Navier-Stokes Project: Minimal $\dot{H}^{1/2}/L^3$ Data for Blow Up Solution

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1 Introduction

The initial value problem for the three dimensional Navier-Stokes equations is given by the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u \cdot \nabla u - \nabla p \ \text{ in } \mathbb{R}^3 \times (0,\infty) & \text{(1a)} \\ \nabla \cdot u = 0 \qquad\qquad \text{in } \mathbb{R}^3 \times (0,\infty) & \text{(1b)} \\ u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}^3 & \text{(1c)} \end{cases}$$

where p is defined so that $u \cdot \nabla u + \nabla p = \mathbb{P}(u \cdot \nabla u)$, where $\mathbb{P}$ is the Leray projection operator, projecting vector fields onto the space of divergenceless vector fields with respect to the $L^2(\mathbb{R}^3)$ inner product. By the Helmholtz decomposition theorem any sufficiently rapidly decaying $C^2(\mathbb{R}^3)$ vector field u can be decomposed as $u = \nabla\phi + \nabla \times A$ for some scalar function ϕ and vector field A, so since the divergence of a curl is zero we can formally write $\mathbb{P}u \equiv u - \nabla(\Delta^{-1}\nabla \cdot u)$. Of course the above system implies that the initial condition must be divergenceless. Note that this we have $-\Delta p = \operatorname{div}(u \cdot \nabla u)$, which will be used later. This paper will give a brief survey of the main results.

We recall the definition of $H^{1/2}(\mathbb{R}^3)$:

$$H^{1/2}(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \|(1+|\xi|^2)^{1/4}\hat{f}(\xi)\|_{L^2(\mathbb{R}^3)} < \infty\}. \tag{2}$$

$\dot{H}^{1/2}(\mathbb{R}^3)$ is then the space $H^{1/2}(\mathbb{R}^3)$ with the seminorm

$$|u|_{\dot{H}^{1/2}(\mathbb{R}^3)} \equiv \||\xi|^{1/2}\hat{f}(\xi)\|_{L^2(\mathbb{R}^3)}. \tag{3}$$

We can write the Navier-Stokes system as an integral equation:

$$u(\cdot,t) = e^{\Delta t}u_0 - \int_0^t e^{\Delta(t-s)}\,\mathbb{P}\left(u\cdot\nabla u(\cdot,s)\right)ds\,. \tag{4}$$

A solution to the above equation is called a mild solution. Morally one should consider this equation instead as integral operators have much better behaviour than differential operators, ie. are continuous, compact under appropriate conditions. It is known that the Navier-Stokes equations have a unique mild solution for small $\dot{H}^{1/2}(\mathbb{R}^3)$ initial data, where $\dot{H}^{1/2}(\mathbb{R}^3)$ is the space of functions with finite Sobolev seminorm induced by $H^{1/2}(\mathbb{R}^3)$. What we want to prove is that the set of real numbers small enough is open, more precisely (note this problem is known to be locally well-posed (Fujita-Kato; 1962)):

Given a function $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, let $T_{\max}(u_0)$ be the maximal time of existence for the mild solution u of the Navier-Stokes equations with initial condition u_0. Set

$$\epsilon_{\max} \equiv \sup\{\epsilon : T_{\max}(u_0) = \infty \text{ for all } \ u_0 \text{ such that } |u_0|_{H^{1/2}(\mathbb{R}^3)} < \epsilon\}\,. \tag{5}$$

It is known that $\epsilon_{\max} > 0$. Set

$$C \equiv \{u_0 : T_{\max}(u_0) < \infty\,,\ |u_0|_{H^{1/2}(\mathbb{R}^3)} = \epsilon_{\max}\}\,, \tag{6}$$

where C stands for critical.

Theorem 1.1: Suppose $\epsilon_{\max} < \infty$, then C is nonempty. □

A priori there could be several reasons why, for a given initial condition, the maximal time of existence is finite. However it can be shown that this can only be the case if the solution develops a singularity, which will be defined shortly.

2 Suitable Weak Solutions and Singularities

Let $O \subset \mathbb{R}^3 \times \mathbb{R}$ be open, let u be an $\mathbb{R}^3$ valued function and p a real valued function, both on O. Then (u,p) will be called a suitable weak solution on O to the Navier-Stokes equations if the following conditions hold:

1) $u \in L_t^\infty L_x^2 \bigcap L_t^2 \dot{H}_x^1$

2) $p \in L_t^{3/2} L_x^{3/2}$

3) (u,p) solve $(1a), (1b)$ in the sense of distributions

4) The following inequality holds for all nonnegative test functions with support contained in O:

$$\int_{\text{spacetime}} |\nabla u|^2 \phi \leq \frac{1}{2} \int_{\text{spacetime}} \left[|u|^2 (\phi_t + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi \right]. \qquad (7)$$

Note that (4) can be motivated from the Navier-Stokes equations by multiplying by $u\phi$ and integrating by parts.

