A semigroup treatment of the Navier-Stokes equation on bounded domains in $\mathbb{R}^3$

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Abstract

In this paper we will address the abstract problem for the Navier-Stokes differential equation

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \Delta u = (u \cdot \nabla) u & t > 0 \\ \nabla \cdot u = 0 \quad \text{and} \quad u(0) = a, \end{cases} \tag{1}$$

where u belongs to the Hilbert space of divergence free functions vanishing up to first order on the boundary $\partial\Omega$ of a bounded domain Ω in $\mathbb{R}^3$. The boundary will be assumed to be of class $\mathscr{C}^3$.

1 Projection onto divergence-free subspace

Let $L^2(\Omega)$ be set of vector-valued square integrable functions on Ω,

$$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)),$$

equipped with $(,)$ L^2 inner-product, makes Hilbert space. Let $\varphi \in \mathscr{C}_\sigma(\Omega)$ be vector-valued continuously differentiable functions with $\operatorname{div} \varphi = 0$ vanishing at up to first order on $\partial\Omega$. Take $\mathcal{H}_\sigma$ to be the L^2-closure of $\mathscr{C}_\sigma(\Omega)$. If $u \in \mathscr{C}^1(\Omega)$ with $\operatorname{div} u = 0$ and $\frac{\partial u}{\partial \nu} = 0$, the vector ν being normal to $\partial\Omega$, then $u \in \mathcal{H}_\sigma$. Moreover, denote

$$\mathcal{M}_\pi = \{u \in L^2(\Omega) : u = \nabla h,\ h \in \mathscr{C}^1(\Omega)\},$$

and take $\mathcal{H}_\pi$ as the L^2-closure of $\mathcal{M}_\pi$. We claim $\mathcal{H}_\sigma = \mathcal{H} \ominus \mathcal{H}_\pi$. To show the first inclusion $\mathcal{H}_\sigma \subset \mathcal{H} \ominus \mathcal{H}_\pi$, we easily see that if $u = \nabla h$ for some scalar-valued smooth function h, we have $\nabla h \equiv 0$ by uniqueness of Neumann problem

$$\begin{cases} \Delta h = 0 & \text{in } \Omega \\ \frac{\partial h}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, for $w \in \mathcal{H} \ominus \mathcal{H}_\pi$ with $w \perp \mathcal{H}_\sigma$. We have

$$(w, \nabla h) = 0 \quad \text{and} \quad (w, \operatorname{curl} \varphi) = 0 \quad \forall h, \varphi \in \mathscr{C}_0^1(\Omega) \tag{2}$$

so partial integration respectively implies that $\operatorname{div} w = 0$ and $\operatorname{curl} w = 0$ weakly. By the first implication we therefore see that $w \in \mathscr{C}^\infty(\Omega)$ by Weyl's Lemma since

$$(\Delta w, \varphi) = -(\nabla w, \nabla \varphi) = 0 \quad \forall \varphi \in \mathscr{C}_0^\infty(\Omega).$$

Moreover, since $\operatorname{curl} w = 0$, w is a conservative vector-field so $w = \nabla h$ for some smooth function h and thence $w \in \mathcal{M}_\pi$ thus proving the orthogonal decomposition.

Let $\mathbb{P}$ be the orthogonal projection from $\mathcal{H} \to \mathcal{H}_\sigma$ and consider the related problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = Fu, \quad t > 0, \\ u(0) = a, \end{cases} \tag{3}$$

where we took $A = \mathbb{P}\Delta$ and $Fu = \mathbb{P}(u \cdot \nabla)u$. In addition, assume that $a \in \mathcal{H}_\sigma$.

Define bilinear form $\tilde{a}(u, v) : [\mathcal{H}_\sigma(\Omega)]^2 \to \mathbb{R}$ given by

$$\tilde{a}(u, v) = (\nabla u, \nabla v), \quad (u, v \in \mathcal{H}_\sigma(\Omega)). \tag{4}$$

The defined form is symmetric, coercive and bounded in $[\mathcal{H}_\sigma(\Omega)]^2$; for simplicity we denote $V = \mathcal{H}_\sigma(\Omega)$. Using $\tilde{a}(\cdot, \cdot)$, we define an operator $\tilde{A}$ on V: for $u \in V$, if there is an element $f \in L^2(\Omega)$ such that $a(u, v) = (f, v)$ for every $v \in V$, then $u \in \mathcal{D}(A)$ and $Au = f$. The bilinear form in (4) is continuous in the V-topology so it it is continuous in the topology induced by $L^2(\Omega)$ since $\|u\|_{L^2} \leq M\|u\|_V$ for a universal M depends solely on the domain Ω; in effect, this is due to Friedrich's Sobolev type inquality. Consequently, we can extend $\tilde{a}(\cdot, \cdot)$ to be defined on $[L^2(\Omega)]^2$ and in turn may extend $\tilde{A}$ similarly in a manner which runs as follows.

Let $\Lambda|_V$ denote the restriction of a member Λ belonging to $(L^2)^*$, the dual of $L^2(\Omega)$ to V. Then for any $v \in V$,

$$|(\Lambda|_V)(v)| = |\Lambda(v)| \leq M\|\Lambda\|_*\|v\|_V, \tag{5}$$

thus making $\Lambda|_V \in V^*$. Here, $V \subset L^2(\Omega)$, densely, so the correspondence $\Lambda \to \Lambda_V$ is an injection and thus we may identify Λ with $\Lambda|_V$ hence concluding the continuous embedding $L^2(\Omega)^* \subset V^*$ since

$$\|\Lambda|_V\|_{V^*} \leq \|\Lambda\|_*,$$

i.e., $(L^2)^*$ has the stronger topology. We have the continuous sequence of topological inclusions

$$V \subset L^2(\Omega) \subset V^*.$$

Moreover, if for $v \in V$ with $(u, v) = 0$ for every $u \in V^*$ making $v = 0$ and that V is densely embedded in V^*. Since V is already a dense subspace of L^2, we infer that the $L^2(\Omega)$ is densely embedded in V^*. Consequently, $f \in L^2(\Omega)$ induces a linear functional expressed by $f(v) = (f, v)$. On the other hand when $u \in V$ fixed, $a(u, \cdot) \in V^*$ and thus we have the following expression for $f \in V^*$

$$a(u, v) = f(v). \tag{6}$$

As a result, we extend $\tilde{A}$ by defining A via

$$a(u, v) = (Au, v) \quad (u, v \in \mathcal{H}_\sigma), \tag{7}$$

and

$$\mathcal{D}(A) = \{u \in \mathcal{H}_\sigma : Au \in L^2(\Omega)\}. \tag{8}$$

In other words, operator A is the Friedrich extension of $\tilde{A}$ and from the definition it follows that

$$(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) = (\nabla u, \nabla v) \quad \forall u, v \in \mathcal{H}_\sigma, \quad \text{and} \quad \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{H}_\sigma. \tag{9}$$

2 Abstract Cauchy problem

2.1 Preliminaries: Hille-Yosida

In most generality, let X be a Banach space and let A be a densely defined linear operator on X from $\mathcal{D}(A) \subset X$ into X. Given $x \in X$, the homogenous abstract Cauchy problem for A with initial data x consists of finding a solution u to the operator equation

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0, \quad t \in [0, \infty), \\ u(0) = x. \end{cases} \tag{10}$$

Here, the sought solution is understood to be a continuous X-valued function $u(t)$ for $t \geq 0$, $u(t) \in \mathcal{D}(A)$ with a continuous derivative for $t > 0$. Evidently, under the continuity requirement on $t \geq 0$, the problem (10) cannot not admit a solution in the sense expressed above whenever $x \notin \overline{\mathcal{D}(A)}$.

The solvability of the Cauchy problem (10) will first be addressed in the Hilbert setting where $X = \mathcal{H}$ is a Hilbert space equipped with inner-product $(\cdot, \cdot)$, A unbounded linear operator sending $\mathcal{D}(A) \to \mathcal{H}$. It is not yet clear that A is densely defined in $\mathcal{H}$. It is sufficient to assume that A enjoyed the properties that $(Av, v) \geq 0$ for every $v \in \mathcal{D}(A)$ and that $\mathrm{Ran}(I + A) = \mathcal{H}$. We will refer to these, respectively, by monotonicity and maximality; as a consequence such operators as said to be *maximal monotone*. As a result, given any $f \in \mathcal{H}$ with $f \perp \mathcal{D}(A)$ we conclude that $f = 0$; this means that the domain $\mathcal{D}(A)$ is dense in the topology of $\mathcal{H}$ induced by the inner-product. Indeed, the monotonicity of A implies that

$$0 = (f, v) = \|v\|^2 + (Av, v) \geq \|v\|^2,$$

for any v belonging to the range $\mathrm{Ran}(I+A)$ whose existence is ensured by the maximality of A. It is noteworthy to remind the reader that for a linear operator A, the *resolvent set* $\rho(A)$ of A is the set of all complex numbers λ for which the operator $I+\lambda A$ is invertible. The family $J_A(\lambda) = (1+\lambda A)^{-1}$ or simply J_λ, for $\lambda \in \rho(A)$, of bounded linear operators is called the *resolvent* of A. Returning to our earlier discussion, we have in addition to the density of $\mathcal{D}(A)$ in X the following:

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space. If A is a linear operator sending $\mathcal{D}(A) \to \mathcal{H}$ is maximal monotone, then A is a closed operator and for every $\lambda > 0$, the resolvent $J_\lambda = (I + \lambda A)^{-1}$ is a contractive bijection from $\mathcal{D}(A)$ onto $\mathcal{H}$.*

