The Ricci flow equation

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1 Introduction

In 1982, Richard Hamilton introduced the Ricci flow equation to homogenize metrics on a manifold [3]. He successfully used the Ricci flow to produce an other proof of the Poincaré conjecture in two dimensions. Unfortunately, the three dimensional case appeared more difficult and he was unable finish the proof. In 2002, Perelman successfully showed the conjecture by using Hamilton's concept of surgery for the Ricci flow.

We first review the key concepts of differential geometry: manifolds, (co)tangent bunde, metrics, connections, curvature, Lie derivative, as well as a brief word on space-time manifolds. This last concept is fundamental to define Ricci flow with surgery, although it will not be presented it here.

Instead, we develop the proof of the local existence for the Ricci flow equation via deTurck's trick [2] in the form presented later on by Hamilton [4]. In short, the idea is to couple the Ricci flow equation to remove the diffeomorphism invariance and obtain a parabolic equation, called Ricci-deTurck equation, from which one can retrieve a solution to the Ricci flow equation. For uniqueness, introducing the harmonic map Laplacian allows us to go from the Ricci flow equation to the Ricci-deTurck equation where in the latter setting the uniqueness is established.

Then, we look at well-known solutions, called solitons, given by self-similar manifolds. Typically, these solutions go beyond the local existence and may be classified in the three types: shrinking, steady and expending; some concrete examples of solitons will be given for each type.

The work is organized as follows: Section 2 prepares the framework for the Ricci flow equation. Section 3 introduces the Ricci flow equation and present the proof of the local existence of the solutions. Finally, Section 4 discusses the solitons and gives explicit examples.

2 Elements of differential geometry

This section provides a concise introduction to differential geometry based on the book [5], by presenting the following notions: manifolds, tangent/cotangent bundle, metrics, connections, curvature, Lie derivative and space-time manifold.

2.1 Manifold, tangent and cotangent bunde

A n-manifold M is an abstract object which *locally looks like* the Euclidean space $\mathbb{R}^n$, that is to say

- there are open sets U_λ yielding a covering of M;
- $\varphi_\lambda : U_\lambda \to \varphi_\lambda(U_\lambda) \subset \mathbb{R}^n$ are homeomorphisms.

Locally, this gives to M all the nice features of $\mathbb{R}^n$, in particular M is locally metrizable. However, M might not be globally metrizable. This is an issue if we try to define objects over the whole manifold. As a matter of fact, we use partitions of unity to pass from local geometry to global geometry.

It is sufficient to require the following two properties:

- M is second-countable;
- M is Hausdorff.

Indeed, since every point has a neighbourhood that is homeomorphic to some open subset of $\mathbb{R}^n$, so since $\mathbb{R}^n$ is locally compact, every manifold is locally compact and a locally compact Hausdorff space is (completely) regular.

Finally, the fact that M is second-countable means that there exists a countable basis for the topology of M and by Urysohn metrization theorem we know that a regular space with a countable base is metrizable.

Furthermore, since we are interested in developing calculus on manifolds, we add a smooth structure to M:

$$\text{for all } \alpha, \beta, \text{ the map } \varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \text{ is } C^\infty.$$

The diffeomorphisms φ_λ provides local coordinates and the pair $(U_\lambda, \varphi_\lambda)$ is referred as a local chart. Altogether, $(\varphi_\lambda)_\lambda$ is an atlas of the manifold.

If we consider M to be a surface, it is clear that the tangent space at a point $p \in M$ is given by all the velocity vectors at a point p. In fact, it is the quotient space of the set of curves on M passing through p with the equivalence relation: $\gamma_1 \sim \gamma_2$ if and only if they have the same velocity vector at p.

Since the velocity vectors are independent of the choice of the curve, we can view tangent vectors as directional derivatives on functions $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$, that is X is an operator $C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ defined by

$$(Xf)_p = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma(t), \qquad \text{where } \gamma(0) = p, \quad \dot{\gamma}(0) = X_p.$$

One can then show that the set of all the velocity vectors at a point p and $T_p\mathbb{R}^n$ are isomorphic.

We call these maps derivations and the set of all of them at a point p is denoted by T_pM. Note that they satisfy the product rule $X(fg) = gXf + fXg$, so for an arbitrary manifold a derivation $X \in T_pM$ is a map $C^\infty(M, \mathbb{R}) \to \mathbb{R}$ satisfying the product rule.

Then, we consider the push-forward $f_* : T_pM \to T_{f(p)}N$ of a smooth map $f : M \to N$, where M and N are smooth manifolds, defined by

$$(f_*X)g = X(g \circ f).$$

We can now relate what we know from $\mathbb{R}^n$ to a n-dimensional smooth manifold M: if φ is a local chart, then $\varphi_* : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism.

To summarize the ideas:

Definition 2.1 (Tangent and cotangent space)**.** *The tangent space T_pM at a point p of a manifold M is the set of all derivations, that is to say all maps $X : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ satisfying $X(fg) = gXf + fXg$.*

In local coordinates $\varphi = (x^1, \dots, x^n)$, the tangent space admits the basis

$$\left.\frac{\partial}{\partial x^i}\right|_p \stackrel{\text{def}}{=} (\varphi^{-1})_* \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}.$$

The set $TM \stackrel{\text{def}}{=} \bigcup\limits_{p\in M} T_pM$ *defines the tangent bundle, from which* T_pM *is a fibre. We denote by* $T(M)$ *the smooth section of* TM*, in other words,* $T(M)$ *is the set of vector fields over* M*.*

The dual of tangent space $(T_pM)^* \stackrel{\text{def}}{=} T_p^*M$ *is called the cotangent space. Given local coordinates* $(x^1, \dots, x^n)$*, it is characterized by the relation*

$$\left((x^i)_* \left.\frac{\partial}{\partial x^j}\right|_p\right) \mathrm{id} = \left.\frac{\partial}{\partial x^j}\right|_p \mathrm{id} \circ x^i = \delta^i{}_j,$$

which yields the basis

$$dx^1|_p, \dots, dx^n|_p, \qquad \text{where } dx^i(\bullet) \stackrel{\text{def}}{=} \Big((x^i)_*\bullet\Big)\,\mathrm{id} : T_p^*M \to \mathbb{R}.$$

The set $T^*M \stackrel{\text{def}}{=} \bigcup_{p\in M} T_p^*M$ *is the cotangent bundle, from which* T_p^*M *is a fibre.*

2.2 Metrics and connections

We know wish to add an additional object to our smooth manifold M: the metric. The purpose is to define an inner product for T_pM, so that we can do introduce some geometry (i.e. angle, distances,...).

