

$$u'(t) = -|D|^\theta u(t), \ \text{ for } t \ge 0, \tag{A.0.2}$$

where $\theta > 0$ and the operator $|D|^\theta u(t)$ is given in Fourier space by $\widehat{|D|^\theta f}(k) = |k|^\theta \hat{f}(k)$. Note that $\theta = 2$ corresponds the the standard hear equation. Let

$$u(t) = e^{-t|D|^\theta} g = \sum_{k\in\mathbb{Z}^n} e^{-|k|^\theta t}\hat{g}(k)e_k, \ \ t \ge 0, \tag{A.0.3}$$

where $g \in L^2(\mathbb{T}^n)$. Prove the following.

(a) Let $u, v \in C([0,T), L^2(\mathbb{T}^n))$ with $T > 0$ be two functions satisfying (1) as an equality in $L^2(\mathbb{T}^n)$ for all $0 < t < T$, and let $u(0) = v(0)$. Then $u = v$ on $[0,T)$.

(b) For any $s \ge 0$ and $t > 0$, the propagator $e^{-t|D|^\theta} : L^2(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n)$ is bounded, with

$$|e^{-t|D|^\theta} g|_s \le C_{s,\theta} t^{-s/\theta}\|g\|_{L^2}, \ \ g \in L^2(\mathbb{T}^n),$$

where C is a constant depending only on s and θ.

(c) The function $u : (0,\infty) \longrightarrow H^s(\mathbb{T}^n)$ given in (2) satisfies $u \in C^\infty((0,\infty), H^s(\mathbb{T}^n))$ for any $s \ge 0$.

(d) If $g \in H^\sigma(\mathbb{T}^n)$ for some $\sigma \ge 0$, then $u(t) \longrightarrow g$ in $H^\sigma(\mathbb{T}^n)$ as $t \searrow 0$. Moreover, u satisfies equation (1) as an equality in $H^s(\mathbb{T}^n)$ for $t > 0$ and $s \ge 0$.

(e) The quantity $u(t)(x)$, considered as a function of $(x,t) \in \mathbb{T}^n \times (0,\infty)$, is smooth.

Solution.

(a) This is shown by the fact that $\|u(0)-v(0)\|=0$ and

$$\begin{aligned}\frac{d}{dt}\|u(t)-v(t)\|^2 &= \frac{d}{dt}\langle u(t)-v(t),u(t)-v(t)\rangle \\ &= \langle (u-v)'(t),(u-v)(t)\rangle+\langle (u-v)(t),(u-v)'(t)\rangle \\ &= 2Re\left(\langle (u-v)'(t),(u-v)(t)\rangle\right) \\ &= 2Re\left(\langle -|D|^{\theta}(u-v)(t),(u-v)(t)\rangle\right) \\ &= -2Re\left(\langle -|D|^{\frac{\theta}{2}}(u-v)(t),-|D|^{\frac{\theta}{2}}(u-v)(t)\rangle\right) \\ &= -2Re\|-|D|^{\frac{\theta}{2}}(u-v)(t)\|^2\leq 0 \quad \text{ for } t>0.\end{aligned}$$

(b) We have

$$|e^{-t|D|^{\theta}}g|_s^2=\sum_{k\in\mathbb{Z}^n}|k|^{2s}e^{-2|k|^{\theta}t}|\hat{g}(k)|^2.$$

We want to get an upper bound on the middle multiplicand in the terms of the above summation, and since it is readily seen that that this multiplicand doesn't acheive a maximum at $|k|=0$, we may consider $0=q'(\xi)=\xi^{2s}e^{-2\xi^{\theta}t}$, $\xi>0$, to find the needed bound, i.e. we can find its maximum by redefining $|k|$ as a real variable $\xi>0$ and differentiating classicaly away from 0. So we solve

$$\begin{aligned}0 &= q'(\xi)=2s\xi^{2s-1}e^{-2\xi^{\theta}t}-2t\theta\xi^{\theta-1}\xi^{2s}e^{-2\xi^{\theta}t} \\ &= 2e^{-2\xi^{\theta}t}\left(s\xi^{2s-1}-t\theta\xi^{2s+\theta-1}\right)\end{aligned}$$

which holds if $s=t\theta\xi^{\theta}$. Hence, q attains its maximum on the reals when $\xi=\sqrt[\theta]{\frac{s}{t\theta}}$. We conclude that

$$|e^{-t|D|^{\theta}}g|_s^2\leq\sum_{k\in\mathbb{Z}^n}q\left(\sqrt[\theta]{\frac{s}{t\theta}}\right)|\hat{g}(k)|^2=\sum_{k\in\mathbb{Z}^n}\underbrace{(\frac{s}{\theta})^{\frac{2s}{\theta}}e^{\frac{-2s}{\theta}}}_{C^*_{s,\theta}}t^{\frac{-2s}{\theta}}|\hat{g}(k)|^2\leq C_{s,\theta}t^{-\frac{2s}{\theta}}\|g\|_{L^2}.$$

(c) First observe that we easily find from the properties of the exponential that $|e^y-1-y|\leq 2|y|^2$ when we suppose $|y|\leq\frac{1}{2}$, and that $\left|\frac{e^y-1}{y}\right|\leq e^{|y|}$ for any $y\in\mathbb{R}$. Indeed, it is readily seen that

$$|e^y-1-y|\leq\sum_{n=2}^{\infty}|y|^n=|y|^2\frac{1}{1-|y|},\ \ |y|\leq\frac{1}{2} \tag{A.0.4}$$

and

$$\left|\frac{e^y-1}{|y|}\right|\leq 1+\frac{|y|}{2}+\frac{|y|^2}{3!}+...\leq e^{|y|}. \tag{A.0.5}$$

We will use those inequalities to show that $\widehat{u'(t)}=-|k|^{\theta}\hat{g}(k)e^{-|k|^{\theta}t}$.

Let

$$\eta_k(h)=\frac{\hat{g}(k)e^{-|k|^{\theta}(t+h)}-\hat{g}(k)e^{-|k|^{\theta}(t)}}{h}+|k|^{\theta}\hat{g}(k)e^{-|k|^{\theta}t}=\hat{g}(k)e^{-|k|^{\theta}t}\underbrace{\left(\frac{e^{-|k|^{\theta}h}-1}{h}+|k|^{\theta}\right)}_{A}.$$

By (3), $|A| < 2|k|^{2\theta}h$ if $|k|^{\theta}h \le \frac{1}{2}$, and it follows from (4) that $|A| \le |k|^{\theta}\left(e^{|k|^{\theta}|h|}+1\right)$. Hence,

$$\begin{aligned}
\|\eta_k(h)\|^2_{\ell^2_s} &= \sum_{|k|\in\mathbb{Z}^n}(1+|k|^{2s})|\hat{g}(k)|^2 e^{-2|k|^{\theta}t}\left(\frac{e^{-|k|^{\theta}h}-1}{h}+|k|^{\theta}\right)^2 \\
&\le \underbrace{\sum_{|k|>N}(1+|k|^{2s})|\hat{g}(k)|^2 e^{-2|k|^{\theta}t}|k|^{2\theta}\left(e^{|k|^{\theta}|h|}+1\right)^2}_{B} + \underbrace{\sum_{|k|\le N}2(1+|k|^{2s})|\hat{g}(k)|^2 e^{-2|k|^{\theta}t}|k|^{2\theta}h}_{C}.
\end{aligned}$$