Proof. Let $(u_n) \subset \mathcal{H}$ be a sequence such that $u_n \to u$ with $Au_n \to f$. We claim that the limit u belongs to $\mathcal{D}(A)$ and satisfies $Au = f$. First of all, observe that for $f \in H$, the equation $u + Au = f$ admits a solution $u \in \mathcal{H}$ by the assumed maximality whereas uniqueness of u is a consequence of monotonicity; if $u_1 - u_2 + A(u_1 - u_2) = 0$ then

$$0 = \|u_1 - u_2\|^2 + (A(u_1 - u_2), u_1 - u_2) \geq \|u_1 - u_2\|^2.$$

From

$$\|u\|^2 + (Au, u) = (f, u) \geq \|u\|^2,$$

we conclude that $\|u\| \leq \|f\|$ making the map $f \mapsto v$, which we write as $(I + A)^{-1}$, continuous and invertible. Now for every $n \geq 1$ we may write $u_n = (I + A)^{-1}(u_n + Au_n)$ so by continuity

$$\|u_n + (I + A)^{-1}(u + f)\| \leq \|(I + A)^{-1}\|(\|u_n - u\| + \|Au_n + f\|),$$

concluding that $u + Au = u + f$; we have shown $u \in \mathcal{D}(A)$ and $Au = f$ proving that the operator A is closed. We now address desired results on the resolvent, in particular, we will show that for any $\lambda > 0$, the map $(I + \lambda A)$ is a bijection $\mathcal{D}(A) \to \mathcal{H}$ and $\|(I + \lambda A)^{-1}\| \leq 1$. Suppose that $\lambda_0 \in \rho(A)$ and let $\lambda > \frac{\lambda_0}{2}$. We will show that for any given $f \in \mathcal{H}$ the equation $\lambda u + \lambda Au = f$ admits a unique solution in $\mathcal{D}(A)$. Rewriting

$$\begin{aligned} u + \lambda Au = f &\iff \lambda(u + \lambda_0 Au)\lambda_0 = f + \lambda u - \lambda_0 u \\ &\iff u + \lambda_0 Au = \frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) u, \end{aligned}$$

so by defining

$$Tu = (I + \lambda_0 A)^{-1} \left[\frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) u \right],$$

we see that for any $u, v \in \mathcal{H}$,

$$\|Tu - Tv\| \leq \|u - v\| \left| 1 - \frac{\lambda_0}{\lambda} \right|.$$

In particular, the operator T is a contraction whenever λ satisfies $|1 - \frac{\lambda_0}{\lambda}| < 1$. Recalling that maximality of A ensures $\mathrm{Ran}(I + A) = \mathcal{H}$, then $1 \in \rho(A)$ and therefore we inductively have conclude that $\lambda \in \rho(A)$ for $\lambda > \frac{1}{2^n}$ for any $n \geq 0$. $\square$

Recalling that A need not be bounded making desirable analysis difficult, however, using the resolvent one may circumvent this obstacle by defining for A a regularized operator

$$A_\lambda \equiv \frac{1}{\lambda}(I - J_A(\lambda)), \tag{11}$$

which is bounded for every $\lambda > 0$ and inherits many of the crucial information of A. Among the most noteworthy is the fact that $A_\lambda : \mathcal{H} \to \mathrm{Ran}(A)$ with the property that

$$\|A_\lambda v - Av\| \to 0 \quad \text{as } \lambda \to 0 \quad \forall v \in \mathcal{D}(A), \tag{12}$$

Equally as important is that resolvent $J_\lambda : \mathcal{H} \to \mathcal{D}(A)$ acts like an approximation to the identity:

$$\|J_\lambda v - v\| \to 0 \quad \text{as } \lambda \to 0 \quad \forall v \in \mathcal{H}. \tag{13}$$

An intimate relationship between the resolvent and A_λ is revealed by the identities

$$A_\lambda v = A(J_\lambda v) \quad \forall v \in \mathcal{H} \quad \text{and that} \quad A(J_\lambda v) = J_\lambda(Av) \quad \forall v \in \mathcal{D}(A). \tag{14}$$

Note that the family $(A_\lambda)_{\lambda > 0}$ of bounded operators approximating A inherits the monotonicity from A and $\|A_\lambda v\| \leq \|Av\|$ for all $v \in \mathcal{D}(A)$. However, the $\|A_\lambda\|$ need not be uniformly bounded for in general we will show below that $\|A_\lambda\| \leq \frac{1}{\lambda}$. The operator A_λ is said to be a *Yosida approximation* or *regularization of* A. We proceed by proving (14), (13) and (12) respectively.

Proof. We begin by expressing for $v \in \mathcal{H}$

$$\begin{aligned} J_\lambda v + \lambda A(J_\lambda v) = (I + \lambda A) J_\lambda v = v \\ \implies A(J_\lambda v) = \frac{1}{\lambda}(v - J_\lambda v) = \frac{1}{\lambda} A_\lambda v. \end{aligned}$$

Again, if $v \in \mathcal{D}(A)$, then

$$Av = A_\lambda v + \lambda A(A_\lambda v) = (I + \lambda A)A_\lambda v,$$

proving that $A_\lambda v = J_\lambda(Av)$ thus completing proof for (14) . Now let $v \in \mathcal{D}(A)$ and we readily see that

$$\|v - J_\lambda v\| \leq \lambda \|A_\lambda v\| \leq \lambda \|Av\| \to 0 \quad \text{as } \lambda \to 0.$$

We extend this argument to $\mathcal{H}$ by using the density of $\mathcal{D}(A)$. Namely, for $v \in \mathcal{H}$, pick $\tilde{v} \in \mathcal{D}(A)$ for which $\|v - v\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\|J_\lambda v - v\| &\leq \|J_\lambda v - J_\lambda \tilde{v}\| + \|J_\lambda \tilde{v} - \tilde{v}\| + \|\tilde{v} - v\| \\ &\leq \|J_\lambda \tilde{v} - \tilde{v}\| + \varepsilon.\end{aligned}$$

Now using the identity in (14) we see that for $v \in \mathcal{D}(A)$

$$\|A_\lambda v - Av\| = \|J_\lambda(Av) - Av\| \to 0 \quad \text{as } \lambda \to 0,$$

by virtue of (13). Finally by adding and subtracting $J_\lambda v$ from the second argument of $(A_\lambda v, v)$ we use monotonicity of A to see that for any $v \in \mathcal{H}$,

$$(A_\lambda v, v) = \lambda |A_\lambda v|^2 + (A(J_\lambda v), J_\lambda v) \geq \lambda \|A_\lambda v\|^2, \quad (\lambda > 0).$$

Finally, an application of Cauchy-Schwarz on the forgoing relation yields for $\lambda > 0$

$$\|A_\lambda v\|^2 \leq \frac{1}{\lambda} \|A_\lambda v\| \|v\| \quad \forall v \in \mathcal{H}.$$

□

2.2 Homogenous initial value problem

We now have the necessary tools for the treatment of the following problem. For $x \in \mathcal{D}(A)$ we define the graph norm

$$|x|_G^2 = \|x\|^2 + \|Ax\|^2,$$

which induces a Banach topology on $\mathcal{D}(A)$ since A is closed under the aforementioned assumptions. In other words, A has closed graph.

Theorem 2.2. *Let A be a maximal monotone operator. Then, given any $x \in \mathcal{D}(A)$, there exists a unique function*

$$u \in \mathscr{C}^1([0, \infty) : \mathcal{H}) \cap \mathscr{C}([0, \infty) : \mathcal{D}(A))$$

satisfying

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \quad \text{on } [0, \infty) \\ u(0) = x. \end{cases} \tag{15}$$

Moreover,

$$\|u(t)\| \leq \|x\| \quad \text{and} \quad \|Au(t)\| \leq \|Ax\| \quad \forall t \geq 0. \tag{16}$$

Proof. Uniqueness follows from monotonicity. Indeed,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-\tilde{u}(t)\|^2 = -(A(u-\tilde{u}), u-\tilde{u}) \leq 0 \quad \forall t \geq 0$$

and $\|u(0)-\tilde{u}(0)\| = 0$. We turn our attention to the matter of existence. The strategy is to make use of the boundedness of the $(A_\lambda)_{\lambda>0}$ to obtain a family of solutions $(u_\lambda)_{\lambda>0}$ belonging to the class $\mathscr{C}^1([0,\infty),\mathcal{H})$. This is ensured by the boundedness of Yosida approximations A_λ. Following to that we will complete the proof by showing that a limit of u_λ for $\lambda \to 0$ exists and satisfies (15) and (16).

Define a family of related equations

$$\frac{\mathrm{d}u_\lambda(t)}{\mathrm{d}t} + A_\lambda u_\lambda(t) = 0 \quad \text{for } t \geq 0 \quad \text{with} \quad u_\lambda(0) = x \in \mathcal{D}(A). \tag{17}$$

We will derive an analogous statement to (16) for the related family of equations (17) by showing that the functions $t \mapsto |u_\lambda(t)|$ and $t \mapsto |A_\lambda u_\lambda(t)|$ are nonincreasing for $t \geq 0$. Note that for any $w \in \mathscr{C}^1([0,\infty),\mathcal{H})$, the function $|w|^2 \in \mathscr{C}^1([0,\infty),\mathbb{R})$ with derivative $\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 = 2(\frac{\mathrm{d}w}{\mathrm{d}t}, w)$ owing to the symmetry of the inner-product. Applying this to the solutions $(u_\lambda)_{\lambda>0}$, together with the monotonicity of A_λ, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\lambda\|^2 \leq \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}, u_\lambda\right) + (A_\lambda u_\lambda, u_\lambda) = 0, \quad (\lambda > 0).$$

Note that $\frac{\mathrm{d}u_\lambda(t)}{\mathrm{d}t}$ also satisfies (17) so inductively we conclude that $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \in \mathscr{C}^1([0,\infty) : \mathcal{H})$ and consequently

$$\|u_\lambda(t)\| \leq \|x\| \quad \text{and} \quad \|(\mathrm{d}u_\lambda/\mathrm{d}t)(t)\| \leq \|Ax\| \quad \forall t \geq 0,\ \forall \lambda > 0.$$

We will now prove that for every $t \geq 0$, the solutions $u_\lambda(t)$ converges to some limit, denoted by $u(t)$. Given any $\lambda, \mu > 0$ we have from (17)

$$\frac{\mathrm{d}(u_\lambda - u_\mu)(t)}{\mathrm{d}t} + A_\lambda u_\lambda(t) - A_\mu u_\mu(t) = 0, \quad (t > 0).$$

We claim that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\lambda - u_\mu\|^2 \leq 2(\lambda+\mu)\|Au_0\|^2, \quad (\lambda, \mu > 0). \tag{18}$$

Indeed,

$$\begin{aligned}
&(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) \\
&\qquad = (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu) \\
&\qquad = (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) + (A(J_\lambda u - J_\mu u_\mu), J_\lambda u - J_\mu u_\mu) \\
&\qquad \geq (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu).
\end{aligned} \tag{19}$$

Owing to (18) we integrate to obtain

$$\|u_\lambda(t) - u_\mu(t)\| \leq 2\sqrt{(\lambda+\mu)t}\|Ax\|, \quad (t \geq 0). \tag{20}$$