Definition 2.2 (Metric)**.** *A Riemannian metric on a smooth manifold* M *is a continuous map* g *of* M *in the set of symmetric positive definite bilinear form on* T_pM*. In other words,* g *associates to each* $p \in M$ *an inner product of* T_pM*. Thus,* g *is a* $(0,2)$*-tensor.*

In local coordinates $(x^1, \dots, x^n)$*, given* $X, Y \in T_pM$*, we have*

$$g(X,Y) = g(\partial_i, \partial_j)X^iY^j \qquad \Longrightarrow \qquad g = g_{ij}dx^i \otimes dx^j, \quad \text{where } g_{ij} \stackrel{\text{def}}{=} g(\partial_i, \partial_j).$$

We denote by $g^{ij} := (g^{-1})^{ij}$ *the components of the inverse of* g*.*

Moreover, consider the isomorphism $T_pM \simeq T_p^*M$ *given by* $v \mapsto g(v, \cdot)$*. Let* (e_i) *be a local basis of* T_pM *and* (ε^i) *a local basis of* T_p^*M*. Then, the representation of* $v = v^ie_i \in T_pM$ *in the dual space is given by* $g(v,\cdot) = v_i\varepsilon^i \in T_p^*M$ *such that:*

$$v_i = g(v, e_i) = v^j g(e_j, e_i) = v^j g_{ji} \text{ and } v^i = v^k g_{kj} g^{ji} = v_j g^{ji}.$$

Lemma 2.3. *We have the following properties:*

1. *Given a manifold* M *and a Riemannian manifold* (N, g^N) *together with a smooh map* $f : M \to N$*, we can define a metric on* M *by the pull-back* f^**:*

$$(f^*g^N)(X,Y) \stackrel{\text{def}}{=} g^N(f_*X, f_*Y).$$

2. *Moreover, every smooth manifold admits a Riemannian metric.*

Idea of the proof. The first property is straightforward from the definition; for the second, use a partition of unity. □

Velocity vectors evolve in *a priori* very different tangent fibres. As a result, the acceleration cannot be computed as we did for the velocity (all positions belonged to the same space). To fix this, we introduce the notion of connections to link bundles (we will in particular consider the tangent bundle).

Definition 2.4 (Affine connection)**.** *An affine connection ∇ is a map $\nabla : T(M) \times T(M) \longrightarrow T(M)$ satisfying*

1. *(Left linearity) for $f, g \in C^\infty(M, \mathbb{R})$, $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$;*
2. *(Right $\mathbb{R}$-linearity) for $a, b \in \mathbb{R}$, $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$;*
3. *(Right Leibnitz-linearity) for $f \in C^\infty(M, \mathbb{R})$, $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.*

For a general $(0, k)$-tensor field τ and $Y_1, \ldots, Y \in T(M)$, we have

$$\begin{aligned}(\nabla_X \tau)(Y^1, \ldots, Y^k) =& X\left(\tau(Y^1, \ldots, Y^k)\right) - \tau(\nabla_X Y^1, Y^2, \ldots, Y^k) - \ldots \\ & \ldots - \tau(Y^1, \ldots, Y^{k-1}, \nabla_X Y^k).\end{aligned}$$

Lemma 2.5. *Let ∇ be an affine connection on M, $p \in M$ and $X, Y \in T(M)$. The quantity $\nabla_X Y(p)$ depends only on $X(p)$ and Y in a neighbourhood of p. Moreover, in local coordinates,*

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij})\partial_k, \qquad \text{where } \Gamma^k_{ij}\partial_k \stackrel{\text{def}}{=} \nabla_{\partial_i}\partial_j.$$

Consequently,

1. *for any manifold, there exists an affine connection.*
2. *there is a unique operator $D_t : T(\gamma) \to T(\gamma)$, where $T(\gamma)$ is the space of vector field along a curve γ, satisfying*
   (a) *($\mathbb{R}$-linearity) D_t is linear;*
   (b) *(Leibnitz-linearity) for $f \in C^\infty(I, \mathbb{R})$, $D_t(fV) = \dot{f}V + fD_tV$;*
   (c) *(Compatibility with ∇) for $V \in T(\gamma)$ with extension $\tilde{V} \in T(M)$, $D_tV(t_0) = \nabla_{\dot{\gamma}(t_0)}(\tilde{V} \circ \gamma)(t_0)$ for all $t_0 \in I$.*

Sketch of the proof. Let $p \in M$ and $X_1, X_2, Y_1, Y_2 \in T(M)$. If $X_1 = X_2$ and $Y_1 = Y_2$ on a neighbourhood of p, then

$$\nabla_{X_1} Y_1(p) = \nabla_{X_2} Y_2(p).$$

This implies that $\nabla_X Y(p)$ depends only on X and Y in a neighbourhood of p. Given a system of coordinates, and supposing $X(p) = 0$, we have

$$\nabla_X Y(p) = \nabla_{X^i\partial_i} Y(p) = X^i(p)\nabla_{\partial_i} Y(p) = 0.$$

The formula in local coordinates is obtained by unravelling the definitions, and the existence of the affine connection on M follows by using a partition of unity.

Finally, let $t_0 \in I$ and consider a local basis for the tangent space in a neighbourhood of $\gamma(t_0)$ a vector field V along a curve γ is written as $\tilde{V} = \tilde{V}^i\partial_i$. The unicity and existence follows by studying

$$\begin{aligned}D_tV &= \nabla_{\dot{\gamma}}(\tilde{V} \circ \gamma) \\ &= \left(\dot{\gamma}(\tilde{V} \circ \gamma)^k + \dot{\gamma}^i(\tilde{V} \circ \gamma)^k\Gamma^k_{ij}\right)\partial_k \\ &= \left(\ddot{\gamma}^k + \dot{\gamma}^j\dot{\gamma}^i(\gamma^k_{ij} \circ \gamma)\right)\partial_k.\end{aligned}$$

□

This last Lemma tells us something fundamental: ∇ is determined by the Christoffel symbols Γ_{ij}^k which should be understood in the same spirit as an operator in linear algebra: it is determined by its action over the basis vectors.