Definition: Let $x_0 \in \mathbb{R}^3$, and let $z_0 = (x_0, t_0)$. We define $B_{x_0,r}$, $Q_{z_0,r}$ by

$$B_{x_0,r} = \{x \in \mathbb{R}^3 : |x - x_0| < r\}$$

$$Q_{z_0,r} = B_{x_0,r} \times (t_0 - r^2, t_0).$$

Definition: Given a suitable weak solution (u,p) on O, a point $(x,t) \in O$ is called a regular point if (u,p) is Hölder continuous in a neighborhood of (x,t). A singular point is one which is not regular.

The following propositions will be stated without proof.

Proposition 2.1: There exists an $\epsilon > 0$ such that, if (u,p) is a suitable weak solution on O such that

$$\frac{1}{r^2} \int_{Q_{z,r}} |u|^3 + |p|^{3/2} < \epsilon \qquad (8) \quad \square$$

for some $Q_{z,r} \subset\subset O$, then all points in $Q_{z,r/2}$ are regular, and for all points in $Q_{z,r/2}$ we have

$$|\nabla^k u| \leq C_k r^{-1-k} \qquad k = 0, 1, \ldots \tag{9}$$

$$|u(x,t) - u(x,t')| \lesssim |t - t'|^{1/3}. \tag{10}$$

Proposition 2.2: Let (u^k, p^k) be suitable weak solutions on O such that u^k are uniformly bounded in $L^\infty_t L^2_x \bigcap L^2_t \dot{H}^1_x$ on compact subsets of O and p^k are uniformly bounded in $L^{3/2}_t L^{3/2}_x$ on compact subsets of O. Then u^k has a convergent subsequence in $L^3_t L^3_x$ on each compact subset of O. Furthermore if $u^k \to u$ in $L^3_t L^3_x$ on compact subsets of O and $p^k \rightharpoonup p$ in $L^{3/2}_t L^{3/2}_x$ on compact subsets of O, then (u, p) is a suitable weak solution. □

Definition: We define the tensor product $u \otimes v : \mathbb{R}^3 \to\colon \mathbb{R}^3$ of two functions $u, v : \mathbb{R}^3 \to \mathbb{R}^3$ to be the linear map defined by $(u \otimes v)(w) \equiv u\langle v, w\rangle$. In matrix form this is equivalent to writing $(u \otimes v)_{ij} = u_i v_j$.

Lemma 2.3: Let (u^k, p^k) be as proposition 2.2, and let $z^k \in O$ be corresponding singular points such that $z^k \to z \in O$. Then z is a singular point of (u, p).

Proof: (Sketch) We have $-\Delta p^k = \operatorname{div}\operatorname{div}(u^k \otimes u^k)$. By hypothesis the sequence $u^k \otimes u^k$ has a convergent subsequence in $L^{3/2}_t L^{3/2}_x$ on compact subsets of O. Hence we can write $p^k = \tilde{p}^k + h^k$ where $\tilde{p}^k$ has a convergent subsequence in $L^{3/2}_t L^{3/2}_x(Q_{z,r})$ and h^k is bounded in $L^{3/2}_t L^{3/2}_x(Q_{z,r})$ and harmonic in space in $Q_{z,r}$. Without loss of generality assume u^k, p^k converge in the respective norms. Now if z is a regular point of (u, p), then in particular u is bounded near z, and so we have that

$$\frac{1}{r^2} \int_{Q_{z,r}} |u|^3 = O(r^3). \tag{11}$$

Since $u^k \to u$ in $L^3_t L^3_x$ we have that, for r small,

$$\frac{1}{r^2} \int_{Q_{z,r}} |u^k|^3 \tag{12}$$

is small for large enough k, and the same reasoning implies that

$$\frac{1}{r^2}\int_{Q_{z,r}} |\tilde{p}^k|^{3/2} \tag{13}$$

is small for small r and k large enough. Now the h^k term can be handled by standard estimates for harmonic funcitons, ie.