We are differentiating with respect to fixed t, so $B \longrightarrow 0$ uniformly in h as $N \to \infty$ if $|h| < \frac{t}{2}$ by (b). Thus for any $\epsilon > 0$ given, take N large so that $B < \frac{\epsilon}{2}$ given any $|h| < \frac{t}{2}$ and h small enough so that $|h| < \frac{t}{2}$, that (5) holds with $N^{\theta}h$, and that $C < \frac{\epsilon}{2}$. Then $\|\eta_k(h)\|^2_{\ell^2_s} < \epsilon$ and this shows that $\|\eta_k(h)\|^2_{\ell^2_s} \longrightarrow 0$ as $h \to 0$. Hence, u is differentiable with

$$u'(t) = \sum_{k\in\mathbb{Z}^n} -|k|^{\theta}e^{-|k|^{\theta}t}\hat{g}(k)e_k,$$

and repeating the argument,

$$u''(t) = \sum_{k\in\mathbb{Z}^n} |k|^{2\theta}e^{-|k|^{\theta}t}\hat{g}(k)e_k,$$

$$\vdots$$

$$u^{(n)} = \sum_{k\in\mathbb{Z}^n} (-1)^n|k|^{n\theta}e^{-|k|^{\theta}t}\hat{g}(k)e_k.$$

Notice that applying (b) amounts to recognizing that an appropriate exponential growth overrule a polynomial growth of any order. We conclude that $u \in C^{\infty}\left((0,\infty), H^s(\mathbb{T})\right)$.

(d) We want to show that $\|\widehat{u(t)-g}\|_{\ell^2_\sigma} \longrightarrow 0$ as $t \searrow 0$. From the same argument used in (c), $|e^y - 1| \le 2|y|$ for $|y| \le \frac{1}{2}$, and so

$$|e^{-|k|^{\theta}t}\hat{g}(k) - \hat{g}(k)| = |\hat{g}(k)\left(e^{-|k|^{\theta}t}-1\right)| \le 2|k|^{\theta}t, \text{ for } |k|^{\theta}t \le \frac{1}{2}.$$

Hence, under the same conditions,

$$\begin{aligned}
\|\widehat{u(t)-g}\|^2_{\ell^2_\sigma} &= \sum_{k\in\mathbb{Z}^n}(1+|k|^{2\sigma})|\hat{g}(k)\underbrace{\left(e^{-|k|^{\theta}t}-1\right)}_{\le 1}|^2 \\
&\le \underbrace{\sum_{k>N}(1+|k|^{2\sigma})|\hat{g}(k)|^2}_{D} + 4t^2\underbrace{\sum_{k\le N}(1+|k|^{2\sigma})|k|^{2\theta}|\hat{g}(k)|^2}_{E}.
\end{aligned}$$

For any $\epsilon > 0$ given, we can find, by assumption, N large enough so that $D < \frac{\epsilon}{2}$, and it follows from the finitness of E that the second term may be made less than $\frac{\epsilon}{2}$ with t sufficiently small.

Now we claim that (1) holds as an equality in $H^s(\mathbb{T}^n)$. We have already proven everything we need, so it is sufficient now to assemble the above results in a coherent form. We have shown in (c) that $u \in C^{\infty}\left((0,\infty), H^s(\mathbb{T})\right)$, but more importantly, we have further demonstrated that

$$u'(t)(x) = \sum_{k\in\mathbb{Z}^n} -|k|^{\theta}\hat{g}(k)e^{-|k|^{\theta}t}e_k(x) \tag{A.0.6}$$

in $H^s(\mathbb{T}^n)$. Hence, u' is equal to $-|D|^\theta u(t)$, because the latter operator is given in Fourier space so that $-|D|^\theta u(t)$ is defined as the right hand side of (5).

(e) It follows from (b) that $u(t)$ is in $H^s(\mathbb{T}^n)$ for any $s \geq 0$, and this ensures, from Berstein's Theorem, regularity in space. We need now to show regularity in time and show that taking any mixed partial derivative exist to complete the proof. By abuse of notation, we will let $u(t)(x) := u(x,t)$ be defined on $(0,T) \times \mathbb{T}^n$. We know from (c) that

$$\frac{u(x,t+h)-u(x,t)}{h} - u'(t)(x) = \frac{u(x,t+h)-u(x,t)}{h} - u'(t)(x) \longrightarrow 0$$

as $h \to 0$ in $H^s(\mathbb{T}^n)$, but it follows from Berstein's Theorem that this convergence also occurs in $C^0(\mathbb{T}^n)$, i.e. we get uniform convergence as a function of $x \in \mathbb{T}^n$, and so $\partial_t u(t,x)$ exists everywhere on $\mathbb{T}$. The same argument and (c) allows us to conclude that $\partial_t^m u(x,t) = u^{(m)}(t)(x)$ exists for any $m \in \mathbb{N}$. Moreover, for any $m, n \in \mathbb{N}$ given,

$$\begin{aligned}
&\lim_{h,\epsilon\to 0} \frac{|\partial_t^n \partial_x^m u(x+\epsilon, t+h) - \partial_t^n \partial_x^m u(x,t)|}{h} \\
&\quad \leq \lim_{h,\epsilon\to 0} \frac{|\partial_t^n \partial_x^m u(x+\epsilon, t+h) - \partial_t^n \partial_x^m u(x+\epsilon,t)|}{h} + \lim_{h,\epsilon\to 0} \frac{|\partial_t^n \partial_x^m u(x+\epsilon, t) - \partial_t^n \partial_x^m u(x,t)|}{h}
\end{aligned}$$

shows that $\partial_t^n \partial_x^m u(x,t) \in C^1(\mathbb{T}^n \times (0,\infty))$. Hence, u has partial derivatives of all order, i.e. u is smooth.

Question 3

Consider the inhomogeneous hyperdissipative heat equation

$$u'(t) = -|D|^\theta u(t) + f(t), \quad \text{for } 0 < t < T, \tag{A.0.7}$$

where $\theta > 0$ and $f \in C((0,T), H^\sigma(\mathbb{T}^n))$ for some $\sigma \geq 0$ and $0 < T \leq \infty$. We will impose the initial condition $\lim_{t\searrow 0} u(t) = g$ in $L^2(\mathbb{T}^n)$, and will understand that u' is the derivative of u considered as a function $u : (0,T) \longrightarrow H^s(\mathbb{T}^n)$ for some suitable $s \geq 0$. In other words, we look for strong H^s-solutions of $\partial_t u = -|D|^\theta u + f$. Prove the following.

(a) Let $u \in C\left([0,T), L^2(\mathbb{T}^n)\right)$ and $f \in C\left((0,T), L^2(\mathbb{T}^n)\right)$ satisfy (9) with $u(0) = g$. In particular, for each $t \in (0,T)$, $u'(t)$ and $|D|^\theta u(t)$ both exist in $L^2(\mathbb{T}^n)$. We also assume that

$$\lim_{\epsilon\searrow 0} \int_\epsilon^a \|f(t)\| dt < \infty, \ \text{ for some } 0 < a < T.$$

Then we have

$$u(t) = e^{-t|D|^\theta} g + \lim_{\epsilon\searrow 0} \int_\epsilon^t e^{(\tau-t)|D|^\theta} f(\tau) d\tau, \quad (0 < t < T),$$

where the limit $\epsilon \to 0^+$ can be replaced by the evaluation $\epsilon = 0$ if $f \in C\left([0,T), L^2(\mathbb{T}^n)\right)$. In particular, (1) has uniquenss in the considered class.

(b) Let $g \in H^\alpha(\mathbb{T}^n)$ and let $f \in C((0,T), H^\sigma(\mathbb{T}^n))$, with $0 \leq \alpha \leq \sigma$. Also let

$$\lim_{\epsilon\searrow 0} \int_\epsilon^a \|f(t)\|_\sigma dt < \infty, \ \text{ for some } 0 < a < T. \tag{A.0.8}$$

Then the function

$$u(t) = e^{-t|D|^\theta} g + \lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon} e^{(\tau - t)|D|^\theta} f(\tau) d\tau, \ \ (0 < t < T)$$

is in $C\left((0,T), H^s(\mathbb{T}^n)\right)$ for $\alpha \le s < \sigma + \theta$. Furthermore, we have $\lim_{t \searrow 0} u(t) = g$ in $H^\alpha(\mathbb{T})$, and u is a strong H^s-solution of (9) for $s < \sigma$.