On might wish to obtain a connection close to the one of $\mathbb{R}^n$, that is to say $\nabla g \equiv 0$.

Definition 2.6 (Levi-Civita connection)**.** *A Levi-Civita connection is an affine connection ∇ satisfying*

1. *(Compatibility with the metric) for any $X \in T(M)$, $\nabla_X g = 0$;*
2. *(Symmetry) for $X, Y \in T(M)$, $[X, Y] = \nabla_X Y - \nabla_Y X$, where the latter is the Lie bracket, defined as $[X, Y]_p f \stackrel{\text{def}}{=} X_p(Yf) - Y_p(Xf)$.*

Lemma 2.7. *For any Riemannian manifold, there exists a unique Levi-Civita connection. Moreover, in local coordinates, we have*

$$\Gamma_{ij}^k = \frac{1}{2} g^{sk}(\partial_j g_{si} + \partial_i g_{sj} - \partial_s g_{ij}).$$

Sketch of the proof. Using the definitions, one can show

$$(Z, \nabla_Y X) = \frac{1}{2}(X(Y, Z) + Y(Z, X) - Z(X, Y) + ([X, Z], Y) - ([Y, Z], X) - ([X, Y], Z)).$$

Since this is independent of ∇, this equality uniquely determine $\nabla_X Y$. In particular, this identity yields

$$(\partial_s, \nabla_{\partial_i} \partial_j) = \Gamma_{ij}^k g_{sk} = \frac{1}{2}(\partial_j g_{si} + \partial_i g_{sj} - \partial_s g_{ij}),$$

which shows the existence. □

The covariant derivative allows us to define the Hessian of a smooth function at any point, not just a critical point.

Definition 2.8 (Gradient, Hessian, Laplacian)**.** *Let $f : M \to \mathbb{R}$ be a smooth function and ∇ denote the Levi-Civita connection.*

From Definition 2.4, the gradient of f is

$$\nabla_\bullet f = df(\bullet) = f_*(\bullet)\mathrm{id} \qquad \text{(cf. Definition 2.1)},$$

that is

$$\nabla_X f = df(X) = Xf, \qquad \text{for } X \in T(M).$$

The hessian of f is

$$\mathrm{Hess}(f)(X, Y) = \nabla_X(\nabla_Y f) = X(Yf) - (\nabla_X Y)(f) = g(\nabla_X(\nabla f), Y),$$

and in local coordinates, $df = \partial_k f dx^k$ and $\nabla(dx^k) = -\Gamma_{ij}^k dx^i \otimes dx^j$, yielding

$$\mathrm{Hess}(f)_{ij} = \partial_i \partial_j f - (\partial_k f)\Gamma_{ij}^k.$$

The Laplacian of f is

$$\Delta f = \mathrm{Hess}(f)^i{}_i.$$

The gradient is a $(0, 1)$-tensor and the hessian is a symmetric $(0, 2)$-tensor over $C^\infty(M, \mathbb{R})$ functions.

2.3 Curvature

The Riemannian tensor measures the non commutativity of second derivatives:

Definition 2.9 (Riemann tensor)**.** *The Riemann curvature tensor is a $(1,3)$-tensor R defined by*

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

or in local coordinates (x^i)

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \qquad \text{where } R(\partial_i,\ \partial_j)\partial_k = R_{ijk}{}^l \partial_l.$$

More precisely,

$$R_{ijk}{}^l = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}.$$

If $\Pi \subset T_pM$ is a 2-plane, the sectional curvature is the contraction of the Riemann tensor:

$$K(\Pi) = g(R(\partial_1, \partial_2)\partial_2, \partial_1).$$

The Ricci curvature tensor is a $(0,2)$-tensor obtained by taking the contraction of the second and fourth indices of the Riemann tensor:

$$\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j) \overset{\text{def}}{=} R_{ikj}{}^k = g^{pq} R_{ipjq}, \qquad \mathrm{Ric} = \mathrm{Ric}(\partial_i, \partial_j) dx^i \otimes dx^j.$$

2.4 Lie derivative

The idea behind the Lie derivative, is to generalize the notion of directional derivatives for vector fields in $\mathbb{R}^n$.

Definition 2.10 (Lie derivative)**.** *Given a covariant tensor field τ on M, we define the Lie derivative of τ with respect to $X \in T(M)$ as*

$$(\mathcal{L}_X \tau)(p) = \left(\frac{d}{dt}\bigg|_{t=0} (\theta_t)^* \tau \right)(p) = \lim_{t\to 0} \frac{(\theta_t)^* \tau_{\theta_t(p)} - \tau_p}{t},$$

where θ_t is the flow of X.

We quote without proof elementary results of the Lie derivative (cf. Chapter 13 in [5] for more details).

Lemma 2.11. *The Lie derivative takes the form:*

1. *for $f \in C^\infty(M, \mathbb{R})$ (i.e a 0-tensor), $\mathcal{L}_X f = Xf$;*
2. *for $Y \in T(M)$ (i.e a $(0,1)$-tensor), $\mathcal{L}_X Y = [X,Y]$.*
3. *for a general $(0,k)$-tensor τ on M and any vector fields $Y_1, \ldots, Y_k$,*

$$(\mathcal{L}_X \tau)(Y_1, \ldots, Y_k) = X(\tau(Y_1, \ldots, Y_k)) - \tau(\mathcal{L}_X Y_1, Y_2, \ldots, Y_k) - \ldots - \tau(Y_1, \ldots, Y_{k-1}, \mathcal{L}_X Y_k).$$

We conclude this section by showing a property that will be useful in the proof of local existence of the Ricci flow.

Lemma 2.12. *Let (M, g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of the metric g. For any vector field X, we have*

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

Proof. Let $X, Y \in T(M)$ and define $X^\flat = g(X, \cdot)$. We compute

$$\begin{aligned}
\mathcal{L}_X g(Y, Z) &= X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - (Y, [X, Z]) \\
&= g(\nabla_X Y - [X, Y], Z) + g(Y, \nabla_X Z - [X, Z]) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\
&= Y(g(X, Z)) - g(X, \nabla_Y Z) + Z(g(Y, X)) - g(\nabla_Z Y, X) \\
&= Y(X^\flat(Z)) - g(X, \nabla_Y Z) + Z(X^\flat(Y)) - g(\nabla_Z Y, X) \\
&= (\nabla_Y X^\flat)(Z) + (\nabla_Z X^\flat)(Y).
\end{aligned}$$

□

2.5 Space-time

This section introduces formally the notion of space-time manifold. It is an essential concept if one wants to define the Ricci flow *with surgery*, however we will not address this concept in this report.