On every finite interval $[0,T]$, the family of solutions $(u_\lambda)_{\lambda>0}$ converge uniformly to a function $u \in \mathscr{C}([0,\infty),\mathcal{H})$. Assuming, in addition, that $x \in \mathcal{D}(A^2)$; it will be shown that $\mathcal{D}(A^2)$ is dense in the topology of $\mathcal{D}(A)$ induced by the graph norm. We claim that $(\mathrm{d}u_\lambda/\mathrm{d}t)(t)$ will converge uniformly on $[0,T]$ as $\lambda \to 0$ to some continuous limit in $\mathcal{H}$ on $[0,T]$ making u continuously differentiable in $\mathcal{H}$ on $[0,T]$. This will be done in the same spirit done for the convergence of $(u_\lambda)_{\lambda>0}$. Set, for $t \geq 0$, $v_\lambda(t) = (\mathrm{d}u_\lambda/\mathrm{d}t)(t)$; the function v_λ satisfies (**??**) and

$$\frac{1}{2}\|v_\lambda - v_\mu\|^2 \leq (\|A_\lambda v_\lambda\| + \|A_\mu v_\mu\|)(\lambda\|A_\lambda v_\lambda\| + \mu\|A - \mu v_\mu\|),$$

by the same procedure made in (19). Following the same logical path we arrive at

$$\|A_\lambda v_\lambda(t)\| \leq \|A_\lambda A_\lambda x\| \quad \text{and} \quad \|A_\mu v_\mu(t)\| \leq \|A_\mu A_\mu x\|.$$

Recall that under the added assumption we have $Ax \in \mathcal{D}(A)$ from which we obtain the chain of relations

$$A_\lambda A_\lambda x = J_\lambda A J_\lambda A x = J_\lambda J_\lambda A A x = J_\lambda^2 A^2 x,$$

which makes

$$\|A_\lambda A_\lambda x\| \leq \|A^2 x\| \quad \text{and} \quad \|A_\mu A_\mu x\| \leq \|A^2 x\|.$$

With these estimates on $\|A_\nu A_\nu x\|$ and $\|A_\nu A_\nu x\|$ for $\nu = \lambda$ and μ, we are led to the familiar estimate

$$\|v_\lambda(t) - v_\mu(t)\| \leq 2\sqrt{(\lambda+\mu)t}\|A^2 x\|, \quad (t \geq 0), \tag{21}$$

proving the claim that $(\mathrm{d}u_\lambda/\mathrm{d}t)_{\lambda>0}$ converges uniformly $[0,T]$. Moreover, this limit is in fact equal to $\mathrm{d}u/\mathrm{d}t$ in $\mathcal{H}$ since $u \in \mathscr{C}^1([0,\infty),\mathcal{H})$. We now proceed by proving that under the assumption that $x \in \mathcal{D}(A^2)$, the limit $u(t)$ of the family of solutions $(u_\lambda)_{\lambda>0}$ satisfies the operator differential equation in (15). By re-writing (17) as

$$\frac{\mathrm{d}u_\lambda(t)}{\mathrm{d}t} + A(J_\lambda u_\lambda(t)) = 0,$$

and observing that $J_\lambda u_\lambda(t) \to u(t)$ as $\lambda \to 0$, we conclude from the fact that A has closed graph that $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$ and satisfies the ordinary differential equations (15) in $\mathcal{H}$. Recall that $u \in \mathscr{C}^1([0,\infty),\mathcal{H})$ so the function $t \mapsto Au(t)$ is continuous from $[0,\infty) \to \mathcal{H}$ making $u \in \mathscr{C}([0,\infty);\mathcal{D}(A))$.

We conclude the proof by turning our attention to the initial condition. This will be proven with the aid of the following result:

> *Let $x \in \mathcal{D}(A)$. Then for every $\varepsilon > 0$ there exists an element $\bar{x} \in \mathcal{D}(A^2)$ such that $|x - \bar{x}|_G < \varepsilon$. In other words, $\mathcal{D}(A)$ is $|\cdot|_G$-dense in $\mathcal{D}(A)$.*

We now have all the necessary ingredients to relate family of problems (17) to original problem (15). Construct using the Lemma above a sequence $(x_n)_{n\geq 1} \in \mathcal{D}(A^2)$ with $x_n \to x$ with respect to the graph topology of A to obtain a sequence of solutions to

$$\frac{\mathrm{d}u_n}{dt} + Au_n = 0 \text{ on } [0,\infty) \quad \text{and} \quad u_n(0) = x_n.$$

We have for all $t \geq 0$, owing to the estimates (16),

$$\begin{aligned} |u_n(t) - u_m(t)|_G &= \|u_n(t) - u_m(t)\| + \left\|\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}u_m(t)}{\mathrm{d}t}\right\| \\ &\leq \|x_n - x_m\| + \|Ax_n - Ax_m\| \to 0 \quad \text{as } n,m \to \infty. \end{aligned}$$

The sequences $(u_n)_{n\geq 1}$ and $(\mathrm{d}u_n/\mathrm{d}t)_{n\geq 1}$ uniformly converges, respectively, to $u(t)$ and $(\mathrm{d}u_n/\mathrm{d}t)(t)$ on $[0,T]$ for every $T > 0$,i.e., convergence is uniform on $[0,\infty)$ making $u \in \mathscr{C}^1([0,\infty),\mathcal{H})$. The fact A is a closed operator makes $u(t) \in \mathcal{D}(A)$ and that $u(t)$ satisfying original problem (15) with $u \in \mathscr{C}([0,\infty),\mathcal{D}(A))$ as well. $\square$

Remark. The requirement $x \in \mathcal{D}(A)$ cannot be relaxed to $x \in \mathcal{H}$ without risking that the obtained solution $u(t)$ would not exhibit sufficient regularity to be a solution in the classical sense. One can circumvent this obstacle by requiring A to be self-adjoint as well. While self-adjoint operators are symmetric in general, the converse fails to hold whenever operators are unbounded.

3 Infinitesimals generator for C_0 semigroups

In view of Theorem 2.2, for each $t \geq 0$, consider the map sending any initial data $x \in \mathcal{D}(A) \mapsto u(t) \in \mathcal{D}(A)$, where $u(t)$ is the solution to (15). This map, denoted by $T_A(t)$, is bounded in $\mathcal{D}(A)$ because $|u(t)|_G \leq |x|_G$, and since $\mathcal{D}(A)$ is dense in the topology of $\mathcal{H}$ induced by the inner-product, $T_A(t) \in \mathcal{L}(\mathcal{H})$ with $\|T_A(t)\| \leq 1$. Moreover by uniqueness,

$$T_A(t+s) = T_A(t) \circ T_A(s) \quad \forall s,t \geq 0,$$

with $T_A(0) = I$ and

$$\lim_{t\to 0} \|T_A(t)x - x\| = 0.$$

The family of bounded operators $\{T_A(t)\}_{t\geq 0}$, depending on a nonnegative parameter t, satisfying the aforementioned properties are said to be a family of *continuous semigroup of contractions*. It is remarkable to see how unbounded operators induces on a family of bounded contractive operators; A needs only to be maximal monotonicity so as to ensure that A has a closed graph. In this section we develop the theory of semigroups for linear operators.

Let X be a Banach space and let $\{T(t)\}_{t\geq 0}$, or simply $T(t)$, be a family of operators. A linear operator A on X defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \tag{22}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} =: D^+T(0). \tag{23}$$

is called the *infinitesimal generator* of the semigroup $T_A(t)$ with $\mathcal{D}(A)$ being the domain of linear operator A. For notational simplicity, we will drop the subscript A from $T_A(t)$ wherever there lies no confusion. We say a semigroup $T(t)$ of bounded linear operators is *strongly continuous* if

$$\lim_{t\to 0} T(t)x = x \quad \forall x \in X. \tag{24}$$

Strongly continuous semigroups are said to be a *semigroup of class* C_0, or simply C_0 *semigroup*; this type of semigroup will be central in our exposition, and for this, we will derive some of the important properties C_0 semigroups enjoy. Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. The following properties hold.

Lemma 3.1. *There exists $\alpha \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\alpha t}$ for $t \geq 0$.*

Proof. Suppose that $\|T(t)\|$ is unbounded on $0 \leq t \leq T$. Then for some $t_n \geq 0$ with $t_n \to 0$ we have $\|T(t)\| > n$. But for every $x \in X$, strong continuity of $T(t)$ implies that $T(t)x \to x$ as $t \to 0$ in X so by Uniform Boundeness property, $\|T(t)\| < \infty$ uniformly in t which is contrary to unboundedness assumption made above. Let now $M > 0$ be a uniform bound for $T(t)$ and without loss of generality, let $\|T(0)\| = 1$, then $M \geq 1$. Let $\alpha = \frac{1}{T} \log M$. Given any $t \geq 0$, write $t = nT + \delta$ for some $\delta \in [0, T)$ and $n \in \mathbb{N}$. Then by the semigroup property we are led to the estimate

$$\|T(t)\| = \|T(\delta)T(t)^n\| \leq M^{n+1} \leq MM^{t/T} = Me^{\alpha t}.$$

□

Remark. If $\alpha = 0$, the semigroup $T(t)$ is uniformly bounded and if moreover $M = 1$ then we say that semigroup $T(t)$ is a C_0 *semigroup of contractions*. This will be addressed in more detail the subsequent section.

Lemma 3.2. *For every $x \in X$, $t \mapsto T(t)x$ belongs to $\mathscr{C}([0, \infty), X)$.*

Proof. Claim readily follows from strong continuity and Lemma 3.1; for any $h > 0$

$$\|T(t+h)x - T(t)x\| \leq \|T(t)\|\|T(h)x - x\| \leq Me^{\alpha t}\|T(h)x - x\|.$$

Now send $h \to 0$. □

Lemma 3.3. *For $x \in X$,*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x \, \mathrm{d}s = T(t)x. \tag{25}$$

Proof. Let $x \in X$ and $h > 0$. By strong continuity of $T(t)$ in $t \geq 0$, pick $\delta > 0$ for which $\|T(s)x - T(t)x\| < \varepsilon$ whenever $|s - t| < \delta$ so that whenever h satisfies $0 < h < \delta$ we have

$$\left\| \frac{1}{h} \int_t^{t+h} T(s)x - T(t)x \, \mathrm{d}s \right\| \leq \frac{1}{h} \int_t^{t+h} \|T(s)x - T(t)x\| \, \mathrm{d}s < \frac{\varepsilon}{h} h.$$

□

Lemma 3.4. *For $x \in X$,*

$$\int_0^t T(s)x \, \mathrm{d}s \in \mathcal{D}(A) \quad \textit{and} \quad A\left(\int_0^t T(s)x \, \mathrm{d}s \right) = T(t)x - x. \tag{26}$$

Proof. Let $x \in X$. For $h > 0$, we have by semigroup property

$$\frac{T(h) - I}{h} \int_0^t T(s)x \, \mathrm{d}s = \frac{1}{h} \int_t^{t+h} T(s)x \, \mathrm{d}s - \frac{1}{h} \int_0^h T(s)x \, \mathrm{d}s.$$

Now apply Lemma 3.3 to the equality above. □

Lemma 3.5. *For $x \in \mathcal{D}(A)$, $T(t)x \in \mathcal{D}(A)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = AT(t)x = T(t)Ax. \tag{27}$$

In other words, $T(t)x$ is differentiable and A commutes with $T(t)$.