Solution.

By abuse of notation, we will often write "$\int_0^t$" when we mean "$\lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon}$".

(a) The proof of part (a) is completely identitical to the proof we can find in the notes, with $e^{t\Delta}$ replaced by $e^{-|D|^\theta}$ and Δ by $|D|^\theta$. We refer the reader to the first theorem of section 2.2.3.

(b) We first want to show that $u(t) \in H^s(\mathbb{T}^n)$ whenever $0 < t < T$. Repeating the argument that we have used in question 2 part (b), we find that

$$\|e^{t|D|^\theta} g\|_s \le C_{\theta,s} \left(1 + t^{-\frac{s-\alpha}{\theta}}\right) \|g\|_\alpha \tag{A.0.9}$$

and

$$\|e^{(\tau - t)|D|^\theta} f(\tau)\|_s \le C^*_{\theta,s} \left(1 + |\tau - t|^{-\frac{s-\sigma}{\theta}}\right) \|f\|_\sigma. \tag{A.0.10}$$

Hence, we have

$$\|u(t)\|_s \lesssim \left(1 + t^{-\frac{s-\alpha}{\theta}}\right) \|g\|_\alpha + \lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon} \left(1 + |\tau - t|^{-\kappa}\right) \|f\|_\sigma d\tau,$$

where $\kappa = \frac{s-\sigma}{\theta} < 1$, and it is sufficient to show that the last terms of this inequality are bounded. The first is by assumption, and we are left to consider

$$\int_0^t \left(1 + |t-\tau|^{-\kappa}\right) \|f\|_\sigma d\tau \le \underbrace{\int_0^{\frac{t}{2}} \left(1 + |t-\tau|^{-\kappa}\right) \|f\|_\sigma d\tau}_{F} + \underbrace{\int_{\frac{t}{2}}^t \left(1 + |t-\tau|^{-\kappa}\right) \|f(\tau)\|_\sigma d\tau.}_{G}$$

It is immediate from (2) and the continuity of f that $F < \infty$. We now turn to the second term on the right hand side and observe that since f is continuous,

$$G \le \|f\|_{C((0,T),H^\sigma(\mathbb{T}^n))} \int_{\frac{t}{2}}^t (1 + t^{-\kappa}) < \infty.$$

We will now prove the convergence of u to the initial condition. We know from question 2 that $e^{-|D|^\theta} g(t) \longrightarrow g$ in $H^\alpha(\mathbb{T}^n)$ as $t \to 0^+$. Thus, since

$$\lim_{\epsilon \searrow 0} \| \int_\epsilon^{t-\epsilon} e^{(t-\tau)|D|^\theta} f(\tau) d\tau \|_\alpha \le \lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon} \|f(\tau)\|_\alpha d\tau \le \lim_{\epsilon \searrow 0} \int_\epsilon^{t-\epsilon} \|f(\tau)\|_\sigma d\tau \longrightarrow 0$$

as $t \to 0^+$, then $\lim_{t \to 0^+} u(t) = g$ in $H^\alpha(\mathbb{T}^n)$.

We claim that $u : (0,T) \longrightarrow H^s(\mathbb{T}^n)$ is continuous for $s < \sigma + \theta$. Let $h := \bar{t} - t$, assume w.l.o.g. that $h < 0$, and observe that it follows from the results obtained in question 2 that

$$\begin{aligned} e^{-|D|^\theta \bar{t}} g - e^{-|D|^\theta t} g &= h \left(e^{-|D|^\theta t} g\right)'_t + \eta_g(h) \\ &= -h|D|^\theta e^{-|D|^\theta t} g + \eta_g(h) \end{aligned} \tag{A.0.11}$$

in $H^s(\mathbb{T}^n)$, and analogously that

$$e^{(\tau-\bar{t})|D|^\theta} f(\tau) - e^{-|D|^\theta(\tau-t)} f(\tau) = -h|D|^\theta e^{(\tau-t)|D|^\theta} f(\tau) + \eta_{f(\tau)}(h) \tag{A.0.12}$$

in $H^\sigma(\mathbb{T}^n)$. From (5), we have

$$\begin{aligned} |\eta_g(h)|_s^2 &= \sum_{k\in\mathbb{Z}^n} |k|^{2s} |e^{-|k|^\theta(t+h)} - e^{-|k|^\theta t} + h|k|^\theta e^{-|k|^\theta t}|^2 |\hat{g}(k)|^2 \\ &= \sum_{k\in\mathbb{Z}^n} |k|^{2s} e^{-2|k|^\theta t} \Big(\underbrace{|e^{-|k|^\theta h} - 1 + |k|^\theta h|}_{J} \Big)^2 |\hat{g}(k)|^2. \end{aligned}$$

Using the properties of the exponential again, we use the inequality

$$|e^y - 1 - y| \leq \frac{|y|^2}{2} + \frac{|y|^3}{3!} + \ldots = \frac{|y|^2}{2}\left(1 + \frac{|y|}{3} + \frac{|y|^2}{3\cdot 4} + \ldots\right) \leq |y|^2 e^{|y|}$$

with $y = |k|^\theta h$ to find that $J \leq ||k|^\theta h|^2 e^{|k|^\theta |h|}$, which further implies that

$$\begin{aligned} |\eta_g(h)|_s^2 &\leq |h|^4 \sum_{k\in\mathbb{Z}^n} |k|^{2s+4\theta} e^{-2|k|^\theta(t-|h|)} |\hat{g}(k)|^2 \\ &\leq |h|^4 C_{s,\theta} t^{-\frac{2(s-\sigma)}{\theta}-2} \|g\|_\sigma \quad \text{for } |h| < \frac{t}{2}, \end{aligned}$$

where we the last inequality follows from (3). Now, if we let $v(\tau) = e^{(\tau-t)|D|^\theta} f(\tau)$, then it follows from (6) that

$$\begin{aligned} \left| \int_0^{\bar{t}} e^{(\tau-\bar{t})|D|^\theta} f(\tau)d\tau - \int_0^t v(\tau)d\tau \right|_s &= \left| -\int_{\bar{t}}^t v(\tau)d\tau + \int_0^{\bar{t}} -h|D|^\theta v(\tau)d\tau + \int_0^{\bar{t}} \eta_{f(\tau)}(h)d\tau \right|_s \\ &\leq -C_1|h|^{-\kappa+1} \|f\|_{C((\bar{t},t),H^\sigma(\mathbb{T}^n))} + C_2|h|^{-\kappa+1} \int_0^{\bar{t}} \|f\|_\sigma d\tau + o(1) \end{aligned} \tag{A.0.13}$$

where the last inequality follows from the continuity of v and $-|D|^\theta v$ in H^σ and (6). It follows from (2) that we can take the limit in the above inequality, because the bounds depend solely on s and θ. Since $s < \sigma + \theta$, $1 - \kappa > 1$, we find, taking $h \to 0$, that $u \in C((0,T), H^s(\mathbb{T}^n))$.