Definition 2.13. *A* space-time *manifold is a smooth $n+1$-dimensional manifold $\mathcal{M}$ (possibly with boundary) equipped with a time coordinate $t : \mathcal{M} \to \mathbb{R}$ and a time vector field $X_t \in T(\mathcal{M})$ such that:*

1. *$t(\mathcal{M})$ is a possibly infinite interval and the boundary of $\mathcal{M}$ is the preimage under t of the boundary of $t(\mathcal{M})$;*
2. *for each $p \in \mathcal{M}$, there is an open neighbourhood $U \subset \mathcal{M}$ of p and a diffeomorphism $f : U \to V \times J$ where V is an open subset of $\mathbb{R}^n$ and J an interval;*
3. *$t = \pi_J \circ f$, where π_J is the projection of $V \times J$ onto J;*
4. *X_t is the image under f^{-1} of the unit vector field in the positive direction tangent to the foliation by $\{v\} \times J$ of $V \times J$, i.e $X_t = (f^{-1})_* \partial_t$.*

Remark 2.14. 1. *We have*

$$X_t(t) = ((f^{-1})_* \partial_t)(t) = \partial_t(t \circ f^{-1}) = \partial_t(\pi_J) = 1.$$

2. *The level sets of the time coordinate t are smooth n-dimensional manifolds whose tangent bundle is a section of $\ker dt \subset T\mathcal{M}$ which is a n-dimensional subbundle. Likewise, the metrics $g(t)$ are a section of $((\ker dt)^*)^{\otimes 2}$.*

Since X_t preserves the horizontal foliation, we can form the Lie derivative of a horizontal metric with respect to X_t.

3 Local existence of the Ricci flow

A n-dimensional generalized Ricci flow is a one-parameter family $(M(t), g(t))$ satisfying

$$\begin{cases} \mathcal{L}_{X_t} g = -2\operatorname{Ric}(g), \\ (M(0), g(0)) = (M_0, g_0). \end{cases} \tag{1}$$

Lemma 3.1. *For a space-time cylinder $\mathcal{M} = M \times I$, the Ricci flow equation reads*

$$\frac{\partial}{\partial t} g = -2\operatorname{Ric} g.$$

Proof. In this context, $X_t = \partial/\partial_t$, and since $\mathcal{L}_{\partial_t}\partial_k = [\partial_t, \partial_k] = 0$, we obtain

$$\mathcal{L}_{X_t} g(\partial_i, \partial_j) = \frac{\partial}{\partial t} g_{ij} - g(\mathcal{L}_{\partial_t}\partial_i, \partial_j) - g(\partial_i, \mathcal{L}_{\partial_t}\partial_j) = \frac{\partial}{\partial t} g(\partial_i, \partial_j).$$

□

The importance of (1) is that the topology of M is allowed to change. This allows *surgery* on the Ricci flow, a concept first introduced by Hamilton. For further readings, refer to [6].

For the purpose of this report, we will not treat this notion. In fact, we will always consider the settings of Lemma 3.1.

Thus, given a fixed background manifold M, we are looking for a one-parameter family of metrics $g(t)$ solving

$$\begin{cases} \frac{\partial}{\partial t} g = -2\operatorname{Ric}(g), \\ g(0) = g_0. \end{cases} \tag{2}$$

To better understand the meaning of the Ricci flow equation, it is useful to consider harmonic coordinates $(x^1, \ldots, x^n)$ about p, that is to say $\Delta x^i = 0$ for all i.

Lemma 3.2 (Lemma 3.32, [1])**.** *In harmonic coordinates $(x^1, \ldots, x^n)$, the Ricci flow equation (2) becomes*

$$\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij}) + Q_{ij}(g^{-1}, \partial g), \tag{3}$$

where Q denotes a sum of terms which are quadratic in the metric inverse g^{-1} and its first derivatives ∂g.

Sketch of the proof. Given a point $p \in M$, we wish to prove the existence of harmonic coordinates. In geodesic coordinates (i.e. $\Gamma_{ij}^k = 0$), by Definition 2.8 it follows that $\Delta x = \partial^i \partial_i x = 0$ and by the theory on elliptic PDEs we have obtained our set of coordinates.

Then,

$$\begin{aligned} -2\operatorname{Ric}_{jk} &= -2\operatorname{Rm}^q{}_{qjk} \\ &= -2\left(\partial_q \Gamma_{jk}^q - \partial_j \Gamma_{qk}^q + \Gamma_{jk}^p \Gamma_{qp}^q - \Gamma_{qk}^p \Gamma_{jp}^q\right) \\ &= -\partial_q g^{qr}(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}) + \partial_j g^{qr}(\partial_q g_{kr} + \partial_k g_{qr} - \partial_r g_{qk}) + \Gamma_{jk}^p \Gamma_{qp}^q - \Gamma_{qk}^p \Gamma_{jp}^q \\ &= g^{qr}(\partial_q \partial_r g_{jk} - \partial_q \partial_k g_{jr} + \partial_j \partial_k g_{qr} - \partial_j \partial_r g_{qk}) + Q_{ij}(g^{-1}, \partial g) \\ &= \Delta(g_{jk}) - g^{qr}\left(\partial_k \left(\Gamma_{qr}^s g_{sj}\right) + \partial_j \left(\Gamma_{qr}^s g_{sk}\right)\right) + Q_{ij}(g^{-1}, \partial g) \\ &= \Delta(g_{jk}) + Q_{ij}(g^{-1}, \partial g). \end{aligned}$$

□

Morally, this lemma shows that the Ricci flow equation (2) is the heat equation for metrics.

To prove the short-time existence of Ricci flows, we recall the definition of parabolic equations:

Definition 3.3 (Principal symbol and parabolic PDEs). *Let E, F be bundles over a manifold M and $\Gamma(E)$ a smooth section of E.*

The principal symbol, denoted $\widehat{\sigma}$, of a differential operator $P : \Gamma(E) \to \Gamma(F)$ in the direction $\xi \in \Gamma(E)$ is defined to be the bundle homomorphism:

$$\widehat{\sigma}[DP(u_0)](\xi) = \sum_{|\alpha|=k} DP_\alpha(u_0)\xi^\alpha,$$

where $u_0 \in \Gamma(E)$ is a given solution and $DP : \Gamma(E) \to \Gamma(F)$ is the linearisation of P at u_0 defined by

$$DP|_{u_0}(v) = \frac{\partial}{\partial t}\bigg|_{t=0} P(u(t)),$$

where $u(0) = u_0$ and $u'(0) = v$.