Proof. For $x \in X$,

$$D^+ T(t)x = \lim_{h \to 0} \frac{T(t+h)x - T(t)x}{h} = T(t)\left(\frac{T(h) - I}{h}\right)x = T(t)Ax.$$

Moreover, applying the same argument to $T(h) - T(t-h)$ yields the limit $AT(t)x$ and thus $T(t)x \in \mathcal{D}(A)$.

$$\begin{aligned}\lim_{h \to 0} &\left[\frac{T(t) - T(t-h)}{h} - T(t)Ax\right] \\ &= \lim_{h \to 0^+} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h \to 0^+} (T(t-h)Ax - T(t)Ax).\end{aligned}$$

The first term goes to zero since $\|T(t-h)\|$ is bounded uniformly on $0 \leq h \leq t$ meanwhile the second term vanishes by strong continuity of $T(t)$. □

Lemma 3.6. *For $x \in \mathcal{D}(A)$,*

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax \,\mathrm{d}\tau = \int_s^t AT(\tau)x \,\mathrm{d}\tau. \tag{28}$$

Proof. In view of (27) in Lemma 3.5, we integrate from s to t. □

In the context of the previous section, it was revealed that the density of the domain $\mathcal{D}(A)$ in the X, where X was taken to be a Hilbert space, is essential in ensuring that a solution $u(t)$ to the homogenous initial value problem admits sufficient regularity. Moreover, it was also revealed that part of the sufficient condition for solvability is the closedness of A. A sufficient condition for an unbounded operator in a Hilbert space to admit both properties is for the operator to be maximal monotone, so for us to extend the theory to a general Banach setting, it would not be natural to believe that a dense domain and closedness of the operator would also play an important role in solvability. We have the following result which is in effect a corollary of the properties above.

Theorem 3.7. *If A is an infinitesimal generator of a C_0 semigroup $T(t)$ then $\mathcal{D}(A)$, the domain of A, is dense in X and A is a closed linear operator.*

Proof. Let $x \in X$ and set $x_t = \frac{1}{t}\int_0^t T(s)x \,\mathrm{d}s$. In view of Lemma 3.4 we have $x_t \in \mathcal{D}(A)$ for any $t > 0$ and by Lemma 3.3, x_t converges in X to x as t tends to 0 from above; the closure of $\mathcal{D}(A)$ therefore includes X. The operator A is linear due to its definition in (23). To prove closedness let $x_n \in \mathcal{D}(A)$ with $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From (28) in Lemma 3.6 we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n \,\mathrm{d}s.$$

The integrand $T(s)Ax_n$ converges uniformly on bounded intervals to $T(s)y$, so upon diving by t on both sides of the above equality, sending $t \to 0^+$ we see from Lemma 3.3 that $x \in \mathcal{D}(A)$ and $Ax = y$. □

We address the uniqueness of semigroups.

Theorem 3.8. *Let $T(t)$ and $S(t)$ be C_0 semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then $T(t) = S(t)$.*

Proof. For $x \in \mathcal{D}(A) = \mathcal{D}(B)$, the map $s \mapsto T(t-s)S(t)$ is differentiable by property and

$$\frac{\mathrm{d}}{\mathrm{d}s}T(t-s)S(t) = T(t-s)(B-A)S(s)s = 0$$

so in particular for $s = 0$ and $s = t$ we conclude $T(t)x = S(t)x$ so $T(t) = S(t)$ for all $x \in X$ by density property. □

3.1 The Hille-Yosida Theorem

In this section we will prove a key result by Hille-Yosida on the characterization of infinitesimal generators of C_0 semigroups of contractions. This result will be in the spirit of Theorem 2.1 generalized to the Banach setting.

Theorem 3.9 (Hille-Yosida)**.** *A linear (unbounded) operator A in a Banach space X is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$, $t \geq 0$ if and only if*

(i) A is closed and $\overline{\mathcal{D}(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$

$$\|R_A(\lambda)\| \leq \frac{1}{\lambda}.$$

Remark. We will address the parallelism between this theorem and it's analog discussed previously in the Hilbert setting. Let $L(\lambda) = I + \lambda A$ and let $S(\mu) = \mu I - A$. By a scaling argument we can see that

$$-\lambda S(-\tfrac{1}{\lambda}) = -\lambda(-\tfrac{1}{\lambda}I - A) = L(\lambda).$$

Meaning, if $\lambda \in \rho(A)$, then $\mu = -\frac{1}{\lambda} \in \rho(A)$. In other words, if $\lambda \in \rho(A)$ with $\lambda > 0$, then $S(\mu)$ is invertible for $\mu < 0$ and the operator A considered in Theorem 3.9 A is *dissipative*. An operator A in a Hilbert space is dissipative whenever $-A$ is monotone or *accretive*. While noticing that $J_A(\lambda) = [L(\lambda)]^{-1}$, we have

$$J_A(\lambda) = -\tfrac{1}{\lambda}[S(-\tfrac{1}{\lambda})]^{-1} = \mu R_A(\mu), \quad \lambda = -\tfrac{1}{\mu} \in \rho(A).$$

Simply put, A in the context of Theorem 3.9, with X being a Hilbert space, plays the role of $-A$ in Theorem 2.1.

Proof. We will prove that conditions (i) and (ii) are necessary for contractive C_0 semigroups; sufficiency will require some additional lemmas. The density of $\mathcal{D}(A)$ in X follows immediately from Theorem 3.9. To prove necessity of (ii), define for $x \in X$ and $\lambda > 0$

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\,\mathrm{d}t,$$

Recall that $t \mapsto T(t)x$ is continuous and uniformly bounded so the integral exists so $R(\lambda)$ is a bounded linear operator on X with

$$\|R(\lambda)x\| \leq \int_0^\infty e^{-\lambda t}\|T(t)x\| \, \mathrm{d}t \leq \frac{1}{\lambda}\|x\|.$$

For $h > 0$

$$\left(\frac{T(h)x - x}{h}\right)R(\lambda)x = \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t}T(t)x \, \mathrm{d}t - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}T(t)x \, \mathrm{d}t,$$

yielding $AR(\lambda)x = \lambda R(\lambda)x - x$ under the limit $h \to 0$. So for all $x \in X$, $R(\lambda) \in \mathcal{D}(A)$ and

$$(\lambda I - A)R(\lambda) = I.$$

Consequently, for $x \in \mathcal{D}(A)$, $R(\lambda)$ commutes with A. Indeed,

$$R(\lambda)Ax = \int_0^\infty e^{-\lambda t}T(t)Ax \, \mathrm{d}t = \int_0^\infty e^{-\lambda t}AT(t)x \, \mathrm{d}t = A\left(\int_0^\infty e^{-\lambda t}T(t)x \, \mathrm{d}t\right) = AR(\lambda x),$$

where we used the fact that A and $T(t)$ commute and that A is closed. We have for any $\lambda > 0$

$$R(\lambda)(\lambda I - A)x = x \quad \forall x \in \mathcal{D}(A),$$

thus making $R(\lambda)$ the inverse of $\lambda I - A$ for all positive λ. $\square$

Remark. Notice by this theorem we my express the resolvent of the operator A characterized in Theorem 3.9 as

$$R_A(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x \, \mathrm{d}t, \quad x \in \mathcal{D}(A). \tag{29}$$

In order to address the sufficiency of conditions (i) and (ii) to characterize contractive C_0 semigroups, recall the Yosida approximation: For $\lambda = -\frac{1}{\mu} > 0$, the Yosida approximation (11) for dissipative operator $-A$ is given by

$$A_\mu = \frac{1}{\mu}(I - J_A(\mu)) = -\lambda(I - \lambda R_A(\lambda)) = \lambda^2 R_A(\lambda) - \lambda I \equiv A_\lambda. \tag{30}$$

With this definition, it follows from Theorem 3.9 that if A is dissipative, we have for any $x \in \mathcal{D}(A)$

$$\begin{aligned}\|\lambda R_A(\lambda)x - x\| &= \|AR_A(\lambda)x\| \\ &= \|R_A(\lambda)Ax\| \leq \frac{1}{\lambda}\|Ax\| \to 0 \quad \text{as } \lambda \to \infty.\end{aligned}$$

Consequently, for any $x \in \mathcal{D}(A)$, we have

$$\|A_\lambda x - Ax\| \to 0 \quad \text{as } \lambda \to \infty, \tag{31}$$

which mirrors the behaviour described in (12). A Yosida approximation A_λ of a perhaps unbounded operator A is bounded. We will use this boundedness of A_λ to obtain an explicit representation for $T(t)x$ for $x \in \mathcal{D}(A)$ and $T(t)$ being A's semigroup. In order to do this, we will first claim that semigroups generated by bounded functions are *uniformly continuous* in the sense that

$$\lim_{s\to t} \|T(s) - T(t)\| = 0.$$

Notice that this is a stronger property than of strong continuity property in (24). By defining, for a bounded linear operator B,

$$e^{tB} \equiv \sum_{n=0}^{\infty} \frac{(tB)^n}{n!}, \quad (t \geq 0). \tag{32}$$

The operator e^{tB} converges uniformly in X for every $t \geq 0$ and satisfies semigroup properties. Moreover, by the Taylor approximation,

$$\|e^{tB} - I\| \leq t\|B\|e^{t\|B\|} \to 0 \quad \text{as } t \to 0,$$

and therefore

$$\left\|\frac{e^{tB} - I}{t} - B\right\| \leq \|B\|\|e^{tB} - I\| \to 0 \quad \text{as} \quad t \to 0.$$

thus $T(t) \equiv e^{tB}$ is a uniformly bounded semigroup with B as its infinitesimal generator. By letting $B = A_\lambda$ in (32), A_λ being the Yosida approximations for an unbounded infinitesimal generator A, we realize the following:

Lemma 3.10. *Let A satisfy the hypothesis of Theorem 3.9. If A_λ is the Yosida approximation of A, then A_λ is the infinitesimal generator of a uniformly continuous semigroup of contractions e^{tA_λ}.*