We will complete the proof by proving that u is a strong H^s-solution of the hyperdissipative inhomogeneous heat equation. We use the FTC. Since $v(\tau) \in C^\infty((0,t], H^s)$ from question 2, then in H^s,

$$\int_{\bar{t}}^t v(\tau)d\tau = V(t) - V(\bar{t}) = hv(\tau) + o(h) = hf(t) + o(h).$$

Moreover, $\int_\epsilon^{\bar{t}} |D|^\theta v(\tau)d\tau$ in continuous in H^s with respect to $\bar{t} \in (0,t]$ from the same inequalities that were used in (7), and so

$$\int_0^{\bar{t}} |D|^\theta v(\tau)d\tau \overset{H^s}{=} \int_0^t |D|^\theta v(\tau)d\tau + o(1).$$

Hence,

$$\begin{aligned} u_t'(t) &= \lim_{h\to 0} \frac{u(\bar{t}) - u(t)}{h} \\ &= |D|^\theta e^{|D|^\theta t} g + \frac{hf(t) + h\int_\epsilon^t |D|^\theta v(\tau)d\tau + h\cdot o(1)}{h} \\ &= |D|^\theta \left(e^{|D|^\theta t} g + \int_0^t v(\tau)d\tau \right) + f(t) \\ &= |D|^\theta u + f(t). \end{aligned}$$

Appendix B

Assignment 2

Question 1

Consider the initial value problem

$$\partial_t u = \Delta u + f(u), \quad u|_{t=0} = g,$$

where $f : H^s(\mathbb{T}^n) \to H^{s-1}(\mathbb{T}^n)$ is locally Lipschitz, and $g \in H^s(\mathbb{T}^n)$, for some constant $s \geq 1$. We know that there exists a unique maximal mild solution $u_g \in C\left([0, T_g), H^s(\mathbb{T}^n)\right)$, with the maximal time of existence $0 < T_g \leq \infty$ possibly depending on the initial datum $g \in H^s(\mathbb{T}^n)$ For $t \geq$ fixed, let $\Omega_t = \{g \in H^s(\mathbb{T}^n) : T_g > t\}$, and define the flow map $\Phi_t : \Omega_t \longrightarrow H^s(\mathbb{T}^n)$ by $\Omega_t g = u_g(t)$. Prove that the solution depends on the initial data continuously in the following sense. For any $g \in H^s(\mathbb{T}^n)$ and $t \in [0, T_g)$, there exists $\delta > 0$ such that $B_\delta(g) = \{h \in H^s(\mathbb{T}^n) : \|h - g\|_s < \delta\} \subset \Omega_t$, and that $\Phi_t : B_\delta(g) :\longrightarrow H^s(\mathbb{T}^n)$ is Lipschitz continuous.

Solution.

<u>Step 1</u>
For $t \in [0, T_g) \cap [0, T_h)$ and $\max\{\|u_g\|_s, \|u_h\|_s\} \leq r$, since

$$\|E(t)(g-h)\|_s = \frac{\|E(t)(g-h)\|_s}{\|g-h\|_s}\|g-h\|_s \leq \|E(t)\|_s \|g-h\|_s$$

with $\|E(t)\|_s \leq 1$, we have

$$\begin{aligned}
\|u_g(t) - u_h(t)\|_s &= \|E(t)(g-h) + \int_0^t E(t-\tau)\left[f(u_g(t)) - f(u_h(t))\right] d\tau\|_s \\
&= \|g-h\|_s + \underbrace{\|\int_0^t E(t-\tau)\left[f(u_g(t)) - f(u_h(t))\right] d\tau\|_s}_{A}.
\end{aligned}$$

Now, as $\|g-h\|_s \in \mathbb{R}$ is constant and

$$\begin{aligned}
A &\leq \|\int_0^t E(t-\tau)\left[f(u_g(t)) - f(u_h(t))\right] d\tau\|_s \\
&\leq \int_0^t C\left(1 + (t-\tau)^\kappa\right) \|f(u_g(\tau)) - f(u_h(t))\|_{s-1} d\tau \\
&\leq \int_0^t CC_r\left(1 + (t-\tau)^\kappa\right) \|u_g(\tau) - u_h(\tau)\|_s d\tau,
\end{aligned}$$

it follows from Gronwall's inequality that

$$\|u_g(t) - u_h(t)\|_s = \|g - h\|_s \underbrace{\exp\left\{\int_0^t CC_r\left(1 + (t-\tau)^\kappa\right) dt\right\}}_{\overline{C_{r,t}}},$$

where $\overline{C_r}$ depends on $r > 0$ and $t \in [0, T_g) \cap [0, T_h)$. Thus, Φ_t is Lipschitz on $\{h \in b_r(0) : t < T_h\}$ and taking $r \geq \|g\|_s + \delta$ is sufficient to ensure it is on $\{h \in B_\delta(g) : t < T_h\}$ for any given $\delta > 0$.

Step 2
Now, let $\alpha > 0$ s.t. $B_\alpha(0) \supset \mathcal{O}$, where $\mathcal{O}$ is an open subset of $H^s(\mathbb{T}^n)$ s.t. $\mathcal{O} \supset \{u_g(\bar{t}) : 0 \leq \bar{t} \leq t\}$, which we can do since $t < T_g$. Using the local existence theorem, we find $\epsilon_\alpha > 0$ indepedent of g s.t. for any initial data $h \in H^s(\mathbb{T}^n)$ s.t. $\|h\|_s < \alpha$, $u_h(\bar{t})$ is a solution of the above inhomogeneous heat equation with initial datum h on $[0, \epsilon_\alpha]$.

The idea is to observe this implies that for any $t' \in [0, t]$, $u_{u_g(t')}(\bar{t})$ is a well-defined solution of the main initial value problem on $[t', t' + \epsilon_\alpha]$. Hence, choose an integer $n \in \mathbb{N}$ s.t. we may define a partition $P = \{0 = t_1, t_2, ..., t_n = t\}$ of $[0, t]$ with $|P| < \epsilon_\alpha$.

Now, from step 1, if $\bar{t} \in [0, \epsilon_\alpha]$, then $\|u_h(\bar{t}) - u_g(\bar{t})\|_s \leq \sup_{\bar{t} \in [0, \epsilon_\alpha]} C_{\alpha, \bar{t}} \|h - g\|$, $C_\alpha \in \mathbb{R}$. We can thus find $\delta_n > 0$ s.t. for any $h \in H^s(\mathbb{T}^n)$ s.t. $\|h - u_g(t)\|_s < \delta_n$, $\|u_h(\bar{t})\|_s \leq \alpha \ \forall \bar{t} \in [0, \epsilon_\alpha]$.

Step 3
Recursively, we find $\delta_1, ..., \delta_{n-1}$, $\delta_i > 0$, s.t. for any $h \in H^s(\mathbb{T}^n)$ for which $\|h - u_g(t_i)\|_s < \delta_i$, we have $\|h - u_g(t_{i+1})\| < \delta_{i+1} \ \forall \bar{t} \in [0, \epsilon_\alpha]$, i.e. we have found $\delta = \delta_1 > 0$ s.t. if $h \in B_\delta(g)$, then h has by uniqueness a maximal extension $u_h \in H^s(\mathbb{T}^n)$ for which $T_h > t$.

Question 2

Consider the following nonlinear reaction-diffusion equation

$$\partial_t u = \Delta u + \delta e^{\delta u},$$

with initial datum $g \in H^s(\mathbb{T}^n)$, where $\delta \in \mathbb{R}$ and $s > \frac{n}{2}$ are constants. We assume that g is positive everywhere. Prove the following, and in the case of (d), provide upper and lower bounds for the blow-up time T.
(a) The problem has a unique maxmal mild solution $u \in C([0, T), H^s(\mathbb{T}^n))$.
(b) The mild solution is in fact smooth in $\mathbb{T}^n \times (0, T)$.
(c) If $\delta > 0$, then the solution blows up in a finite time, i.e., $T < \infty$.

Solution.