A differential operator P of order $2m$ is elliptic if for every $\xi, v \in \Gamma(E)$ there exists $c > 0$ such that

$$g(\widehat{\sigma}[DP(u_0)](\xi), v) \geq c|\xi|^{2m}|v|^2.$$

A linear equation $\partial_t u = Pu$ is parabolic if DP is elliptic.

Lemma 3.4. *Suppose $g(t)$ is a one-parameter family of metrics on M such that $\frac{\partial}{\partial t} g = h$. Then, the linearisation of the Ricci curvature tensor is*

$$[D\,\mathrm{Ric}(g)(h)]_{ik} = \frac{1}{2} g^{jp} \left(\nabla_j \nabla_k h_{ip} - \nabla_i \nabla_k h_{jp} + \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_p h_{ik} \right),$$

so the principal symbol is

$$\widehat{\sigma}[D\,\mathrm{Ric}(g)(h)_{ik}](\xi) = \frac{1}{2} g^{jp} \left(\xi_j \xi_k h_{ip} - \xi_i \xi_k h_{jp} + \xi_j \xi_i h_{kp} - \xi_j \xi_p h_{ik} \right).$$

Proof. Recall the local expression of the Riemann tensor given in Definition 2.9. In geodesic coordinates, i.e. $\Gamma_{ij}^k(p) = 0$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ijk}{}^l &= \frac{\partial}{\partial t}(\partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l) + \left(\left(\frac{\partial}{\partial t} \Gamma_{ik}^m \right) \Gamma_{jm}^l + \Gamma_{ik}^m \left(\frac{\partial}{\partial t} \Gamma_{jm}^l \right) \right) - \left(\left(\frac{\partial}{\partial t} \Gamma_{jk}^m \right) \Gamma_{im}^l + \Gamma_{jk}^m \left(\frac{\partial}{\partial t} \Gamma_{im}^l \right) \right) \\
&= \frac{\partial}{\partial t}(\partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial t} \Gamma_{ij}^k &= \frac{1}{2} \left(\frac{\partial}{\partial t} g^{kl} \right) (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) + \frac{1}{2} g^{kl} \frac{\partial}{\partial t} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) \\
&= \frac{1}{2} g^{kl} \frac{\partial}{\partial t} (\nabla_j h_{il} + \nabla_i h_{jl} - \nabla_l h_{ij}),
\end{aligned}$$

where we used $\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij} = 0$ and the commutativity of the differentials as well as the fact that $\nabla_j \partial_i = 0$ to obtain the last equality.

Hence,

$$\partial_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right)=\frac{1}{2}\partial_j g^{lp}\left(\nabla_k h_{ip}+\nabla_i h_{kp}-\nabla_p h_{ik}\right)+\frac{1}{2}g^{lp}\left(\partial_j(\nabla_k h_{ip}+\nabla_i h_{kp}-\nabla_p h_{ik})\right)$$
$$=\frac{1}{2}g^{lp}\left(\nabla_j\nabla_k h_{ip}+\nabla_j\nabla_i h_{kp}-\nabla_j\nabla_p h_{ik}\right).$$

Consequently,

$$\frac{\partial}{\partial t}R_{ijk}{}^l=\frac{1}{2}g^{lp}\left(\nabla_j\nabla_k h_{ip}+\nabla_j\nabla_i h_{kp}-\nabla_j\nabla_p h_{ik}-\nabla_i\nabla_k h_{jp}-\nabla_i\nabla_j h_{kp}+\nabla_i\nabla_p h_{jk}\right),$$

and the desired result follows by contraction:

$$\frac{\partial}{\partial t}\mathrm{Ric}_{ik}=\frac{\partial}{\partial t}R_{ijk}{}^j$$
$$=\frac{1}{2}g^{jp}\left(\nabla_j\nabla_k h_{ip}-\nabla_i\nabla_k h_{jp}+\nabla_j\nabla_i h_{kp}-\nabla_j\nabla_p h_{ik}\right).$$

□

Firstly, let us be cautious that (3) does not make the Ricci flow a strongly parabolic equation. Indeed, there is no reason for harmonic coordinates to remain harmonic at latter time.

Moreover, the Ricci flow equation (1) is invariant under diffeomorphisms: given any diffeomorphism $\Phi : M \to M$ and a Ricci flow $g(t)$, we have

$$\frac{\partial}{\partial t}\Phi^* g_t = -2\,\mathrm{Ric}(\Phi^* g_t).$$

It can be shown that the diffeomorphism invariance of the Ricci flow equation breaks the parabolicity (cf. Chapter 3 in [1] for more details). Hence, the strategy is to couple the Ricci flow equation with an evolving diffeomorphism.

While the proof of existence does not require any more tool than the one already introduced, the uniqueness requires one more operator:

Definition 3.5 (Harmonic map Laplacian)**.** *Let (M,g), (N,h) be two Riemannian manifolds and let $f : M \to N$ be a smooth map. The harmonic map Laplacian is*

$$\Delta_{g,h} f \overset{\text{def}}{=} g^{ij}(\nabla_{\partial_i} f_*)(\partial_j).$$

It is constructed as follow. The pull-back bundle f^*TN over M of TN by f is the smooth vector bundle defined by

$$f^*TN = \{(p,\xi)\,:\, p\in M,\, \xi\in TN,\, \pi(\xi)=f(p)\}.$$

The restriction X_f of $X \in T(N)$ to f is a smooth section of f^*TN given by

$$X_f(p)=X(f(p)),\qquad \forall p\in M.$$

Then, $f_*(p) : T_pM \to T_{f(p)}N = (f^*TN)_p$, that is $f_*(p) \in T_p^*M \otimes (f^*TN)_p$, so f_* is a smooth section of $T^*M \otimes f^*TN$.