Furthermore, for every $x \in X$, $\lambda, \mu > 0$, we have

$$\|e^{tA_\lambda}\| = e^{-t\lambda}\|e^{t\lambda^2 R_A(\lambda)}\| \leq e^{-t\lambda}e^{t\lambda^2\|R_A(\lambda)\|} \leq e^{-t\lambda}e^{t\lambda I} = 1,$$

and

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(e^{tsA_\lambda}e^{t(1-s)A_\mu}x)\,\mathrm{d}s = \int_0^1 te^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)\,\mathrm{d}s,$$

so we deduce that

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq \int_0^1 t\|e^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)\|\,\mathrm{d}s \leq t\|A_\lambda x - A_\mu x\|. \tag{33}$$

We will now use the preceding results to complete the proof of Theorem 3.9; in particular, we prove sufficiency of condition (i) and (ii) in characterizing infinitesimal generators of strongly continuous semigroups of contractions. In view of (31), we define for $x \in \mathcal{D}(A)$, $x_\lambda = e^{tA_\lambda}x$ and conclude from (33) that $(x_\lambda)_{\lambda>0}$ converges in X as $\lambda \to \infty$ so we may write

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x, \tag{34}$$

for every $x \in \mathcal{D}(A)$ which by density of the range extends to every $x \in X$. The defined $T(t)$ in (34) satisfies the semigroup properties and $T(0) = I$ with $\|T(t)\| \leq 1$ since $\|e^{tA_\lambda}\| \leq 1$ uniformly. Moreover, $t \mapsto T(t)x$ is continuous since the limit in (34) is uniform on bounded intervals (in the t variable). Consequently, $T(t)$ is indeed a strongly continuous semigroup of contractions on X. We will now conclude by the assertion that A is the infinitesimal generator for $T(t)$. For $x \in \mathcal{D}(A)$, viewing implication (34) in light of (28) we deduce that

$$T(t)x - x = \lim_{\lambda\to\infty}(e^{tA_\lambda}x - x) = \lim_{\lambda\to\infty}\int_0^t e^{sA_\lambda}A_\lambda x\,\mathrm{d}s = \int_0^t T(s)Ax\,\mathrm{d}s,$$

with uniform convergence of $e^{tA_\lambda}A_\lambda x$ to $T(t)Ax$ justifies the interchanging of limits. Let B be the infinitesimal generator of $T(t)$ and let $x \in \mathcal{D}(A)$. Using the reasoning made in Lemma 3.5 and an application of Lemma 3.3 we arrive at $Bx = Ax$ for all $x \in \mathcal{D}(B)$; $\mathcal{D}(B) \supseteq \mathcal{D}(A)$. Recall that $1 \in \rho(B)$ necessarily and that $1 \in \rho(A)$ holding by assumption.

$$(I - B)\mathcal{D}(A) = (I - A)\mathcal{D}(A) = X \implies \mathcal{D}(B) = (I - B)^{-1}X = D(A).$$

In other words, $A = B$.

Corollary 3.11. *Let A be the infinitesimal generator of a C_0 semigroup of contractions $T(t)$. If A_λ is the Yosida approximation of A, then*

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda x} \quad \forall x \in X.$$

Remark. An interpretation of this corollary is that $T(t)$ in a sense is "equal" to e^{tA}.

We infer an equivalence between maximal monotone operators and infinitesimal generators of strongly continuous semigroup of contractions. In light of Theorem 3.9 we observe that for a Hilbert space $X = \mathcal{H}$ a densely defined linear operator $A : \mathcal{D}(A) \to \mathcal{H}$ which is maximal monotone has a closed range with a contractive and bijective resolvent sending $\mathcal{D}(A)$ to $\mathcal{H}$. It is revealed by Lumer-Philips that maximal monotone operators form continuous semigroups of contractions. In particular, if A is dissipative then there exists $\lambda_0 > 0$ such that $\mathrm{Ran}(\lambda_0 I - A) = X$ implying that A is an infinitesimal generator of a C_0 semigroup of contractions.

We conclude this section by the following example. Let X be the space of uniformly continuous bounded functions on $\mathbb{R}$. Let

$$(T(t)f)(x) \equiv f(x + t), \quad (x \in \mathbb{R},\ t \geq 0).$$

Clearly, the operator $T(t)$ is a C_0 semigroup of contractions on X and in view of (22) and (23), the infinitesimal generator for $T(t)$ is given by $Af = f'$ with domain

$$\mathcal{D}(A) = \{f : f \in X,\ f' \text{ exists and } f' \in X\}.$$

For this semigroup we have

$$\big(A_h f\big)(x) = \frac{f(x + h) - f(x)}{h} \equiv (\Delta_h f)(x),$$

from which we can write

$$(A_h^k f)(x) = \frac{1}{h^k}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m} f(x + mh) \equiv (\Delta_h^k f)(x).$$

In view of the following result. We have on bounded intervals of t:

$$\forall x \in X, \quad A_h x = \frac{T(h)x - x}{h} \implies T(t)x = \lim_{h\to 0^+} e^{tA_h}x \quad \text{uniformly},$$

a generalized Taylor's formula for uniformly continuous and bounded functions f

$$f(x+t) = \lim_{h\to 0^+} \sum_{k=0}^{\infty} \frac{t^k}{k!} (\Delta_h^k f)(x). \tag{35}$$

3.2 Inhomogeneous abstract Cauchy problem

In this section we extend the results presented in the previous section to the inhomogeneous setting in Banach spaces. Here, we consider the initial value problem for $x \in \mathcal{D}(A)$ given by

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t), \quad t > 0 \\ u(0) = x \end{cases} \tag{36}$$

where $f : [0, T) \to X$ and, throughout this section, we assume that $-A$ is infinitesimal generator of C_0 semigroup $T(t)$. The most natural notion of solvability for (36) is characterized by the following:

Definition 1. *We say that $u : [0, T) \to X$ is a classical operator solution of the operator equation (36) if $u(t) \in X$ satisfies the inhomogeneous problem on $[0, T)$,*

$$u \in \mathscr{C}^1(0, T : \mathcal{D}(A)) \cap \mathscr{C}([0, T) : X),$$

and $u(t) \in \mathcal{D}(A)$ for $(0, T)$.

We ensure uniqueness of solvability by the requirement that $f \in L^1(0, T; X)$. Indeed, if we set $v(s) = T(t-s)u(s)$, u being a solution to (36),

$$\frac{\mathrm{d}v}{\mathrm{d}s} = AT(t-s)u(s) + T(t-s)u'(s) = T(t-s)f(s),$$

so by (28),

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s. \tag{37}$$

In other words, if $f \in L^1(0, T; X)$ then any solution $u(t)$ of (36) will be given by (37). For example, setting $f \equiv 0$ in (36) we recover (15) and from (28) and infer that the unique solution is given by $u(t) = T(t)x$ and this is true for every initial value $x \in \mathcal{D}(A)$. It is important to note that $f \in L^1(0, T; X)$ only ensures that the integral expression (37) satisfies the initial value problem *formally*, that is, the representation (37) does not necessarily admit the necessary regularity to be an operator solution in the classical sense. This motivates to define a weaker notion for solvability.

Definition 2. *Let $-A$ be the infinitesimal generator of a C_0 semigroup $T(t)$. let $x \in X$ and $f \in L^1(0, T : X)$. The function $u \in C([0, T] : X)$ given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s, \quad 0 \le t \le T, \tag{38}$$

is the mild solution *of the initial value problem (36) on $[0, T]$.*

Naturally, we are interested in imposing further conditions on $f \in L^1(0,T;X)$ so that for $x \in \mathcal{D}(A)$, the mild solution given by (38) becomes a classical operator solution.

Theorem 3.12. *Assume that $f \in L^1(0,T:X)$ be continuous on $[0,T)$ and let*

$$v(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s, \quad 0 \le t \le T.$$

Problem (36) has a classical operator solution $u(t)$ on $[0,T)$ for all $x \in \mathcal{D}(A)$ if one of the following is satisfied:

(i) If $v(t)$ belong to class $\mathscr{C}^1(0,T;X)$, or

(ii) if $v(t) \in \mathcal{D}(A)$ for $0 < t < T$ and $Av(t) \in \mathscr{C}(0,T:X)$.

Conversely, if problem (36) admits a classical operator solution on $[0,T)$ for some $x \in \mathcal{D}(A)$ then v satisfies (i) and (ii) necessarily.

Proof. Suppose that u is a solution to (36). Then

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s \equiv T(t)x + v(t).$$

The solution $u(t)$ is differentiable by assumption and $T(t)x \in \mathcal{D}(A)$ is differentiable in X since $x \in \mathcal{D}(A)$ and $v'(t) = u'(t) + AT(t)x$ is a continuous on $(0,T)$ since u is a solution and we have proved condition (i) holds. On the other hand, $v(t) \in \mathcal{D}(A)$ for $t > 0$ since $u(t) \in \mathcal{D}(A)$ for $t > 0$, and

$$Av(t) = Au(t) - AT(t)x = -u'(t) + f(t) - AT(t)x, \tag{39}$$

where all terms are continuous for $t \ge 0$ thus proving (ii). Conversely, express

$$\begin{aligned}\frac{v(t+h)-v(t)}{h} &= \frac{1}{h}\int_0^{t+h} T(t+h-s)f(s)\,\mathrm{d}s - \frac{1}{h}\int_0^t T(t-s)f(s)\,\mathrm{d}s \\ &= \frac{T(h)-I}{h}\int_0^t T(t-s)f(s)\,\mathrm{d}s + \int_t^{t+h} T(t+h-s)f(s)\,\mathrm{d}s.\end{aligned}$$

Look at the identity for $h > 0$

$$\frac{T(h)-I}{h}v(t) = \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)\,\mathrm{d}s. \tag{40}$$

If we assume (i), then $\frac{v(t+h)-v(t)}{h} \to v'(t) \in \mathscr{C}(0,T;X)$ and $\frac{1}{h}\int_t^{t+h} \cdot\,\mathrm{d}s \to f(t)$ on $(0,T)$ which makes $\lim_{h\to 0} \frac{T(h)-I}{h}v(t)$ exist and therefore $v(t) \in \mathcal{D}(A)$ on $(0,T)$ by definition; thence we have the equality $-Av(t) = v'(t) - f(t)$. By setting $u(t) = T(t)x + v(t)$ we see that $u(0) = x$ since $v(0) = 0$ and that $u'(t) = -AT(t)x + v'(t)$ is a sum of continuous functions on $(0,T)$ and by equalities concerning $Av(t)$ we see from

$$u'(t) = -AT(t)x + v'(t) = -AT(t)x - Av(t) + f(t) = -A(T(t)x + v(t)) + f(t),$$

that $u(t)$ given above solves the initial value problem for $x \in \mathcal{D}(A)$. On the other hand if $v(t) \in \mathcal{D}(A)$, $\lim_{h\to 0^+} \frac{T(h)-I}{h}v(t)$ exists so (40) tends to $-Av(t) = D^+v(t) - f(t)$. The righthand derivative $D^+v(t)$ is continuous by assumption (ii) so $D^+v(t) \equiv v'(t)$ and the result follows from the same reasoning made earlier. $\square$

We will now consider another notion of solution of the initial value problem (**??**), namely the *strong* solution.

