MATH 581 : PDE 2

Periodic solutions and reproductive property of Navier and Stokes equations

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Introduction

Navier and Stokes equation is a millennium problem that has been studied by a tremendous number of researchers. Besides, a point of view that is very interesting from the physics side is the existence of periodic solutions. This intuitive kind of behavior (waves in fluid dynamics) led to a wide range of articles around the proof of such solutions.

Thus, this paper presents the original ideas of Serrin ([1], [2]) and then of Kaniel and Shinbrot ([3], [4]). These authors developed some natural methods based on energy decrease to prove the existence of periodic solutions.
Serrin was the first to think about the use of the L^2 energy to obtain criteria about the stability and the uniqueness of periodic solutions in Navier and Stokes. However he used some huge hypotheses such as the existence of a solution. Then Kaniel and Shinbrot introduced the idea of the reproductive property which is even a stronger result than the periodicity. The idea is that the solution is able to reproduce its initial condition at some given time T.

Therefore the idea of this paper is to lighten some dark points left in the original articles by expliciting some proofs and results. Moreover we tried to restore the path taken by the authors and how the different methods evolved.

The problem

We are considering a bounded domain Ω in $\mathbb{R}^n$ $(n = 3)$ with $\partial\Omega$ smooth enough to be able to use the Sobolev embeddings. Ω will represent the spatial domain and is independent of time or periodic in time.

Now we say that u satisfies the Navier and Stokes equations (NSE) if

$(1.1) :$ $\frac{\partial u}{\partial t} = \nu\Delta u - u.\nabla u - \nabla p + f$ in $\Omega \times \mathbb{R}_*^+$

$(1.2) :$ div $u = 0$ in $\Omega \times \mathbb{R}_*^+$

$(1.3) :$ $u(x,t) = g(x,t)$ on $\partial\Omega$, $\forall t > 0$

$(1.4) :$ $u(x,0) = a(x)$ in Ω

p is defined as the pressure but it will not be very useful in our study, f is the forcing term, a is the initial condition and ν is the kinematic viscosity.

The goal is to derive criteria on f, a and ν such that u becomes a periodic solution of NSE.

Serrin's Method

Let $\Omega = \Omega(t)$ be a bounded domain in space, $\Omega \in \mathbb{R}^n$, such that it is periodic in time. Now let g be periodic in time (with the same period as Ω), we prescribe g on $\partial\Omega$:

(1.1) : $\frac{\partial u}{\partial t} = \nu \Delta u - u.\nabla u - \nabla p + f$ in $\Omega \times \mathbb{R}_*^+$

(1.2) : div $u = 0$ in $\Omega \times \mathbb{R}_*^+$

(1.3) : $u(x,t) = g(x,t)$ on $\partial\Omega$, $\forall t > 0$

Serrin also adds two important assumptions on the problem.

(A.1) : To every continuous initial distribution of velocities over Ω there corresponds a solution of the Navier and Stokes equations satisfying the prescribed boundary condition.

(A.2) : There is one solution whose Reynolds number $Re = \frac{Vd}{\nu}$ is less than 5.7. (Here V is the maximum speed of the flow during the whole time interval $t \geq 0$, d is the maximum diameter of Ω, and ν is the kinematic viscosity.)
This solution is equicontinuous in space for all time t.

We can directly notice that Serrin is assuming the existence of a solution of Navier and Stokes, which is still a remaining problem. However, the method he has developed had led to a tremendous study of the existence and uniqueness of periodic solutions.
We can first state the result he obtained under such conditions :

Serrin's Theorem (1959) ([1])**:**
Let Ω and g be defined as above and assume (A.1) and (A.2).
Then there exists a unique, stable, periodic solution (with the same period as g) of the Navier and Stokes equations in Ω which satisfies the boundary condition.

The method

The idea behind Serrin result was to study a solution v along with a perturbation v'. v and v' satisfy (1.1-3). Then he considered $u = v - v'$ and he was searching for conditions such that $\lim_{t\to+\infty} u(x,t) = v(x,t)$ in "L^2", which corresponds to the stability of the solution. Then u is solution of :

(2.1) : $\frac{\partial u}{\partial t} = \nu\Delta u - (u.\nabla)v - (v'.\nabla)u - \nabla p'$ in $\Omega \times \mathbb{R}_*^+$

(2.2) : div $u = 0$ in $\Omega \times \mathbb{R}_*^+$

(2.3) : $u(x,t) = 0$ on $\partial\Omega$, $\forall t > 0$

This is in part why a lot of authors after Serrin have studied the Navier and Stokes equations with Dirichlet boundary conditions as it does not really matter for the stability and uniqueness.