(a) We first need to verify that $f(u) = \delta e^u \in H^s(\mathbb{T}^n)$ for $u \in H^s(\mathbb{T}^n)$, i.e. that $\|f(u)\|_s < \infty$, but this is immediate from the triangle inequality, since using the series expansion of the exponential,

$$\|\delta e^{\delta u}\|_s \leq |\delta| \sum_{k=0}^\infty \frac{\|\delta u\|^n}{n!} = |\delta| e^{|\delta| \cdot \|u\|_s} < \infty,$$

where we found $e^{\|\delta u\|_s} < \infty$ using the assumption that $\|u\|_s \in \mathbb{R}$ and the fact that e^x is a real-valued function on $\mathbb{R}$. If we can prove that $f(u)$ is also locally Lipschitz, then the existence theorems of section

3.1 ensure the desired result. Let $u, v \in H^s(\mathbb{T}^n)$ s.t. $\max\{\|u\|_s, \|v\|_s\} \leq r$, $r > 0$. It follows from the Banach algebra property ($s > \frac{n}{2}$) that

$$\begin{aligned}
\|f(u) - f(v)\|_s &= |\delta| \cdot \|e^{\delta u} - e^{\delta v}\|_s \\
&= |\delta| \cdot \| \sum_{k=0}^{n} \frac{1}{k!} \left((\delta u)^k - (\delta v)^k \right) \|_s \\
&= |\delta| \cdot \| \sum_{k=0}^{n} \delta(u - v) \frac{1}{k!} \left((\delta u)^{k-1} + (\delta u)^{k-2} \delta v + ... + (\delta v)^{k-2} \delta u + ((\delta v)^{k-1} \right) \|_s \\
&\leq C_{s,n} \|u - v\|_s \sum_{k=0}^{n} \frac{k}{k!} (|\delta| r)^{k-1},
\end{aligned}$$

and so because $\sum_{k=0}^{n} \frac{k}{k!} (|\delta| r)^{k-1}$ converges to a real number by the ratio test, letting

$$C_r = |\delta| C_{s,n} \sum_{k=0}^{n} \frac{k}{k!} (|\delta| r)^{k-1}$$

shows that $f(u) : H^s(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n)$ is locally Lipschitz and the proof is complete.

(b) According to section 3.2, it is sufficient to compute the time derivative of $f(u) = \delta e^{\delta u}$ at $t \in (0, T)$ to adapt the proof of the regularity of the mild solution u to the case of the nonlinear reaction-diffusion equation. Hence, observing that

$$\begin{aligned}
\lim_{h \to 0} \frac{\delta e^{\delta u(t+h)} - \delta e^{\delta u(t)}}{h} &= \lim_{h \to 0} \frac{\delta}{h} \left[\sum_{k=0}^{\infty} \frac{(\delta u)^k (t+h) - (\delta u)^k (t)}{k!} \right] \\
&= \lim_{h \to 0} \delta^2 \frac{u(t+h) - u(t)}{h} \sum_{k=0}^{\infty} \delta^{k-1} \frac{u^{k-1}(t+h) + u^{k-2}(t+h) u(t) + ... + u^k(t)}{k!} \\
&= \delta^2 u'(t) \sum_{k=0}^{\infty} \frac{k (\delta u)^{k-1}(t)}{k!} \\
&= \delta^2 u'(t) \sum_{k=0}^{\infty} \frac{(\delta u)^k (t)}{k!} \\
&= \delta^2 u'(t) e^{u(t)}
\end{aligned}$$

implies the required result.

(c) The ODE $\partial_t v = \delta e^{\delta v}$ has solution $v(t) = -\frac{1}{\delta} \log\left(\delta(C_0 - \delta t)\right)$, where C_0 is an arbitrary constant that we may choose to our liking. If we let $C_0 = e^{-\min_{x \in \mathbb{T}^n} g(x)}$, then $v(t)$ is defined on $[0, C_0)$, $C_0 > 0$, and $|v(t)| \longrightarrow \infty$ as $t \nearrow C_0$.

We can compare u and v, and apply the maximum principle to bound u by v. Indeed, from the above definition of C_0,

$$v(0) = \min_{x \in \mathbb{T}^n} g(x) \leq g, \ \forall x \in \mathbb{T}^n,$$

so $v \geq u$ on $\mathbb{T}^n \times \{0\}$. Moreover, we have

$$
\begin{aligned}
\partial_t(u-v)+\Delta(u-v) &= \delta \sum_{k=0}^{\infty} \frac{1}{k!}((\delta u)^k-(\delta v)^k) \\
&= (u-v)\underbrace{\delta^2 \sum_{k=1}^{\infty} \frac{1}{k!}((\delta u)^{k-1}+(\delta u)^{k-2}(\delta v)+\ldots+(\delta v)^{k-1})}_{c(u,v)},
\end{aligned}
$$

where c is a real-valued bounded function on $\mathbb{T}^n \times (0,T]$ for $T < \min\{T_u, C_0/\delta\}$, where T_u is the maximal time of existence of u. Hence, the comparision principle implies that $v \leq u$ on $\mathbb{T}^n \times (0,T]$. Since v blows up at C_0/δ, then u also bows up in a finite amount of time at $T_u \leq C_0$.

Question 3

Consider the Allen-Cahn equation

$$\partial_t u = \Delta u + u - u^3,$$

with initial datum $g \in H^s(\mathbb{T}^n)$ for some $s > \frac{n}{2}$. Prove the following.
(a) The problem has a unique maximal mild solution $u \in C([0,T), H^s(\mathbb{T}^n))$.
(b) The mild solution is in fact smooth in $\mathbb{T}^n \times (0,T)$.
(c) The solution is global in time.

Solution.

(a) It is immediate from the Banach algebra property ($s > \frac{n}{2}$) that $f(u) = u - u^3 \in H^s(\mathbb{T}^n)$ if $u \in H^s(\mathbb{T}^n)$:

$$\|u-u^3\|_s \leq \|u\|_s + C_{s,n}\|u\|_s^3 < \infty.$$

Showing that $f : H^s(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n)$ is also locally Lipschitz completes the proof, because then the existence theorems of section 3.1 apply. Let $u, v \in H^s(\mathbb{T}^n)$ s.t. $\max\{\|u\|_s, \|v\|_s\} \leq r$, $r > 0$, then using the Banach algebra property again,

$$
\begin{aligned}
\|f(u)-f(v)\|_s &= \|u-v-u^3+v^3\|_s \\
&= \|(u-v)\left(1+u^2+uv+v^2\right)\|_s \\
&\leq \|u-v\|_s\left(1+3r^2\right)
\end{aligned}
$$

and choosing $C_r = \left(1+3r^2\right)$ shows that $u-u^3$ is locally Lipschitz in $H^s(\mathbb{T}^n)$.

(b) In respect of section 3.2, it is sufficient to observe that in this particular case, regularity in time follows from the fact that

$$
\begin{aligned}
\frac{f(u(t+h))-f(u(t))}{h} &= \frac{u(t+h)-u^3(t+h)-u(t)+u^3(t)}{h}, \\
&= \underbrace{\frac{u(t+h)-u(t)}{h}}_{\longrightarrow u'(t) \text{ in } H^s} - \underbrace{\frac{u(t+h)-u(t)}{h}}_{\longrightarrow u'(t) \text{ in } H^s}\left(\underbrace{u^2(t+h)+u(t+h)u(t)+u^2(t)}_{\longrightarrow 3u^2(t) \text{ in } H^s}\right),
\end{aligned}
$$

while the rest of the proof stays identical.