$$|h(x) - \frac{1}{|B_{x_0,r'}|}\int_{B_{x_0,r'}} h\,| \lesssim \left(\frac{r'}{r}\right)^{3/2} r^{-3}\int_{B_{x_0,r}} |h|^{3/2}\,,$$

and using the fact that we can are free to add any function of only t to the pressure. □

Lemma 2.4: With the assumptions of proposition 2.2, if $K \subset O$ is a compact set of regular points of u, then for large enough k, K is also a set of regular points for u^k, and u^k and all its spatial derivative converge to u and all its spatial derivatives, respectively, on K. □

3 Leray Solutions

The local theory tells us that if $T_{\max} < \infty$, then $\lim_{T\uparrow T_{\max}(u_0)} \|u\|_{L_t^4\dot{H}_x^1(\mathbb{R}^3\times(0,\infty))} = \infty$. It will be shown that the only way for this to happen is if u develops a singularity in finite time.

Definition: A suitable weak solution (u,p) (Kohn, Nirenberg; 1982) will be called a Leray solution if $u(t) \xrightarrow{t\downarrow 0} u_0$ in $L^2(K)$ for all compact $K \subset \mathbb{R}^3$.

Theorem 3.1: For a given initial data, the Leray solution coincides with the mild solution for the time of existence of the mild solution. □

It is not known that the Leray solution is unique passed this time of existence for $T_{\max}(u_0) < \infty$.

Proposition 3.2: Let u be a Leray solution of the Navier-Stokes equations with $u_0 \in \dot{H}^{1/2}$. Then for some compact set $K \subset \mathbb{R}^3 \times (0,\infty)$ we have $\nabla u \in L_t^4 L_x^2(\mathbb{R}^3 \times (0,\infty) - K)$. In particular u is regular at every point of $\mathbb{R}^3 \times (0,\infty) - K$. □

Lemma 3.3: Let $u_0 \in \dot{H}^{1/2}$ and let u be a corresponding Leray solution of the Navier-Stokes equations with this initial condition. Then for each $r > 0$ and $x_0 \in \mathbb{R}^3$ we have that

$$\|u\|^2_{L_t^\infty L_x^2(B_{x_0,r} \times (0,r^2))} + \|\nabla\|^2_{L_t^2 L_x^2(B_{x_0,r} \times (0,r^2))} \lesssim r \|u_0\|_{\dot{H}^{1/2}}\,, \tag{14}$$

and for some function $p_{x_0,r}(t)$ we have that

$$\int_{B_{x_0,r} \times (0,r^2)} |p - p_{x_0,r}(t)|^{3/2} \lesssim r^2 \|u_0\|_{\dot{H}^{1/2}}\,. \tag{15}$$ □

Lemma 3.4: Norms on Banach spaces are weakly lower semicontinuous.

Proof: Let $\|\cdot\|$ be a norm on some Banach space X, and suppose $x_n \rightharpoonup x$. For any $\lambda \in \mathbb{R}$ define a functional on the ray though x by $f(\lambda x) \equiv \lambda \|x\|$. By the Hahn-Banach theorem there is an extention of this linear functional to $f : X \to \mathbb{R}$ such that $\|f\| = 1$. Then $\|x\| = f(x) = \lim\limits_{n\to\infty} f(x_n) \le \liminf\limits_{n\to\infty} \|x_n\|$, proving the lemma. □

Lemma 3.5: In a Hilbert space, weak convergence and convergence of norms implies strong convergence.

Proof: Suppose $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then

$$\|x - x_n\|^2 = \|x\|^2 + \|x_n\|^2 - 2\langle x_n, x\rangle \xrightarrow{n\to\infty} \|x\|^2 + \|x\|^2 - 2\langle x, x\rangle = 0\,,$$

proving the theorem. □

Remark : The previous lemma remains true if Hilbert space is replaced with uniformly convex space, in particular if replaced by L^p space for $1 < p < \infty$.

Lemma 3.6: Let u_0^k be a bounded, divergeneless sequence in $\dot{H}^{1/2}$ converging weakly in $H^{1/2}$ to u_0. Let u_k be corresponding Leray solutioins. Assume that u^k converges to u in the sense of distributions. Then u is a Leray solution of the Navier-Stokes equations with initial condition u_0.