Definition 3. *A function u which is differentiable almost everywhere on $[0,T]$ such that $u' \in L^1(0,T:X)$ is called a strong solution of the initial value problem (36) if $u(0) = x$ and*

$$u'(t) + Au(t) = f(t) \text{ a.e}, \quad (t \in [0,T]). \tag{41}$$

In a manner similar to proving Theorem 3.12 we arrive to an existence theorem in the sense characterized in Definition 3 by simply replacing the criteria (i) and (ii) in Theorem (3.12) to

(i) If $v(t)$ is differentiable a.e on $[0,T]$ and $v'(t) \in L^1(0,T;X)$, or

(ii) if $v(t) \in \mathcal{D}(A)$ on $[0,T]$ and $Av(t) \in L^1(0,T:X)$.

Let now f be Lipschitz continuous on $[0,T]$ with Banach space X be reflexive. Appealing to Lebsegue's theorem, we see that f is differentiable almost everywhere. and $f' \in L^1(0,T;X)$ so by looking at the difference

$$v(t+h) - v(t) = \int_0^{t+h} T(s)[f(t+h-s) - f(t-s)]\,\mathrm{d}s,$$

it is clear that that $v'(t)$ exists a.e on $[0,T]$ and that

$$\|v'(t)\|_{L^1} \leq \int_0^t \|T(t)f(0)\| + \int_0^T \int_0^t \|T(t-s)f'(s)\|\,\mathrm{d}s\mathrm{d}t < \infty.$$

In other words, we have the following corollary.

Corollary 3.13. *Let X be a reflexive Banach space and let $-A$ be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If f is Lipschitz continuous on $[0,T]$ then for every $x \in \mathcal{D}(A)$ the initial value problem (36) has a unique strong solution u on $[0,T]$ given by (38).*

3.3 Extension to semilinear equations

We complete the first part of our exposition by addressing semilinear equations with C_0 semigroups. Consider the initial value problem

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t,u(t)), & t > 0 \\ u(0) = x \end{cases} \tag{42}$$

where the operator $-A$ is an infinitesimal generator of a C_0 semigroup on a Banach space X. Analogous to the reasoning above, we know if u solves (42) then it satisfies the integral equations

$$u(t) = T(t)a + \int_0^t T(t-s)f(s,u(s))\,\mathrm{d}s. \tag{43}$$

Definition 4. *A continuous function $t \mapsto u(t) \in X$ on $[0,T]$ satisfying the integral equation (43) will be called a mild solution of the initial value problem (42).*

We conclude this exposition by saying a sufficient condition when a mild solution is in fact a strong one.

Theorem 3.14. *Let $-A$ be the infinitesimal genitor of a C_0 semigroup $T(t)$ on a reflexive Banach space X. If $f : [0,T] \times X \to X$ is Lipschitz continuous in both variables, $x \in \mathcal{D}(A)$ and u is the mild solution of the initial value problem (42), then u is the strong solution.*

Proof. Assume $\|T(t)\| \leq M$; note that $[0,T]$ is compact. Assume that $\|f(t,u(t))\| \leq N$ for $0 \leq t \leq T$; this can be due to continuity, and let C be associated Lipschitz constant;

$$\|f(s,u) - f(t,v)\| \leq (C|s-t| + \|u-v\|), \quad 0 < s,t < T, \ u,v \in X.$$

Let $t > 0$ be given. For $h > 0$ with $h < t$, appealing to the (43) we express

$$\begin{aligned} u(t+h) - u(t) = T(t+h)x - T(t)x + \int_0^h T(t+h-s) f(s,u(s))\,\mathrm{d}s \\ + \int_0^t T(t-s)[f(s+h,u(s+h)) - f(s,u(s))]\,\mathrm{d}s, \end{aligned}$$

note that $\|T(h)x - x\| \leq h\|Ax\|$ by definition of A so we arrive at

$$\begin{aligned} \|u(t+h) - u(t)\| &\leq hM\|Ax\| + hMN + MC \int_0^t (h + \|u(s+h) - u(s)\|)\,\mathrm{d}s \\ &\leq \mathcal{C}_\Omega h + MC \int_0^t \|u(s+h) - u(s)\|\,\mathrm{d}s, \end{aligned}$$

Finally, by virtue of Gronwall we obtain

$$\|u(t+h) - u(t)\| \leq \mathcal{C}_\Omega e^{TMC} h.$$

The vector-valued function u is Lipschitz continuous in t. It follows that $t \mapsto f(t,u(t))$ is also locally Lipschitz continuous on $[0,T]$ since for any $s,t < T$,

$$\|f(s,u(s)) - f(t,u(t))\| \leq (C + \mathcal{C}_\Omega e^{TMC})|s-t|, \quad |s-t| < h.$$

By appealing to Corollary 3.13, the initial value problem

$$\begin{cases} \frac{\mathrm{d}v(t)}{\mathrm{d}t} + Av(t) = f(t,u(t)) \quad t > 0 \\ v(0) = x, \end{cases} \tag{44}$$

has a unique strong solution v on $[0,T]$ satisfying

$$v(t) = T(t)x + \int_0^t T(t-s) f(s,u(s))\,\mathrm{d}s \equiv u(t).$$

□

4 Navier-Stokes on bounded domains

5 Formulation

We have the abstract initial value problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = -Au + Fu, \quad t > 0, \\ u(0) = a, \end{cases} \tag{45}$$

where $Fu = -\mathbb{P}(u\cdot\nabla)u$, where $\mathbb{P}$ denotes the projector defined above. Here, the map $t \mapsto u(t)$ is regarded as a function on intervals in $\mathbb{R}_+$ into $\mathcal{H}_\sigma$. The derivative $\mathrm{d}u/\mathrm{d}t$ is understood to be the derivative of u in the strong topology of $\mathcal{H}_\sigma$. We are interested in solutions belonging to a particular class defined as follows:

Definition 5. *Let I be a closed interval $[0,T]$ or a semiclosed interval $[0,T)$. By $S(I)$ we denote the class of all functions $u \in \mathscr{C}(I : \mathcal{H}_\sigma)$ such that $A^{\frac{1}{2}}u \in \mathscr{C}(I \sim \{0\} : \mathcal{H}_\sigma)$ with $\|A^{\frac{1}{2}}u(t)\| = o(t^{-\frac{1}{4}})$ for $t \to 0$ and $A^{\frac{3}{4}}u \in \mathscr{C}(I \sim \{0\} : \mathcal{H}_\sigma)$ with $\|A^{\frac{3}{4}}u\| = o(t^{-\frac{1}{2}})$ for $t \to 0$.*

Remark. A particular $K(t)$ can be chosen to be $\sup_{0<s\le t} s^{\frac{1}{4}}\|A^{\frac{1}{2}}u(s)\|$.

We seek a solution $u \in S[0,T]$ to the integral equation

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}Fu(s)\,\mathrm{d}s. \tag{46}$$

6 lemmas

Lemma 6.1. *If $u \in \mathcal{D}(A^{\frac{3}{4}})$, then for an absolute constant $\mathcal{C}_\Omega$ depending on Ω we have*

$$\|Fu\| \le \mathcal{C}_\Omega \|A^{\frac{1}{2}}u\|\|A^{\frac{3}{4}}u\|. \tag{47}$$

Moreover, if $v \in \mathcal{D}(A^{\frac{3}{4}})$

$$\|Fu - Fv\| \le \mathcal{C}_\Omega(\|A^{\frac{3}{4}}u\|\|A^{\frac{1}{2}}(u-v)\| + \|A^{\frac{3}{4}}(u-v)\|\|A^{\frac{1}{2}}v\|). \tag{48}$$

Proof.

$$\|Fu\| \le \left(\int_\Omega |u|^2|\nabla u|^2\right)^{\frac{1}{2}} \le \|u\|_{L^6}\|\nabla u\|_{L^3},$$

owing to Holder with $p = \frac{1}{3}$. Noting that $\|u\|_{L^6} \le \|\mathcal{C}_\Omega A^{\frac{1}{2}}\|$ by Friedrich, it only remains to prove that $\|\nabla u\|_{L^3}$ can be controlled by $\|A^{\frac{3}{4}}u\|$. We will use, without proof, the fact $\mathcal{D}(A) \subset [H^2(\Omega)]^3$; cf. Cattabriga (1961). The exponent refers to the tensor function with number of components. This implies that the ∇ operator sends $[H^2(\Omega)]^3$ into $\mathrm{grad}[H^2(\Omega)]^3 = [H^1(\Omega)]^9 \subset [L^6(\Omega)]^9$. Similarly,

$$\nabla : \mathcal{D}(A^{\frac{1}{2}}) = [H_0^1(\Omega)]^2 \subset [H^1(\Omega)]^9 \to \mathrm{grad}[\mathrm{H}^1(\Omega)]^3 = [L^2(\Omega)]^9.$$

Appealing to interpolation theory of Banach spaces for $1/6 + 1/2 = 2/3$ we conclude that $\nabla : \mathcal{D}(A^{\frac{3}{4}}) \to [L^3(\Omega)]^9$ continuously. $\square$

Lemma 6.2. *Let α be a real number in $0 < \alpha \le e$. Then*

$$\|A^\alpha e^{-tA}\| \le t^{-\alpha} \quad (t > 0). \tag{49}$$

Furthermore, $t^\alpha\|A^\alpha e^{-tA}u\| \to 0$ as $t \to 0$ for every $u \in \mathcal{H}_\sigma$.

Lemma 6.3. *Let α be any real number in $0 < \alpha < 1$. Then, the inequality*

$$\|(e^{-hA} - I)u\| \le \frac{1}{\alpha}h^\alpha\|A^\alpha u\| \quad (h > 0), \tag{50}$$

holds for any $u \in \mathcal{D}(A^\alpha)$.