Then at this point we want to find a condition for $H_t(u) = \frac{1}{2}\int_\Omega u(x,t)^2 dx$ to decrease. In fact Serrin showed that :

$\frac{dH(u)_t}{dt} = -\int_\Omega [u.D.u + \nu(\nabla u)^2] = \int_\Omega [u.\nabla u.v - \nu(\nabla u)^2]$
where D is the deformation matrix with respect to the velocity v : $D_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i})$. To show this equality we just multiply (2.1) by u and integrate it in Ω using the properties of u. We will see later how the pressure term disappears.

Now the goal is to make the L^2-norm of u appear on the right hand side to obtain the decrease.

It comes from the fact that we can find α such that : $\alpha\int_\Omega u^2 \leq d^2\int_\Omega (\nabla u)^2$.
Then we have $u.\nabla u.v \leq \frac{1}{2}(\nu(\nabla u)^2 + \frac{u^2v^2}{\nu})$. Both inequalities come from vector calculus. The details can be found in Serrin's article [2].

Therefore we obtain $\frac{dH(t)}{dt} \leq \frac{1}{\nu}(V^2 - \frac{\alpha\nu^2}{d^2})H(t)$ and the solution will be stable if $Re \leq \sqrt{\alpha}$.

From this point we define $\epsilon = \frac{1}{\nu}(\frac{\alpha\nu^2}{d^2} - V^2)$.

Proof of the theorem

We consider v to be the function given by (A.2) and define the sequence $(\phi_n)_n$ by $\phi_n(x) = v(x,n)$. We also consider the period of the boundary condition to be equal to 1.

By hypothesis $(\phi_n)_n$ is equicontinuous and bounded therefore we can find a subsequence converging to some function ϕ. Serrin is not stating clearly in what space this convergence occurs, but we will see later that this can be seen in L^2. We can actually prove that the whole sequence is converging to ϕ.

If we consider $v'(x,t) = v(x, t-n+m)$, with $m > n$, then v' is clearly a solution of our problem thanks to the hypothesis (periodicity on the boundary).

Therefore from the previous proof we obtain :

$$\boxed{H_t(v'-v) \leq H_0(v'-v)e^{-\epsilon t}} \tag{1}$$

and from our hypothesis (A.2) we have $\epsilon > 0$. Now $||v||_\infty$ and $||v'||_\infty$ are bounded by V by assumption, therefore H_0 is clearly bounded as well (let us say by $C > 0$).

Then we just apply the inequality (1) to $t = n$:

$$H_n(v'-v) \leq Ce^{-\epsilon n} \tag{2}$$

But then $v'(x,n) = \phi_m(x)$ and $v(x,n) = \phi_n(x)$ therefore if an other subsequence converges it should converges to ϕ. As a consequence the whole sequence converges to ϕ in L^2.

Now we define w to be the solution given by (A.1) and associated to ϕ (that is $w(x,0) = \phi(x)$). Therefore w is a solution of Navier and Stokes, we need to prove it is periodic, stable and unique.

- Periodicity

  We first redefine v' to be $v'(x,t) = v(x,t+n)$. Then we use the same reasoning as previously to show that (w is seen as the perturbation) :

$$H_t(v'-w) \leq H_0(v'-w)e^{-\epsilon t} \tag{3}$$

  with $H_0(v'-w) = \frac{1}{2}||\phi_n - \phi||_{L^2}$, therefore we can take $t = 1$ to obtain

$$H_1(v'-w) < H_0(v'-w) \iff ||\phi_{n+1} - w(\bullet,1)||_{L^2} < ||\phi_n - \phi||_{L^2} \tag{4}$$

  Therefore we obtain $w(\bullet,1) = \phi = w(\bullet,0)$ in L^2 which means that the solution is periodic.

- Stability

  We know that w and v are equicontinuous and that $H_t(v-w) \leq H_0(v-w)e^{-\epsilon t}$ (w is seens as the perturbation). As a consequence we have that $||w||_\infty \leq V + o(t)$ but as w is periodic we obtain that $||w||_\infty \leq V$ which gives that $\epsilon > 0$ will be satisfied for w as a perturbed solution.

- Unicity

  If w' is an other solution of 2.1-3 the energy decrease gives
  $H_t(w-w') \leq H_0(w-w') = 0$. This directly leads to the unicity of the solution.

Reproductive property

The work of Kaniel and Shinbrot has been to introduce spaces related to Serrin's method to be able to define where the solutions could live. They also defined a more general approach which is the presence of a reproductive property. In particular, this property implies the periodicity of the solution if the forcing term is periodic as well.

Notations and decomposition

We need to define several functional spaces :

$C^\infty_{0,\sigma}(\Omega) = \{\phi \in D(\Omega)^n \ / \ \mathrm{div}\phi = 0\}$, with $D(\Omega)$ the space of the test functions in Ω.