Let (x^i) be local coordinates on M about p and (y^α) be local coordinates on N about $f(p)$, such that $f^\alpha = y^\alpha \circ f$. Thus,

$$f_*(p)=\left(\left.\frac{\partial}{\partial x^i}\right|_p f^\alpha\right)dx^i|_p\otimes\left.\frac{\partial}{\partial y^\alpha}\right|_{f(p)},\qquad \textit{that is}\quad f_*=(\partial_i f^\alpha)\,dx^i\otimes(\partial_\alpha)_f.$$

Let $\nabla^{(M,g)}, \nabla^{(N,h)}$ be the connections on TM and TN respectively. Moreover, the pull-back connection ${}^f\nabla^{(N,h)}$ on f^*TN is the connection satisfying

$$ {}^f\nabla^{(N,h)}_X Y_f = \nabla^{(N,h)}_{f_*X} Y, \qquad X \in T(M),\, Y \in T(N), $$

and is determined by the Christoffel symbols

$$ {}^f\nabla^{(N,h)}_{\partial_i}(\partial_\alpha)_f = (\partial_i f^\beta)\nabla^{(N,h)}_{(\partial_\beta)_f}\partial_\alpha = \partial_i f^\beta (\Gamma^{(N,h)})^\gamma_{\beta\alpha}(\partial_\gamma)_f, \qquad \text{that is} \quad ({}^f\Gamma^{(N,h)})^\gamma_{i\beta} = \partial_i f^\alpha (\Gamma^{(N,h)})^\gamma_{\alpha\beta}. $$

Then, there is an induced connection ∇ on $T^*M \otimes f^*TN$ over M satisfying

$$ \nabla_X(\xi \otimes Y) = \nabla^{(M,g)}_X(\xi) \otimes Y + \xi \otimes ({}^f\nabla^{(N,h)}_X Y). $$

Then,

$$ \begin{aligned} (\nabla_{\partial_i} f_*)(\partial_j) &= \nabla_{\partial_i}\left[(\partial_k f^\alpha)\, dx^k \otimes (\partial_\alpha)_f\right](\partial_j) \\ &= \left[\left(\nabla^{(M,g)}_{\partial_i} dx^k\right) \otimes \partial_k f^\alpha(\partial_\alpha)_f + dx^k \otimes \left({}^f\nabla^{(N,h)}_{\partial_i}\partial_k f^\alpha(\partial_\alpha)_f\right)\right](\partial_j) \\ &= -(\Gamma^{(M,g)})^k_{ij}\partial_k f^\alpha(\partial_\alpha)_f + \partial_i\partial_j f^\alpha(\partial_\alpha)_f + \partial_j f^\alpha \partial_i f^\beta (\Gamma^{(N,h)})^\gamma_{\alpha\beta}(\partial_\gamma)_f \\ &= \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f^\alpha - (\Gamma^{(M,g)})^k_{ij}\frac{\partial}{\partial x^k} f^\alpha + \frac{\partial}{\partial x^i} f^\beta \frac{\partial}{\partial x^j} f^\gamma (\Gamma^{(N,h)})^\alpha_{\gamma\beta}\right)(\partial_\alpha)_f, \end{aligned} $$

and the harmonic map is obtained by taking the trace.

Lemma 3.6 (Local existence)**.** *Let (M, g) be a compact Riemannian manifold with no boundary. There exists $0 < T \le +\infty$ and a unique solution $g(t)$ to the Ricci flow equation (2) for all $t \in [0, T)$.*

Proof. The proof is done as follow: we compensate the Ricci flow equation with a vector field W to obtain a strongly parabolic equation called the Ricci-de Turck equation. Then, we obtain a diffeomorphism $\Phi_t : M \to M$ such that its pull-back on the solution of the Ricci-de Turck equation, in fact gives us a solution for the Ricci flow equation.

To establish the uniqueness, we prove that if we have a solution of the Ricci flow equation, we can define a diffeomorphism Φ_t using the harmonic map Laplacian such that we obtain a solution to the Ricci-de Turck equation. Then, we have the uniqueness of the diffeomorphism to conclude.

Existence

Let $\tilde{g}$ be some fixed metric on M, and define

$$ W^k = g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}). \tag{4} $$

Consider the Ricci-de Turck equation given by

$$ \partial_t g = Q(g) = -2\,\mathrm{Ric}(g) + \mathcal{L}_W g. \tag{5} $$

We first rewrite the linearisation obtained in Lemma 3.4 in the form

$$ \begin{aligned} -2[D\,\mathrm{Ric}(h)]_{ik} &= \Delta h_{ik} + g^{pq}(\nabla_i\nabla_k h_{qp} - \nabla_q\nabla_i h_{kp} - \nabla_q\nabla_k h_{ip}) \\ &= \Delta h_{ik} - \nabla_i V_k - \nabla_k V_i + S_{ik}, \end{aligned} $$

where

$$\begin{aligned} V_k &= g^{pq}(\nabla_q h_{pk} - \frac{1}{2}\nabla_k h_{pq}), \\ \nabla_i V_k &= g^{pq}(\nabla_i \nabla_q h_{pk} - \frac{1}{2}\nabla_i \nabla_k h_{pq}, \\ S_{ik} &= g^{pq}(2R_{qik}{}^r h_{rp} - R_{ip}h_{kq} - R_{kp}h_{ip}). \end{aligned}$$

Then, by Lemma 2.12 we know that $(\mathcal{L}_W g)_{ij} = \nabla_i W_j + \nabla_j W_i$ from which we deduce

$$[D\mathcal{L}_W(h)]_{ik} = \nabla_i V_k + \nabla_k V_i + T_{ik},$$

where T_{ik} is a linear first order expression in h.

It follows that

$$[DQ(h)]_{ik} = \Delta h_{ik} + T_{ik} - S_{ik} \qquad \Longrightarrow \qquad \widehat{\sigma}[DQ(h)](\xi) = |\xi|^2 h,$$

which imply that the Ricci-deTurck equation (5) is parabolic.

Consequently, for any smooth initial metric g_0 there exists $T > 0$ such that $g(t)$ is a smooth unique solution to (5) for all $t \in [0, T)$. Note that this implies in turn the existence of a one-parameter family of vector fields $W(t)$ as defined in (4) for the same time interval.

Hence, the ODE

$$\begin{cases} \frac{\partial}{\partial t}\Phi_t(p) = -(W(t))_{\Phi_t(p)}, \\ \Phi_0 = \mathrm{id}, \end{cases} \tag{6}$$

has a unique solution (for more details cf. Lemma 12.9 and Lemma 12.11 in [5]), yielding a one parameter family of diffeomorphisms Φ_t for all $t \in [0, T)$.