(c) We proceed by induction. Solving the ODE $\partial_t v_0 = v_0 - v_0^3$, we find $v_0(t) = \frac{e^t}{\sqrt{C_0+e^{2t}}}$. Choosing $C_0 = \frac{1}{(\max_{x\in\mathbb{T}^n} g(x))^2} - 1$, we have $u \leq v$ on $\mathbb{T}^n \times \{0\}$, and thus since

$$\begin{aligned}\partial_t(u-v_0) + \Delta(u-v_0) &= u - u^3 - (v_0 - v_0^3) \\ &= (u-v)\underbrace{\left(1-(u^2+uv_0+v_0^2)\right)}_{c(u,v):\mathbb{T}^n\times(0,T]\longrightarrow\mathbb{R}}\end{aligned}$$

on $\mathbb{T}^n \times (0,T]$ for $T < T_u$ where T_u is the maximal time of existence associated to the solution u, then $u \leq v_0$ on $\mathbb{T}^n \times (0,T]$. Hence, as $v_0(t) : (0,\infty) \longrightarrow \mathbb{R}$, then we have found the basic estimate $|u(x,t)| \leq v_0(t) = \frac{e^t}{\sqrt{C_0+e^{2t}}}$, i.e. $u(x,t)$ is bounded pointwise on $(0,\infty)$. We need to bound the higher order derivatives to complete the proof. First, we have

$$\begin{aligned}\partial_t \underbrace{\partial_k u} \quad &= \Delta \underbrace{\partial_k u} + 3u^2 \underbrace{\partial_k u} + \underbrace{\partial_k u} \\ \Longrightarrow \quad \partial_t \underbrace{\partial_k u} - \Delta\partial_t \underbrace{\partial_k u} \quad &\leq v_1(T) \underbrace{\partial_k u} + \underbrace{\partial_k u},\end{aligned}$$

where $v_1(T) = 3\left[v_0(T)\right]^2$, and we compare $\partial_k u$ with the solution $v_2^k = C_1^k e^{(v_1(T)+1)t}$ of the ODE $\partial_t v_2 = v_1(T)v_2 + v_2$. From the local theory, $\exists \epsilon > 0$ s.t. $u(\cdot,\epsilon) \in H^s$, hence we may choose $C_1^k \geq \|\partial_k u(\cdot,\epsilon)\|_\infty$ and conclude from the maximum principle that $\|\nabla u(t)\|_\infty < \infty$ on $(0,\infty)$. We then repeat the argument on the inequality

$$\begin{aligned}\partial_t \underbrace{\partial_i\partial_k u} - \Delta \underbrace{\partial_i\partial_k u} &= 6u\partial_i u\partial_k u + (3u^2+1)\underbrace{\partial_i\partial_k u} \\ &\leq (\underbrace{3v_1(T)}_{B} + 1)\underbrace{\partial_i\partial_k u} + \underbrace{6v_0(T)v_2^k(T)v_2^i(T)}_{A},\end{aligned}$$

which we compare with the the ODE $\partial_t v_3 = (B+1)v_3 + A$, which has again a solution of the form $C_2^{i,k}e^{(B+1)t} - \frac{A}{B}$, which we can use to bound $\partial_i\partial_k u$ at time $t = \epsilon$. The above shows that the base case holds.

For a genral ∂^α, the induction step reduces to applying the induction hypothesis when comparing our PDE to

$$\partial_t\partial^\alpha v^\alpha - \Delta\partial^\alpha v = (B_\alpha 3v^2 + 1)\partial^\alpha v + A_\alpha,$$

where $A_\alpha, B_\alpha \in \mathbb{R}$. We conclude that $\|u(\cdot,t)\|_s < \infty$ on $(0,\infty)$.

Appendix C

Assignment 3

Question 1

Prove that the solution of the magnetohydrodynamics system

$$
\begin{aligned}
\partial_t u &= \Delta u - u \cdot \nabla u - \nabla p + h \cdot \nabla h \\
\partial_t u &= \Delta h - u \cdot \nabla h + h \cdot \nabla u,
\end{aligned}
$$

with $\operatorname{div}(u) = \operatorname{div}(h) = 0$, in $\mathbb{T}^2$ ia global in time.

Solution.

We want to derive a basic energy identity analog to the BEI we have derived in section 4.2 for the NSE. Using the same arguments we have used to find the latter, we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\|\nabla \otimes u\|^2 + \underbrace{\int u \cdot (h \cdot \nabla h)}_{A}.$$

Integrating by parts and using the product rule, we further establish that

$$
\begin{aligned}
A &= \int u \cdot (h \cdot \nabla h) \\
&= \int u_j h_i \partial_i h_j \\
&= -\int h_j h_i \partial_i u_j - \int h_j u_j \partial_i h_i \\
&= -\int h \cdot (h \nabla u)\,,
\end{aligned} \tag{C.0.1}
$$

where (1) was derived using the fact that $-\int h_j u_j \partial_i h_i = -\int h_j u_j \operatorname{div}(h) = 0$ by hypothesis. This implies that

$$\frac{d}{dt}\|u\|^2 = -2\|\nabla \otimes u\|^2 + A. \tag{C.0.2}$$

In the same way, we have

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 \leq -\|\nabla \otimes h\|^2 + \underbrace{\int h \cdot (u \cdot \nabla h)}_{B} + \underbrace{\int h \cdot (h \cdot \nabla u)}_{-A}.$$

It turns out $B = 0$. In order to arrive at this conclusion, we observed that

$$\begin{aligned} B &= \int h \cdot (u \cdot \nabla h) \\ &= \int h_j u_i \partial_i h_j \\ &= -\int u_i h_j \partial_i h_j - \int h_j h_j \partial_i u_i \\ &= -\int h \cdot (u \cdot \nabla h), \end{aligned}$$

which holds since $-\int h_j h_j \partial_i u_i = -\int h_j h_j \mathrm{div}(u) = 0$ by assumption. Hence, we have

$$\frac{d}{dt}\|h\|^2 = -2\|\nabla \otimes h\|^2 - A,$$

and combining this inequality with (2), we find the desired basic energy idendity

$$\frac{d}{dt}(\|h\|^2 + \|u\|^2) = -2\|\nabla \otimes u\|^2 - 2\|\nabla \otimes h\|^2.$$

The integral form of the above inequality is given by

$$\|u(0)\|^2 + \|h(0)\|^2 = \|u(t)\|^2 + \|h(t)\|^2 + 2\int_0^t |u(s)|_1^2 ds + 2\int_0^t |h(s)|_1^2 ds,$$

which exhibits the fact that knowledge of the initial data yields control on the $L^\infty L^2-$norm and the $L^2 H^1$-norm of $(u,h)^T$. We now proceed to investigate the equations

$$\begin{aligned} \partial_t \partial_k u &= \partial_k \Delta u - \partial_k(u \cdot \nabla u) - \partial_k(\nabla p) + \partial_k(h \cdot \nabla h) \quad \& \\ \partial_t \partial_k u &= \partial_k \Delta h - \partial_k(u \cdot \nabla h) + \partial_k(h \cdot \nabla u). \end{aligned}$$

From the same integrating tools that we have used in the NSE problem, we find that

$$\frac{d}{dt}\|\nabla \otimes u\|^2 \leq -|u|_{2+}^2 \|u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + 2\sum_k \int \partial_k u \cdot \partial_k(h \cdot \nabla h)$$

and

$$\frac{d}{dt}\|\nabla \otimes h\|^2 \leq -|h|_2^2 + \|h\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + 2\sum_k \int \partial_k h \cdot \partial_k(h \cdot \nabla u).$$