Proof: (Sketch) Lemma 3.3, proposition 3.2 theorem 3.1 together imply that we only need to show that $u(t) \xrightarrow{t\to 0} u_0$ in L^2 on compact subsets of $\mathbb{R}^3$. Let $\phi \geq 0$ be a smooth function with compact support in $\mathbb{R}^3 \times [0,\infty)$. We can take a smooth function with compact support in $\mathbb{R}^3 \times (-\epsilon, \infty)$, $\epsilon > 0$ such that $\phi^n \to \phi$ in H^1. From the local energy inequality we then have

$$2\int_{-\epsilon}^{\infty}\int_{\mathbb{R}^3} |\nabla u^k|^2\phi^n \leq \int_{-\epsilon}^{\infty}\int_{\mathbb{R}^3} |u^k|^2(\phi_t^n + \Delta u^k) + (|u^k|^2 + 2p^k)u^k\nabla\phi^n\,.$$

Splitting up the integral on the right we can see that, taking $n \to \infty$, $\epsilon \to 0$ appropriately, we have

$$2\int_{0}^{\infty}\int_{\mathbb{R}^3} |\nabla u^k|^2\phi \leq \int_{\mathbb{R}^3} |u_0^k|\phi(x,0) + \int_{0}^{\infty}\int_{\mathbb{R}^3} |u^k|^2(\phi_t + \Delta u^k) + (|u^k|^2 + 2p^k)u^k\nabla\phi\,.$$

Now sending $k \to \infty$ we get that

$$2\int_{0}^{\infty}\int_{\mathbb{R}^3} |\nabla u|^2\phi \leq \int_{\mathbb{R}^3} |u_0|\phi(x,0) + \int_{0}^{\infty}\int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta u) + (|u|^2 + 2p)u\nabla\phi\,.$$

where p is the corresponding pressure. This is enough to imply the necessary convergence on compact subsets, since the Navier-Stokes equations imply that $u(t) \rightharpoonup u_0$ weakly in L^2 on compact subsets, and the above inequality implies that $\limsup_{t\to 0} \|u(t)\psi\| \leq \|u_0\psi\|$ for all smooth compactly supported ψ. Now by weak lower semicontinuity of norms we have that $\|u_0\phi\| \leq \liminf_{t\to\infty} \|u(t)\psi\|$, hence we have that $\lim_{t\to 0} \|u(t)\psi\| = \|u_0\psi\|$. Now we have weak convergence as well as convergence of norms on compact sets, hence from a previous lemma we have strong convergence. □

The next corollary is implied by lemma 3.3, proposition 3.2 and lemma 3.6, and the one after that by lemma 3.3, proposition 3.2 and lemma 2.4.

Corollary 1: Let u_0^k, u_0, u^k be as in lemma 3.6. Then for any compact set $K \subset \mathbb{R}^3 \times (0, T_{\max}(u_0))$ and for k large enough, depending on K, the solutions are regular at all points K and converge uniformly to u in K, along with all its spatial derivatives. □

Corollary 2: Let u_0^k, u_0, u^k be as in lemma 3.6. Assume $T_{\max}(u_0) < \infty$ for each k and that the singular points z_k of u at $t = T_{\max}(u_0^k)$ stay in a compact subset of $\mathbb{R}^3 \times T_{\max}(u_0^k)$. Then $T_{\max}(u_0) \leq T_{\max}(u_0^k)$. □

4 Proof of Theorem

We will now recall and sketch a proof of the main theorem.

Theorem 4.1: Suppose $\epsilon_{\max} < \infty$, then C is nonempty.

Proof: Let $u_0^k \in \dot{H}^{1/2}$ be a sequence of initial data such that

$$T_{\max}(u_0^k) < \infty, \|u_0^k\|_{\dot{H}^{1/2}} \downarrow \epsilon_{\max},$$

and let u^k be the corresponding Leray solutions. After translating and scaling, we may assume u^k has a singularity at $(0, 1)$ and that $t = 1$ is the first singularity of u^k. After taking a subsequence, we can assume $u_0^k \rightharpoonup u_0^k$ in $\dot{H}^{1/2}$ by Banach-Alaoglu. Hence $T_{\max}(u_0) < \infty$, so that $\|u_0\|_{\dot{H}^{1/2}} \geq \epsilon_{\max}$. But by weak lower semicontinuity, $\|u_0\|_{\dot{H}^{1/2}} \leq \epsilon_{\max}$, hence $\|u_0\|_{\dot{H}^{1/2}} = \epsilon_{\max}$, which means that $M \neq \varnothing$. Also since weak convergence and convergence of norm implies strong convergence, we have $u_0^k \xrightarrow{k \to \infty} u$ □

Note that the proof of the above theorem actually gives that the set C is compact after modding out by scaling and translations.

5 Minimal L^3 Data for Blow Up Solution

For completion we will mention that a similar result holds for the corresponding Navier-Stokes problem but for initial data in L^3 instead of $\dot{H}^{1/2}$. We will briefly describe this case, stating the main definitions, lemmas and theorems.