Lemma 6.4. *Consider*

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \quad (t \in [0,T],\ T > 0), \tag{51}$$

where $f \in \mathscr{C}(0,T] \equiv \mathscr{C}((0,T];\mathcal{H}_\sigma)$ *is assumed to satisfy*

$$\sup_{0<s\le t} s^\lambda \|f(s)\| \le M(t) < \infty \quad (0 < t \le T), \tag{52}$$

for a constant $\lambda \in [0,1)$ *and real-valued function* M*. If* α *is a number in* $0 \le \alpha < 1$*, then* $A^\alpha u(t)$ *exists for each* $t \in (0,T]$ *and satisfies the inequality*

$$\|A^\alpha u(t)\| \le t^{1-\alpha-\lambda} M(t) B(1-\alpha, 1-\lambda), \tag{53}$$

where $B(\cdot,\cdot)$ *represents the beta function* $\int_0^1 s^{\cdot-1}(1-s)^{\cdot-1}\,\mathrm{d}s$*. Moreover,* $A^\alpha u \in \mathscr{C}^\vartheta(0,T]$ *for any* $\vartheta \in (0, 1-\alpha)$*. In particular, we have* $A^\alpha u \in \mathscr{C}^\vartheta[0,T]$ *with* $A^\alpha u(0) = 0$ *if* $0 < \vartheta \le 1-\alpha-\lambda$*.*

Lemma 6.5. *Let* α*,* ϑ *and* μ *be real numbers such that* $0 \le \alpha < 1$ *and* $0 < \mu < \vartheta - \alpha$*. Let*

$$v(t) = \int_0^t e^{-(t-s)A}(f(s) - f(t))\,\mathrm{d}s \quad (t \in [0,T],\ T > 0) \tag{54}$$

where $f \in \mathscr{C}^\vartheta[0,T]$*. Then* $A^{1+\alpha}v(t)$ *exists for each* $t \in [0,T]$ *and can be expressed as*

$$A^{1+\alpha}v(t) = \int_0^t A^{1+\alpha} e^{-(t-s)A}(f(s) - f(t))\,\mathrm{d}s. \tag{55}$$

Moreover, $A^{1+\alpha}v \in \mathscr{C}^\mu[0,T]$*.*

Lemma 6.6. *Again consider* u *given by* (**??**)*, assuming that* $f \in \mathscr{C}^\vartheta[0,T]$ *for some* $\vartheta \in (0,1)$*. Then*

$$u \in \mathscr{C}^{1+\nu}(0,T] \quad \textit{and} \quad Au \in \mathscr{C}^\nu(0,T] \tag{56}$$

for any ν *subject to* $0 < \nu < \vartheta$*. Furthermore,* $u' \equiv \mathrm{d}u/\mathrm{d}t$ *can be expressed as*

$$u' = -Au + f \tag{57}$$

or as

$$u'(t) = e^{-tA} f(t) - \int_0^t A e^{-(t-s)A}(f(s) - f(t))\,\mathrm{d}s. \tag{58}$$

6.1 Existence of mild solution

Let $u \in S[0,T]$ for a positive number T. For some nonnegative continuous function $K = K(t)$ with $K(0) = 0$, owing to (59)

$$\|Fu(s)\| \le \mathcal{C}_\Omega K^2 s^{-\frac{3}{4}}, \quad (0 < s \le t \le T). \tag{59}$$

Now put

$$\Phi(t) \equiv \Phi(u;t) = \int_0^t e^{-(t-s)A} Fu(s)\,\mathrm{d}s. \tag{60}$$

We claim that the function Φ exists and continuous on $[0,T]$.

$$\|\Phi(t)\| \leq \int_0^t \|e^{-(t-s)A}\|\|Fu(s)\|\,\mathrm{d}s \leq \mathcal{C}_\Omega K^2 \int_0^t s^{-\frac{3}{4}}\,\mathrm{d}s,$$

making

$$\|\Phi(t)\| \leq \mathcal{C}_\Omega \tfrac{1}{4} K^2 t^{\frac{1}{4}} < \infty, \quad (0 \leq t \leq T),$$

A similar reasoning will show that $A^{\frac{1}{2}}\Phi$ is also continuous in $(0,T]$ and for any $\alpha \in (0,1)$, the integral $A^\alpha\Phi$, by appealing to the Beta function

$$B(1-\alpha, 1-\beta) = t^{1-\alpha-\beta}\int_0^t (t-s)^{-\alpha}s^{-\beta}\,\mathrm{d}s < \infty \quad \forall \alpha,\beta \in (0,1),$$

yields

$$\begin{aligned} \|A^\alpha\Phi(t)\| &\leq \int_0^t \|A^\alpha e^{-(t-s)A}\|\|Fu(s)\|\,\mathrm{d}s \\ &\leq \mathcal{C}_\Omega K_t^2 \int_0^t (t-s)^{-\alpha}s^{\frac{3}{4}}\,\mathrm{d}s \leq \mathcal{C}_\Omega K_t^2 B_\alpha t^{\frac{1}{4}-\alpha}, \end{aligned} \tag{61}$$

where $B_\alpha = B(1-\alpha, \frac{1}{4})$, valid for $0 \leq t \leq T$. Consequently, we conclude that $\Phi \in S[0,T]$ whenever $u \in S[0,T]$; the integral (60) is well-defined for solutions belonging to class S. Moreover, from

$$\|A^{\frac{1}{2}}e^{-tA}a\| \leq \|A^{\frac{1}{4}}e^{-tA}\|\|A^{-\frac{1}{4}}a\| \leq \|A^{\frac{1}{4}}a\|t^{-\frac{1}{4}}, \quad (0 \leq t \leq T).$$

In other words, $e^{-tA}a \in S[0,T]$ whenever $a \in \mathcal{D}(A^{\frac{1}{4}})$ making the right side of (60) belong to $S[0,T]$ whenever $u \in S[0,T]$ and $a \in \mathcal{D}(A^{\frac{1}{4}})$. With this set up, we may conclude uniqueness of mild solution to the integral equation (60).

Theorem 6.7 (Uniqueness)**.** *If $a \in \mathcal{H}_\sigma$ then the solution to* (60) *is unique and is in the class $S[0,T]$.*

Proof. Supposing that u and v are solutions to (60) in the class $S[0,T]$. Put $w = u - v$ and expressing

$$w(t) \equiv \int_0^t e^{-(t-s)A}(Fu(s) - Fv(s))\,\mathrm{d}s, \quad 0 < t \leq T.$$

Buy definition and employing estimate (48) together with bounding $\|A^\alpha(u-v)\|$ by $\|A^\alpha u\| + \|A^\alpha v\|$ we have

$$\|Fu(s) - Fv(s)\| \leq 4\mathcal{C}_\Omega \beta K(t)D(t)s^{-\frac{3}{4}}, \quad 0 < s \leq t \leq T,\ \beta = B_{1/2}B_{3/4},$$

where we made the choices

$$K(t) \equiv \max_{\alpha=\frac{1}{2},\frac{3}{4}}\left\{\sup_{0<s\leq t} s^{\frac{1}{4}-\alpha}\|A^\alpha u(s)\|,\ \sup_{0<s\leq t} s^{\frac{1}{4}-\alpha}\|A^\alpha v(s)\|\right\},$$

and $D(t) \equiv \max\{D_{1/2}, D_{3/4}\}$ for $D_\alpha \equiv \sup_{0<s\leq t} s^{\frac{1}{4}-\alpha}\|A^\alpha w(s)\|$. Again, we have for

$$\|A^\alpha w(t)\| \leq 4\mathcal{C}_\Omega\beta K(t)D(t)\int_0^t (t-s)^{-\alpha}s^{-\frac{3}{4}}\,\mathrm{d}s = 4\mathcal{C}_\Omega\beta B_\alpha K(t)D(t)t^{\frac{1}{4}-\alpha}, \quad (\alpha = \tfrac{1}{2}, \tfrac{3}{4}).$$

Then

$$D(t) \le 4\mathcal{C}_\Omega B_1 K(t) D(t), \quad (0 < t \le T). \tag{62}$$

due to the increasing nature of $K(t)D(t)$ in t. We claim that $w(t) \equiv 0$ for sufficiently small t. Since $K(0) = 0$, we can choose $\tau > 0$ such that $4\mathcal{C}_\Omega B_1 K(\tau) < 1$, so concluding from (62) above that $D(\tau) = 0$ and also for any $\tau_0 < \tau$. From now on we restrict ourselves to $I_\tau = [\tau, T]$. Since $A^{\frac{1}{2}}u$ and $A^{\frac{1}{2}}v$ are both continuous on I_τ, we find a positive constant K_τ such that

$$\max\{\|A^{\frac{1}{2}}u(s)\|, \|A^{\frac{3}{4}}u(s)\|\} \le K_\tau \quad \text{and} \quad \max\{\|A^{\frac{1}{2}}v(s)\|, \|A^{\frac{3}{4}}\|\} \le K_\tau, \quad (s \in I_\tau).$$

It therefore follows that

$$\|Fu(s) - Fv(s)\| \le 4\mathcal{C}_\Omega K_\tau D_\tau(t), \quad (\tau \le s \le t \le T),$$

where

$$D_\tau(t) \equiv \max\{\sup_{\tau \le s \le t} \|A^{\frac{1}{2}}w(s)\|, \sup_{\tau \le s \le t} \|A^{\frac{3}{4}}w(s)\|\} = D(t),$$

where the equality on the right is true since $D \equiv 0$ on $[0, \tau]$. In order to obtain the desired conclusion, we let $\eta \in I_\tau$ and let $\delta = 1/(16\mathcal{C}_\Omega K_\tau)^2$ and we will proceed by showing that if $w(t) \equiv 0$ on $[0, \eta]$, then $w(t) = 0$ on $[0, \eta + \delta] \cap I_\tau$. Indeed, from the relation

$$w(t) = \int_\eta^t e^{-(t-s)A}(Fu(s) - Fv(s))\,\mathrm{d}s,$$

we see that

$$\|A^{\frac{1}{2}}w(t)\| \le 8\mathcal{C}_\Omega K_\tau D_\tau(t)(t-\eta)^{\frac{1}{2}}, \quad (t \in [\eta, T]).$$

In particular

$$\|A^{\frac{1}{2}}w(t)\| \le 8\mathcal{C}_\Omega K_\tau D_\tau(\eta + \delta)\delta^{\frac{1}{2}} = \frac{1}{2}D_\tau(\eta + \delta),$$

for any $t \in [\eta, \eta + \delta]$, so it follows that $D_\tau(\eta + \delta) \le \frac{1}{2}D_\tau(\eta + \delta)$ so $D_\tau(\eta + \delta) = 0$ and consequently $w \equiv 0$ on $[0, \eta + \delta]$. □

We will justify the following: if u is a solution to the integral problem in $[0, T]$, then u is a solution of

$$u(t) = e^{-(t-\tau)A}u(\tau) + \int_\tau^t e^{-(t-s)A}Fu(s)\,\mathrm{d}s \quad (t \ge \tau) \quad \text{in } [\tau, T] \tag{63}$$

for any $\tau \in (0, T)$. Suppose that u satisfies integral equation (60) then for any $\tau \in (0, t)$ easily verify

$$\begin{aligned}\int_0^t e^{-(t-s)A}Fu(s)\,\mathrm{d}s &= \int_0^\tau e^{-(t-s)A}Fu(s)\,\mathrm{d}s + \int_\tau^t e^{-(t-s)A}Fu(s)\,\mathrm{d}s \\ &= e^{(t-\tau)A}\int_0^\tau e^{-(\tau-s)A}Fu(s)\,\mathrm{d}s + \int_\tau^t e^{-(t-s)A}Fu(s)\,\mathrm{d}s.\end{aligned}$$

So noting that $e^{-tA}a = e^{-(t-\tau)A}e^{-\tau A}a$ we therefore obtain (**??**) from (**??**).