Thus, $\overline{g}(t) = \Phi_t^* g(t)$ is a solution to the Ricci flow equation since $\overline{g}(0) = g(0) = g_0$ and

$$\begin{aligned} \frac{\partial}{\partial t}\overline{g} &= \frac{d}{ds}\bigg|_{s=0} \Phi_{t+s}^* g(t+s) \\ &= \Phi_t^*\left(\frac{\partial}{\partial t} g(t)\right) + \Phi_t^* \frac{d}{ds}\bigg|_{s=0} \Phi_s^* g(t) \\ &= \Phi_t^*(-2\operatorname{Ric}(g) + \mathcal{L}_W g) + \Phi_t^* \mathcal{L}_{-W} g \\ &= -2\Phi_t^* \operatorname{Ric}(g) \\ &= -2\operatorname{Ric}(\Phi_t^* g) \\ &= -2\operatorname{Ric}(\overline{g}). \end{aligned}$$

Uniqueness

We first show that a solution of the Ricci flow equation satisfies the Ricci-de Turck equation.

Let $(M, \overline{g}(t))$ satisfy the Ricci flow equation and consider the Levi-Civita $\tilde{\nabla}$ over M associated with the metric $\tilde{g}$.

Let Φ_t be defined by

$$\partial_t \Phi_t = \Delta_{\overline{g},\tilde{g}} \Phi_t.$$

Then, $g(t) = (\Phi_t^{-1})^* \overline{g}(t)$ solves the Ricci-deTurck equation. Indeed, following the same development as before, we obtain

$$\frac{\partial}{\partial t}g = (\Phi_t^{-1})^* \left(\frac{\partial}{\partial t}\overline{g}\right) + \mathcal{L}_V g = -2\,\mathrm{Ric}(g) + \mathcal{L}_V g,$$

where $V = -(\Delta_{\overline{g},\tilde{g}}\Phi_t)_{\Phi_t^{-1}}$.

Let $p \in M$ and set (x^i) to be the local coordinates near p and (y^α) the ones near $\Phi_t(p)$ given by $y^\alpha = x^\alpha \circ \Phi_t$. Then, $\overline{g}^{ij} = g^{ij}$ so $\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k$ and we have

$$\begin{aligned}
V(p) &= -\Delta_{\overline{g}(t),\tilde{g}}\Phi_t(\Phi_t^{-1}(p)) \\
&= -\overline{g}^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\Phi_t^\alpha - \overline{\Gamma}_{ij}^k\frac{\partial}{\partial x^k}\Phi_t^\alpha + \frac{\partial}{\partial x^i}\Phi_t^\beta\frac{\partial}{\partial x^j}\Phi_t^\gamma\tilde{\Gamma}_{\gamma\beta}^\alpha\right)(\partial_\alpha)_{\Phi_t}(\Phi_t^{-1}(p)) \\
&= -g^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}x^\alpha - \Gamma_{ij}^k\frac{\partial}{\partial x^k}x^\alpha + \frac{\partial}{\partial x^i}x^\beta\frac{\partial}{\partial x^j}x^\gamma\tilde{\Gamma}_{\gamma\beta}^\alpha\right)\partial_\alpha \\
&= g^{ij}\left(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k\right)\partial_\alpha \\
&= W(p).
\end{aligned}$$

We are now ready to prove the uniqueness: suppose there are solutions $\overline{g}_1(t), \overline{g}_2(t)$ of the Ricci flow with $\overline{g}_1(0) = \overline{g}_2(0)$. From the preceding result, we can obtain solutions $g_1(t), g_2(t)$ of the Ricci-deTurck equation such that $g_1(0) = \overline{g}_1(0) = g_2(0) = \overline{g}_2(0)$. By uniqueness of solutions of the Ricci-deTurck equation, $g_1(t) = g_2(t)$ for all t in their common interval of existence.

Thus, $W(t)$ defined by (4) is the same for the two solutions. Then, $\Phi_t^{(1)} = \Phi_t^{(2)}$ since the two diffeomorphisms $\Phi_t^{(1)}, \Phi_t^{(2)}$ generated by the harmonic map Laplacian satisfy the same ODE given by (6).

Then,

$$\overline{g}_1 = (\Phi_t^{(1)})^* g_1 = (\Phi_t^{(2)})^* g_2 = \overline{g}_2$$

which proves the uniqueness. □

4 Ricci solitons

A Ricci soliton is a Ricci flow $(M, g(t))$, $0 \le t < T \le \infty$, with the property that for each $t \in [0, T)$ there is a diffeomorphism $\Phi_t : M \to M$ and a term $\sigma(t)$ such that

$$g(t) = \sigma(t)\Phi_t^* g(0).$$

The following lemma gives a way to produce a Ricci soliton.

Lemma 4.1. *Let $X \in T(M)$, $\lambda \in \mathbb{R}$ and a metric $g(0)$ such that*

$$-\,\mathrm{Ric}(g(0)) = \frac{1}{2}\mathcal{L}_X g(0) - \lambda g(0). \tag{7}$$

Set $T = \infty$ if $\lambda \le 0$ or $T = (2\lambda)^{-1}$ if $\lambda > 0$, and define

$$\sigma(t) = 1 - 2\lambda t, \forall t \in [0, T), \qquad Y_t(x) = \frac{X(x)}{\sigma(t)}.$$

Let Φ_t be the one-parameter family of diffeomorphisms generated by the vector fields Y_t. Then, the flow $(M, g(t))$, $0 \le t < T$, where $g(t) = \sigma(t)\Phi_t^* g(0)$, is a soliton.

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= \sigma'(t)\Phi_t^* g(0) + \sigma(t)\Phi_t^*\mathcal{L}_{Y(t)}g(0)\\ &= \Phi_t^*(-2\lambda + \mathcal{L}_X)g(0)\\ &= \Phi_t^*(-2\operatorname{Ric}(g(0)))\\ &= -2\operatorname{Ric}(\Phi_t^*(g(0))).\end{aligned}$$

Since $\operatorname{Ric}(\alpha g) = \operatorname{Ric}(g)$ for any $\alpha > 0$, it follows that

$$\frac{\partial}{\partial t}g = -2\operatorname{Ric}(g).$$

□

An important class of solitons are the gradient solitons.