Using Young's inequality, we find that

$$\begin{aligned} \frac{d}{dt}\left(\|\nabla \otimes u\|^2 + \|\nabla \otimes h\|^2\right) \leq& -\left(|u|_2^2 + |h|_2^2\right) + \|u\|_{L^4}^2 \left(\|\nabla u\|_{L^4}^2 + \|\nabla h\|_{L^4}^2\right) \\ &+ \epsilon_1 \left(\|\Delta u\|^2 + \|\Delta h\|^2\right) + \frac{1}{\epsilon_1}\|h\|_{L^4}^2 \left(\|\nabla u\|_{L^4}^2 + \|\nabla h\|_{L^4}^2\right) \end{aligned}$$

for any $\epsilon > 0$. Hence, from applying Ladyzhenskaya inequality and Young's inequality again, we conclude, assuming w.l.o.g. that $\hat{u}(0) = \hat{h}(0) = 0$, that

$$\begin{aligned} \frac{d}{dt}\left(\|\nabla \otimes u\|^2 + \|\nabla \otimes h\|^2\right) \lesssim& -\left(|u|_2^2 + |h|_2^2\right) + \frac{1}{\epsilon_2}\left(\|u\|^2 |u|_1^4\right) + \frac{1}{\epsilon_2}\|u\|^2 |u|_1^2 |h|_1^2 + (\epsilon_1 + \epsilon_2)\left(|u|_2^2 + |h|_2^2\right) \\ &+ \frac{1}{\epsilon_1 \epsilon_3}\left(|u|_2^2 + |h|_2^2\right) + \frac{\epsilon_3}{\epsilon_1}\|h\|^2 |h|_1^2 |u|_1^2 + \frac{\epsilon_3}{\epsilon_1}\|h\|^2 |h|_1^4. \end{aligned}$$

Choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ s.t. $\epsilon_1 + \epsilon_2 < 1$, and choose $\epsilon_3 > 0$ accordingly such that $(\epsilon_1 + \epsilon_2) + \frac{1}{\epsilon_1 \epsilon_3} < 1$. By doing so, we may further derive that

$$\begin{aligned} \frac{d}{dt}\left(\|\nabla \otimes u\|^2 + \|\nabla \otimes h\|^2\right) &\lesssim \|u\|^2|u|_1^4 + \|h\|^2|h|_1^4 + \|u\|^2|u|_1^2|h|_1^2 + \|h\|^2|h|_1^2|u|_1^2 \\ &= \|u\|^2|u|_1^4 + \|h\|^2|h|_1^4 + \left(\|u\|^2 + \|h\|^2\right)|u|_1^2|h|_1^2 \\ &\leq \left(\|u\|^2 + \|h\|^2\right)\left(|u|_1^4 + |h|_1^4\right) + \left(\|u\|^2 + \|h\|^2\right)\left(|u|_1^4 + |h|_1^4\right) \\ &\lesssim \left(\|u\|^2 + \|h\|^2\right)\left(|u|_1^4 + |h|_1^4\right) \\ &\lesssim \left(\|u\|^2 + \|h\|^2\right)\left(|u|_1^2 + |h|_1^2\right)^2, \end{aligned}$$

and Gronwall's inequality, along with the L^2H^1-norm control on $(u,h)^T$ obtained from the basic energy identity, yields control on the H^1-norm of $(u,h)^T$. In trying to acheive control on the H^2-norm of the solution. We follow the same arguments we have used in the NSE problem to find that

$$\frac{d}{dt}\|\nabla^2 \otimes u\|^2 \lesssim -\frac{1}{4}|u|_3^2 + |u|_1^2|u|_2^2 + \|u\|^2|u|_1^2|u|_2^2 + \underbrace{|u|_3|h|_1|h|_2}_{C} + \underbrace{|u|_3\|h\|^{\frac{1}{2}}|h|_1^{\frac{1}{2}}|h|_2^{\frac{1}{2}}|h|_3^{\frac{1}{2}}}_{D}.$$

Using Young's inequality, $C \leq \epsilon_1|u|_3^2 + \frac{1}{\epsilon_1}|h|_1^2|h|_2^2$ and

$$D \leq \epsilon_2|u|_3^2 + \frac{1}{\epsilon_2}\|h\||h|_1|h|_2|h|_3 \leq \epsilon_2|u|_3^2 + \frac{\epsilon_3}{\epsilon_1}|h|_3^2 + \frac{1}{\epsilon_2\epsilon_3}\|h\|^2|h|_1^2|h|_2^2.$$

Hence,

$$\frac{d}{dt}\|\nabla^2 \otimes u\|^2 + \frac{d}{dt}\|\nabla^2 \otimes h\|^2 \lesssim |u|_2^2\left(|u|_1^2 + \|u\|^2|u|_1^2\right) + |h|_2^2\left(\frac{1}{\epsilon_1}|h|_1^2 + \frac{1}{\epsilon_2\epsilon_3}\|h\|^2|h|_1^2\right) + E + F$$

where

$$\begin{aligned} E &\leq |h|_3\|\nabla \otimes (u \cdot \nabla h)\| \\ &\leq \epsilon_4|h|_3^2 + \frac{1}{\epsilon_4}\left(\|(\nabla \otimes u)\cdot \nabla h\| + \|u \cdot \nabla^2 h\|\right)^2 \\ &\leq \epsilon_4|h|_3^2 + \frac{1}{\epsilon_4}|u|_1^2|h|_1^2 + \frac{1}{\epsilon_4}\|u\|^2|h|_2^2 \end{aligned}$$

and where analogously,

$$F \leq \epsilon_5|h|_3^2 + \frac{1}{\epsilon_5}|h|_1^2|u|_1^2 + \frac{1}{\epsilon_5}\|h\|^2|u|_2^2,$$

for any $\epsilon_1, ..., \epsilon_5 > 0$. Hence,

$$\frac{d}{dt}\|\nabla^2 \otimes u\|^2 + \frac{d}{dt}\|\nabla^2 \otimes h\|^2 \leq [\cdot] + [\cdot]\left(|u|_2^2 + |h|_2^2\right),$$

where $[\cdot]$ are terms over which we have previously established control. The H^2-norm control over $(u,h)^T$ is now obtained by using Gronswall's inequality or by comparision with the ODE $y' = C + Cy$.

Question 1

Prove that the Navier-Stokes initial value problem

$$\partial_t u = \Delta u - \mathbb{P}\mathrm{div}(u \otimes u),$$

in $\mathbb{T}^4$or $\mathbb{T}^5$,with smooth, divergence-free initial data, has global smooth solution if the H^3-norm of the initial data is small.

Solution.

W.l.o.g., assume $\hat{u}(0) = 0$, and thus that $\|u\|_{L^2} = 0$. Recall that this implies that $|\cdot|_{H^{s_1}} \leq |\cdot|_{H^{s_2}}$ whenever $s_1 \leq s_2$. We need global control on the H^s-norm of the solution u for some $s > \frac{n}{2}$. It is thus sufficient for the NSE in $\mathbb{T}^4$ and $\mathbb{T}^5$ to control the H^3-norm of u. In order to gain this control, we consider the equivalent equation

$$\partial_t (\partial_i \partial_j \partial_k u) = \Delta (\partial_i \partial_j \partial_k u) + \partial_i \partial_j \partial_k \mathrm{div}(u \otimes u) - \partial_i \partial_j \partial_k p,$$

and assume that the H^s-norm of u is controlled for $s < 3$. We use the energy method again. Since $\mathrm{div}(u) = 0$ and

$$\frac{1}{2} \frac{d}{dt} \|\partial_i \partial_j \partial_k u\|^2 = -\|\nabla \otimes \partial_i \partial_j \partial_k u\|^2 - \int \partial_i \partial_j \partial_k u \cdot \partial_i \partial_j \partial_k \mathrm{div}(u \otimes u),$$

we have

$$\begin{aligned} \frac{d}{dt}|h|_3^2 &\leq -2|u|_4^2 + 2 \int \Delta \partial_j \partial_k u \cdot \partial_j \partial_k (u \cdot \nabla u) \\ &\leq -2|u|_4^2 + 2 \int |\Delta \partial_j \partial_k u| |\partial_j \partial_k (u \cdot \nabla u)| \\ &\leq -2|u|_4^2 + C|u|_4 |u \otimes u|_3 \\ &\leq -|u|_4^2 + C|u \otimes u|_3^2 \\ &\leq -|u|_4^2 + C|u|_3^4 \end{aligned}$$

where the product rule and the smoothness of u was used. By the integral assumption on u, it holds that $|u|_3^2 < |u|_4^2$, hence

$$-|u|_4^2 + C|u|_3^4 \leq -|u|_3^2 + C|u|_3^4,$$

and conclusion follows from comparing $\frac{d}{dt}|h|_3^2 \leq -|u|_3^2 + |u|_3^4$ with $y' = -y + Cy^2$, if we take $|u|_3$ small enough for the solution of the ODE to be bounded.