Theorem 6.8 (Existence)**.** *Assume that $a \in \mathcal{D}(A^{\frac{1}{4}})$. Then there exists a solution $u \in S[0, T]$ for the integral equation* (60) *for some $T > 0$ depending on a.*

Proof. We will construct a solution by successive approximation by first setting $u_0(t) = e^{-tA}a$ and iteratively defining

$$u_{n+1}(t) = u_0(t) + \Phi(u_n; t) \quad (n = 0, 1, 2, ...), \tag{64}$$

This (why) iteration can be continued indefinitely within the class $S[0, \infty)$, so the

$$K_n(t) = \sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} u_n(s)\| \quad (n = 0, 1, 2, ...),$$

are continuous and increasing on $[0, \infty)$. For $t > 0$, $n = 0, 1, 2....$, we have

$$\begin{aligned} K_{n+1}(t) &= \sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}}(u_0 + \Phi(u_n; t))\| \\ &\leq K_0(t) + \sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} \Phi(u_n; t)\| \leq K_0(t) + \mathcal{C}_\Omega B_1 K_n^2(t), \end{aligned}$$

by (61) so K_n satisfies the recurrence inequalities

$$K_{n+1}(t) \leq K_0(t) + \mathcal{C}_\Omega B_1 K_n^2(t) \quad (t > 0,\ n = 0, 1, 2, ...),$$

where, as before, $B_1 = B(\frac{1}{4}, \frac{1}{2})$. Note that

$$\sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} \Phi(u_n; s)\| \leq \mathcal{C}_\Omega B_1 K_n^2(t),$$

owing to the monotonicity of $K_n(t)$ in t. By the same reasoning carried earlier, we may choose $T > 0$ such that $4\mathcal{C}_\Omega B_1 K_0(T) < 1$ then using the recursive relation to infer that $\{K_n(T)\}$ is bounded with

$$K_n(T) \leq \chi(T) \quad (n = 0, 1, 2, ...),$$

where

$$\chi(T) = \frac{1 - \sqrt{1 - 4\mathcal{C}_\Omega B_1 K_0(t)}}{2\mathcal{C}_\Omega B_1}.$$

Concluding that $K_n(t) \leq \chi(t)$ for any $t \in (0, T]$. Note also that $\chi(t) \leq 2K_0(t)$. Now by setting $w_n = u_{n+1} - u_n$, let

$$D_n(t) = \sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} w_n(s)\| \quad (n = 0, 1, 2...;\ t \in (0, T]),$$

we see that

$$D_{n+1}(T) \leq 2\mathcal{C}_\Omega B_1 \chi(T) D_n(T) \leq 4\mathcal{C}_\Omega B_1 K_0(T) D_n(T) < D_n(T). \tag{65}$$

So this means that $\sum D_n(T)$ converges (by the ratio test), in particular, $\Sigma t^{\frac{1}{4}} A^{\frac{1}{2}} w_n(t)$ converges uniformly and strongly in $(0, T]$. Consequently, the sequence $\|t^{\frac{1}{4}} A^{\frac{1}{2}} w_n(t)\| \to 0$ as $n \to \infty$ uniformly making $t^{\frac{1}{4}} A^{\frac{1}{2}} u_n(t)$ a uniformly convergent sequence. Appealing to the facts that $A^{-\frac{1}{2}}$ is bounded

$$A^{-\frac{1}{2}} \left(\lim_{n\to\infty} t^{\frac{1}{4}} A^{\frac{1}{2}} u_n(t) \right) = \lim_{n\to\infty} t^{\frac{1}{4}} u_n(t).$$

The operator $A^{\frac{1}{2}}$ is closed making $u_n(t)$ converge to a limit $u(t) \in \mathcal{D}(A^{\frac{1}{2}})$ and $t^{\frac{1}{4}} A^{\frac{1}{2}} u_n(t)$ converges to $t^{\frac{1}{4}} A^{\frac{1}{2}} u(t)$ uniformly in $(0, T]$.

The function

$$K(t) = \sup_{0<s\le t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} u(s)\|,$$

satisfies $K(t) \le \chi(t) \le 2K_0(t)$ on $(0, T]$. Finally,

$$\alpha_n \equiv \sup_{0<s\le T} s^{\frac{1}{2}} \|Hu_n(s) - Hu(s)\| \to 0$$

as seen from (48). We finally verify if this $u(t)$ is indeed a solution in $[0, T]$. Clearly, $\Phi(u_n; t) \to \Phi(u; t)$ as $n \to \infty$ since

$$\|\Phi(u_n; t) - \Phi(u; t)\| \le \int_0^t s^{-\frac{1}{2}} (t-s)^{-\frac{1}{4}} \alpha_n \, \mathrm{d}s \to 0,$$

hence we get

$$u(t) = u_0(t) + \Phi(u; t) \quad (t \in (0, T]).$$

We set $u(0) = a$ and still this above holds for any $t \in [0, T]$ with u is continuous on $[0, T]$. From uniform convergence of $t^{\frac{1}{4}} A^{\frac{1}{2}} u_n(t)$ we have continuity of $A^{\frac{1}{2}} u(t)$. Also, (**??**) implies $\|A^{\frac{1}{2}} u(t)\| = o(t^{-\frac{1}{4}})$ and thus u solves and is in $S[0, T]$. □

Theorem 6.9 (global)**.** *In addition to the hypothesis of theorem above, suppose that*

$$\|A^{\frac{1}{4}} a\| + B_1 M < \frac{1}{4\mathcal{C}_\Omega B_1} \quad (B_1 = B(\tfrac{1}{4}, \tfrac{1}{2})),$$

where $\mathcal{C}_\Omega$ is [] . Then there exists a solution $u \in S[0, \infty)$ of integral equation (**??**)*.*

Proof. It suffices to show existence of a solution in $S[0, T]$ for any positive T. By lemma 2.10 we have

$$t^{\frac{1}{4}} \|A^{\frac{1}{2}} e^{-tA} a\| \le t^{\frac{1}{4}} \|A^{\frac{1}{4}} e^{-tA}\| \|A^{\frac{1}{4}} a\| \le \|A^{\frac{1}{4}} a\|.$$

With $K_0(t)$ used before satisfies $K_0(T) \le \|A^{\frac{1}{4}} a\|$ for any $T > 0$. □

6.2 Existence of strong solution for the Navier-Stokes equations

In view of Theorem 3.14, a sufficient condition to assert that the unique mild solution for (45) is in fact a strong one is by showing that the nonlinear term Fu is locally Lipschitz. This can be achieved by showing that $A^\alpha u$ is Lipschitz in u for all positive $\alpha < 1$ since,

$$\|Fu - Fv\| \le \mathcal{C}_\Omega(\|A^{\frac{3}{4}} u\| \|A^{\frac{1}{2}} (u - v)\| + \|A^{\frac{3}{4}} (u - v)\| \|A^{\frac{1}{2}} v\|), \quad t \in [0, T].$$

Recall that

$$A^\alpha u(t) = A^\alpha e^{-tA} a + \int_0^t A^\alpha e^{-(t-s)A} Fu(s) \, \mathrm{d}s.$$

Lemma 6.10. *The function $A^\alpha u(t)$ is uniformly Holder continuous in any closed interval $[\varepsilon, T]$ with $\varepsilon > 0$ if $\alpha < 1$.*

Proof. Let $\varepsilon > 0$ be given. It is obvious that $t \mapsto e^{-tA}a$ is Holder continuous on $[\varepsilon, T]$ so we focus on the second term of which we denote by $v(t)$. Let t be fixed. For $h < T - t$ we have

$$A^\alpha v(t+h) - A^\alpha v(t) = \int_0^t (e^{-hA} - I)A^\alpha e^{-(t-s)A} Fu(s) + \int_t^{t+h} A^\alpha e^{-(t+h-s)A} Fu(s).$$

Then for any $\gamma > 0$,

$$\begin{aligned}\|A^\alpha v(t+h) - A^\alpha v(t)\| \leq \|(e^{-hA} - I)A^{-\gamma}\| \int_0^t \|A^{\gamma+\alpha} e^{-(t-s)A}\| \|Fu(s)\| \, \mathrm{d}s \\ + \int_t^{t+h} \|A^\alpha e^{-(t+h-s)A}\| \|Fu(s)\| \, \mathrm{d}s.\end{aligned}$$

We appeal to the previously established relation

$$\int_0^t \|A^\alpha e^{-(t-s)A}\| \|Fu(s)\| \, \mathrm{d}s \leq \mathcal{C}_\Omega \left(\sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} u(s)\| \right)^2 B(1-\alpha, \tfrac{1}{4}) t^{\frac{1}{4}-\alpha},$$

to deduce that for any $\alpha + \gamma < 1$ with $\gamma > 0$,

$$\|A^\alpha v(t+h) - A^\alpha v(t)\| \leq \|(e^{-hA} - I)A^{-\gamma}\| \mathcal{C}_\Omega K^2 B t^{\frac{1}{4}-\alpha} + \int_t^{t+h} \frac{1}{(t+h-s)^\alpha} \frac{\mathcal{C}_\Omega K^2}{s^{\frac{3}{4}}} \, \mathrm{d}s$$

arriving at

$$\|A^\alpha v(t+h) - A^\alpha v(t)\| \leq \mathcal{C}_\Omega K^2 B h^\gamma + \frac{\mathcal{C}_\Omega K^2}{1-\alpha} t^{-\frac{3}{4}} h^{1-\alpha} \leq \left(\mathcal{C}_\Omega K^2 B + \frac{\mathcal{C}_\Omega K^2}{1-\alpha} t^{-\frac{3}{4}} \right) h^{1-\alpha}, \tag{66}$$

holding for any $t \in [\varepsilon, T]$, where $K = \sup_{0<s\leq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} u(s)\|$ and $B = B(1-\alpha, \frac{1}{4})$. □