Lemma 4.2. *Suppose we have a complete Riemannian manifold $(M, g(0))$, a smooth function $f : M \to \mathbb{R}$, and a constant λ such that*

$$-\operatorname{Ric}(g(0)) = \operatorname{Hess}(f) - \lambda g(0).$$

Then there is a $T > 0$ and soliton $(M, g(t))$ called gradient soliton.

Sketch of the proof. One can prove $L_{(\nabla f)^\sharp}g(0) = 2\operatorname{Hess}(f)$, from which we recover Equation (7) with the vector field $X = (\nabla f)^\sharp$, where $\sharp : v_i e^i \mapsto v^i e_i$. □

In particular, an Einstein manifold (M, g_0), that is

$$\operatorname{Ric}(g_0) = \lambda g_0,$$

yields a gradient soliton.

By using the fact that the Ricci curvature tensor is invariant under rescaling, i.e. $\operatorname{Ric}(\lambda g_0) = \operatorname{Ric}(g_0)$, the one parameter family $g(t) = \sigma(t)g_0$ is a solution to the Ricci flow if

$$\sigma'(t)g_0 = \frac{\partial}{\partial t}g(t) = -2\operatorname{Ric}(g(t)) = -2\operatorname{Ric}(g_0) = -2\lambda g_0.$$

Hence, we recover the previous result

$$g(t) = (1 - 2\lambda t)g_0.$$

One notes the following three cases:

1. (*shrinking case*) $\lambda > 0$: the solution shrinks to 0 in finite time, it exists up to $t = 1/2\lambda$;
2. (*steady case*) $\lambda = 0$: the solution is constant for all time;
3. (*expanding case*) $\lambda < 0$: the solution grows to infinity.

4.1 Examples

We finish this report by giving some concrete examples of solitons.

Euclidean space

For the Euclidean space, the metric is $g_{\mathbb{R}^n} = \delta_{ij} dx^i \otimes dx^j$, so ${R_{ijk}}^l = 0$ and therefore $\mathrm{Ric} = 0$. Hence, $g(t) = g_{\mathbb{R}^n}$ at all time; it is a steady soliton.

Gaussian soliton

The Gaussian soliton $(\mathbb{R}^n, g_{\mathbb{R}^n}, f = \lambda|x|^2/4)$. It is a shrinking, steady, or expanding gradient soliton if $\lambda > 0, = 0, < 0$ respectively.

Sphere

The two dimensional sphere has the metric $g(t) = r^2(t) g_{\mathbb{S}^2} = r^2 \left(d\theta^{\otimes 2} + \sin^2\theta d\phi^{\otimes 2}\right)$ and we can compute, using the invariance under rescaling of the Ricci curvature tensor, $\mathrm{Ric}(g(t)) = \mathrm{Ric}(g_{\mathbb{S}^2}) = g_{\mathbb{S}^2}$. Consequently, the Ricci flow equation reads

$$2r\dot{r} g_{\mathbb{S}^2} = -2 g_{\mathbb{S}^2} \qquad \Longrightarrow \qquad r(t) = \sqrt{1-t}.$$

Thus, we have a shrinking soliton.

Cylinder

For a cylinder $\mathbb{S}^2 \times \mathbb{R}$, we have the metric $g_0 = g_{\mathbb{S}^2} + g_{\mathbb{R}}$. From the preceding examples, we know that the solution is $g(t) = (1-t) g_{\mathbb{S}^2} + g_{\mathbb{R}}$.

Cigar soliton

The cigar soliton $(\mathbb{R}^2, g_0 = \frac{dx^{\otimes 2} + dy^{\otimes 2}}{1+x^2+y^2}, (\nabla f)^\# = -2x\partial_x - 2y\partial_y)$. The flow Φ_t satisfies $\frac{\partial}{\partial t}\Phi_t = (\nabla f)^\#|_{\Phi_t}$, that is

$$x'(t)\frac{\partial}{\partial x}\bigg|_{(x(t),y(t))} + y'(t)\frac{\partial}{\partial y}\bigg|_{(x(t),y(t))} = -2x(t)\frac{\partial}{\partial x}\bigg|_{(x(t),y(t))} - 2y(t)\frac{\partial}{\partial y}\bigg|_{(x(t),y(t))},$$

so $\Phi_t = (xe^{-2t}, ye^{-2t})$, and by direct computation

$$\begin{aligned}
(\Phi_t)_* \frac{\partial}{\partial x}\bigg|_p &= \frac{\partial}{\partial x}\bigg|_p (xe^{-2t})\frac{\partial}{\partial x}\bigg|_{\Phi_t(p)} + \frac{\partial}{\partial x}\bigg|_p (ye^{-2t}\frac{\partial}{\partial x}\bigg|_{\Phi_t(p)} = e^{-2t}\frac{\partial}{\partial x}\bigg|_{(xe^{-2t}, ye^{-2t})}, \\
(\Phi_t)_* \frac{\partial}{\partial y}\bigg|_p &= e^{-2t}\frac{\partial}{\partial y}\bigg|_{(xe^{-2t}, ye^{-2t})}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi_t^* g_0(\partial_x|_p, \partial_y|_p) &= e^{-4t} g_0(\partial_x|_{\Phi_t(p)}, \partial_y|_{\Phi_t(p)}) = 0, \\
\Phi_t^* g_0(\partial_x|_p, \partial_x|_p) &= \frac{e^{-4t}}{1 + x^2 e^{-4t} + y^2 e^{-4t}} = \frac{1}{e^{4t} + x^2 + y^2}, \\
\Phi_t^* g_0(\partial_y|_p, \partial_y|_p) &= \frac{1}{e^{4t} + x^2 + y^2},
\end{aligned}$$

that is $g(t) = \Phi_t^* g_0 = \frac{dx^{\otimes 2}+dy^{\otimes 2}}{e^{4t}+x^2+y^2}$. It is straightforward to check that this is indeed a solution to the Ricci flow equation, yielding a steady soliton.

Rosenau soliton

The Rosenau soliton $(\mathbb{S}^2, g_\alpha = \frac{1}{1-\alpha^2 x^2} g_{\mathbb{S}^2})$ where $\alpha \in [0,1)$. It can be shown to converge (in the sense of Cheeger-Gromov) to the cylinder (base points away from the ends) as $\alpha \to 1$. Similarly, it converges to the cigar (base points near an end) as $\alpha \to 1$.

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