Definition : We will call $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0,\infty))$ a Leray solution to the Navier-Stokes equations with initial data u_0 if, for all $0 < r < \infty$,

1)

$$\sup_{0\le t<r}\sup_{x_0\in\mathbb{R}^3}\int_{B_r(x_0)}\frac{|u|^2}{2}\,dx + \sup_{x_0\in\mathbb{R}^3}\int_0^r\int_{B_r(x_0)}|\nabla u|^2\,dx\,dt < \infty\,, \tag{16}$$

$$\lim_{|x_0|\to\infty}\int_0^r\int_{B_r(x_0)}|u|^2(x,t)\,dx\,dt = 0\,. \tag{17}$$

2) For some distribution p on $\mathbb{R}^3 \times (0,\infty)$, (u,p) satisfies the Navier-Stokes equations in the sense of distributions and for any compact set $K \subset \mathbb{R}^3$,

$$\lim_{t\to 0^+}\|u(\cdot,t) - u_0\|_{L^2(K)} = 0\,. \tag{18}$$

3) For all test functions $\phi \ge 0$ we have that

$$\int_0^\infty\int_{\mathbb{R}^3}|\nabla u|^2\phi(x,t)\,dx\,dt \le \int_0^\infty\int_{\mathbb{R}^3}\frac{|u|^2}{2}(\partial_t\phi+\Delta\phi+u\cdot\nabla\phi)+pu\cdot\nabla\phi\,dx\,dt\,. \tag{19}$$

We note that (16) is there to ensure uniqueness, and this is a new condition compared with the $H^{1/2}$ case. Also it is a fact that condition (19) allows us to calculate p as so: For all $B_r(x_0)\times(0,t_*) \subset \mathbb{R}^3\times(0,\infty)$ take a smooth cutoff function with $\phi|_{B_{2r}(x_0)} \equiv 1$, there then exists a function $f(t)$ depending only on x_0, r, t, ϕ such that for $(x,t) \in B_r(x_0)\times(0,t_*)$

$$p(x,t) = -\Delta^{-1}\text{div}\,\text{div}(u\otimes u)$$

$$-\int_{\mathbb{R}^3}[k(x-y)-k(x_0-y)]u\otimes u(y,t)(1-\phi(y))\,dy + f(t)\,, \tag{20}$$

where k is the kernel of Δ^{-1}divdiv and $(u\otimes v)(w) \equiv u\langle v,w\rangle$. Another reason for considering Leray solutions is due to the following lemma:

Lemma 5.1: Let u be a Leray solution with divergence free initial data $u_0 \in L^3(\mathbb{R}^3)$. Then There exists a nonnegative function $h(t)$ depending only on $\|u_0\|_{L^3}$, such that $\lim_{t\to 0^+} h(t) = 0$ and

$$\|u(\cdot,t) - e^{\Delta t}u_0\|_{L^2(B_1(x))} \leq h(t)$$

for any $x \in \mathbb{R}^3$ and almost every $0 \leq t < 1$. □

Note that the importance of this lemma will be for compactness; it gives us a kind of local uniform continuity at time zero for Leray solutions with uniformly bounded initial data.

Lemma 5.2: Suppose $(u,p), (v,q)$ are two Leray solutions corresponding to the same initial data and $v \in L^5(\mathbb{R}^3 \times [0,T))$ for any $T < \infty$, then $u \equiv v$ in $\mathbb{R}^3 \times (0,\infty)$. □

Lemma 5.3 (Compactness Lemma): Let (u^k, p^k), $k = 1, 2, \ldots$ be a sequence of suitable weak solutions such that u^k are uniformly bounded in the energy space $L^\infty_t L^2_x \bigcap L^2_t \dot{H}^1_x$ on compact subsets of open sets $O \subset \mathbb{R}^3 \times \mathbb{R}$ and p^k are uniformly bounded $L^{3/2}_t L^{3/2}_x$ on compacts subsets of O. Then the sequence u^k has a convergent subsequence in $L^3_t L^3_x$ on compact subsets of O. Also, if $u^k \to u$ in $L^3_t L^3_x$ on compact subsets O and $p^k \rightharpoonup p$ in $L^{3/2}_t L^{3/2}_x$ on compact subsets of O, then (u,p) is again a suitable weak solution. □

Basically there are many results analogous to the $\dot{H}^{1/2}$ case, but it is a bit more technically difficult.

6 References

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