Appendix D

Rellich's Theorem

A COMPACTNESS THEOREM FOR BOCHNER-SOBOLEV SPACES

TSOGTGEREL GANTUMUR

In the previous lecture, we have used the following result without proof.

Theorem 1. *Let $X \hookrightarrow Y \hookrightarrow Z$ be Hilbert spaces, with X compactly embedded into Y. Then $L^2_T X \cap H^1_T Z$ is compactly embedded into $L^2_T Y$ for any $0 < T < \infty$.*

In this note, we want to prove this theorem. We start with a couple of preliminary lemmata, which are also interesting on their own.

Lemma 2. *Let X be a normed space, and let $\{x_n\} \subset X$ be a sequence converging to 0 weakly in X. Then the sequence $\{x_n\}$ is bounded in X. In addition, if Y is a Banach space, and if $K : X \to Y$ is a compact operator, then $Kx_n \to 0$ strongly in Y.*

Proof. Consider the linear operators $T_n : X^* \to \mathbb{R}$ defined by $T_n x^* = \langle x^*, x_n\rangle$ for $x^* \in X^*$. Then for each $x^* \in X^*$ we have $T_n x^* \to 0$, and in particular, $T_n x^*$ is bounded. Hence by the the Banach-Steinhaus theorem, the sequence $\{T_n\}$ is bounded in X^{**}, i.e., there is a constant $M < \infty$ such that

$$|\langle x^*, x_n\rangle| \leq M\|x^*\|_{X^*}, \tag{1}$$

for all $x^* \in X^*$. Now by the Hahn-Banach theorem, for each n, there exists $x^* \in X^*$ such that $\langle x^*, x_n\rangle = \|x_n\|$ and $\|x^*\|_{X^*} \leq 1$, which shows that the sequence $\{x_n\}$ is bounded in X.

For the second part, take an arbitrary subsequence of $\{x_n\}$, and denote it again by $\{x_n\}$. Since this sequence is bounded, the sequence $\{Kx_n\}$ is precompact in Y, and so passing to a subsequence, which we call again $\{Kx_n\}$, we have $Kx_n \to y$ in Y, for some $y \in Y$. Moreover, $y = 0$ by noting that $Kx_n \to 0$ weakly in Y, i.e.,

$$\langle y^*, Kx_n\rangle = \langle K^* y^*, x_n\rangle \to 0, \qquad y^* \in Y^*, \tag{2}$$

where $K^* : Y^* \to X^*$ is the adjoint of K. To conclude, we have proved that any subsequence of $\{Kx_n\}$ contains a subsequence converging to 0 in Y, which means that the original sequence converges to 0 in Y. $\square$

Lemma 3. *Let $X \hookrightarrow Y \hookrightarrow Z$ be Banach spaces, with X compactly embedded into Y. Then for any $\varepsilon > 0$, there exists a constant C_ε such that*

$$\|x\|_Y \leq \varepsilon\|x\|_X + C_\varepsilon\|x\|_Z, \qquad x \in X. \tag{3}$$

Proof. If it was not true, there would exist a number $\varepsilon > 0$ and a sequence $\{x_n\} \subset X$ with $\|x_n\|_X = 1$, such that

$$\|x_n\|_Y > \varepsilon + n\|x_n\|_Z. \tag{4}$$

By compactness of the embedding $X \hookrightarrow Y$, a subsequence of $\{x_n\}$ converges in Y, to an element $y \in Y$. The convergence is also in Z, by continuity of the embedding $Y \hookrightarrow Z$. We have $y \neq 0$, because (4) implies $\|x_n\|_Y > \varepsilon > 0$. However, since $\|x_n\|_Y \lesssim \|x_n\|_X = 1$, the condition (4) would also imply that $\|x_n\|_Z \to 0$ as $n \to \infty$. $\square$

Proof of Theorem 1. Let $\{u_k\}$ be a bounded sequence in $L^2_T X \cap H^1_T Z$. Then passing to a subsequence, we have $u_k \to u$ weakly in $L^2_T X$ and $u'_k \to u'$ weakly in $L^2_T Z$. Letting $v_k = u_k - u$,

Date: December 15, 2014.

we claim that $v_k \to 0$ strongly in $L^2_T Z$. If the claim is true, fixing an arbitrary $\varepsilon > 0$, by Lemma 3 we have

$$\|v_k(t)\|_Y \leq \varepsilon \|v_k(t)\|_X + C_\varepsilon \|v_k(t)\|_Z, \tag{5}$$

for almost every t, implying that

$$\|v_k\|_{L^2_T Y} \lesssim \varepsilon \|v_k\|_{L^2_T X} + C_\varepsilon \|v_k\|_{L^2_T Z}. \tag{6}$$

Since v_k is bounded in $L^2_T X$, this would imply that $v_k \to 0$ strongly in $L^2_T Y$.

Now we prove the claim. Using that $H^1_T Z \subset C([0,T], Z)$, we can write

$$v_k(t) = v_k(s) + \int_s^t v_k'(\tau)\,\mathrm{d}\tau, \tag{7}$$

and integrating it over s between $t - \sigma$ and t for small $\sigma > 0$, we have

$$v_k(t) = \underbrace{\frac{1}{\sigma}\int_{t-\sigma}^t v_k(s)\,\mathrm{d}s}_{a_k(t)} + \underbrace{\frac{1}{\sigma}\int_{t-\sigma}^t \int_s^t v_k'(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}_{b_k(t)}. \tag{8}$$

The second term can be estimated as

$$\|b_k(t)\|_Z \leq \frac{1}{\sigma}\int_{t-\sigma}^t (\tau + \sigma - t)\|v_k'(\tau)\|_Z\,\mathrm{d}\tau \leq \int_{t-\sigma}^t \|v_k'(\tau)\|_Z\,\mathrm{d}\tau \leq \sqrt{\sigma}\|v_k'\|_{L^2_T Z}, \tag{9}$$

which converges to 0 as $\sigma \to 0$. For the first term, we have

$$\langle a_k(t), x^* \rangle = \frac{1}{\sigma}\int_{t-\sigma}^t \langle v_k(s), x^* \rangle\,\mathrm{d}s \to 0, \qquad x^* \in X^*, \tag{10}$$

for almost every t, by the weak convergence of v_k to 0 in $L^2_T X$. By Lemma 2, this means that $a_k(t) \to 0$ strongly in Z, for almost every t. Overall, we now have $v_k \to 0$ in Z almost everywhere. We already know that the sequence $\{v_k\}$ is bounded in $C([0,T], Z)$, which implies by $T < \infty$ that there is $g \in L^2((0,T))$ satisfying $\|v_k(t)\|_Z \leq g(t)$ for almost every t. Finally, an application of the dominated convergence theorem establishes that $v_k \to 0$ in $L^2_T Z$. □