$H_{k,\sigma}$ = completion of $C^\infty_{0,\sigma}$ in $H^k(\Omega)^n$, $k \geq 1$

L_σ = completion of $C^\infty_{0,\sigma}$ in $L^2(\Omega)^n$

$L^{p;q}_T$ = completion of $D(\Omega)^n$ with the norm $\ _T||u||_{p,q} = [\int\limits_0^T \ (\int_\Omega |u|^p)^{\frac{q}{p}} \ ]^{\frac{1}{q}}$

G = completion of $\{\nabla\phi \ / \ \phi \in C^\infty(\Omega)\}$ in $L^2(\Omega)^n$

We are also going to use the Einstein notation : $\sum_{i=1}^n u_i v_i = u_i v_i$
and the classical inner product in $L^2(\Omega)^n$: $(u, v) = \int_\Omega u_j v_j$.

At this point we need to prove some decomposition theorem that will be hugely helpful :

Decomposition theorem
$L_\sigma \bigoplus G = L^2(\Omega)^n$

Proof of the decomposition
We clearly have the inclusion $L_\sigma \bigoplus G \subset L^2(\Omega)^n$ by construction.

Now let $u \in L^2(\Omega)^n$ and $u \perp L_\sigma \bigoplus G$. If $w \in C^\infty_{0,\sigma}(\Omega)$ then there exists $f \in C^\infty_0(\Omega)^n$ such that $w = \mathrm{curl}(f)$. Then $(u,\mathrm{curl}(f)) = (u, \nabla g) = 0$ for all $f, g \in C^\infty_0(\Omega)^n$

Therefore $u \perp L_\sigma \bigoplus G$ gives that $\mathrm{div}(u) = 0$ and $\mathrm{curl}(u) = 0$ in the weak sense (as curl(grad)=div(curl)=0). But we know that if $\mathrm{div}(u) = 0$ and $\mathrm{curl}(u) = 0$ in the strong sense, then $\Delta u = 0$ in the strong sense.

Then we have $\int_\Omega u\Delta\phi = 0 \ \forall\phi$ in $D(\Omega)^n$, so applying Weyl's lemma (as $u \in L^2(\Omega)^n$) we obtain that $\Delta u = 0$ in Ω and $u \in C^\infty(\Omega)^n$.

Therefore as $u \in C^\infty(\Omega)^n$ we can build ϕ such that $u = \nabla\phi$ which directly gives that $u = 0$, hence the equality between the spaces.

Now Kaniel and Shinbrot introduced a norm that a new space of functions for the solutions to live in :
$\tilde{K_T} = L^2_\sigma \cap D(\bar{\Omega} \times \mathbb{R}^+)$ with respect to the norm
$||u||^2_{K_T} = ||\nabla u(\bullet, 0)||^2_2 + {}_T\,||u_t||^2_{2,2} + {}_T\,||\Delta u||^2_{2,2}$. We call K_T its completion with the previous norm.

Δ has to be seen has the extension of the Laplacian in L_σ with a Dirichlet condition : we suppose that $g = 0$ and $\nu = 1$ in this section for simplification. The value $\nu = 1$ does not have a real importance as we can modify the diameter of the space to obtain the Reynolds number we want.

We need to prove some regularity results of the functions of the space K_T.

Some useful Lemmas

Lemma 1

Let $T > 0$ and $u \in K_T$. Then we can redefined u on a set of values of t of one-dimensional measure zero in such a way that $||\nabla u||_2$ becomes absolutely continuous (as a function of time) and we get

$$\boxed{\frac{d||\nabla u||_2^2}{dt} = -2 \int_\Omega u_t.\Delta u} \tag{5}$$

Proof of Lemma 1

Let $u \in K_T$ then by definition we can find $(u_n)_n$ a sequence in $\tilde{K_T}$ such that $\lim_{n\to\infty} u_n = u$ in K_T.
Now we take t such that $0 < t < T$ and we compute

$$\begin{aligned}
||\nabla(u_m - u_n)||_2^2(t) &= \int_0^t \frac{d}{dt}||\nabla(u_m - u_n)||_2^2 + ||\nabla(u_m - u_n)||_2^2(0) \\
&= \int_0^t 2\int_\Omega \frac{d}{dt}\nabla(u_m - u_n).\nabla(u_m - u_n) + ||\nabla(u_m - u_n)||_2^2(0) \\
&= -2\int_0^t \int_\Omega \frac{d}{dt}(u_m - u_n)\Delta(u_m - u_n) + ||\nabla(u_m - u_n)||_2^2(0) \\
&\leq c[\ _T||(u_m - u_n)_t||_{2,2}^2 + \ _T\ ||(u_m - u_n)||_{2,2}^2 + ||\nabla(u_m - u_n)||_2^2(0)] \\
&= c||u_n - u_m||_{K_T}^2
\end{aligned}$$

where c is a positive constant. We used Green's identity and Schwarz inequality to obtain the result.

Therefore, as L^2 is complete, there exists v in L^2 such that $\lim_{n\to\infty} \nabla u_n(t) = v(t)$ in L^2 for almost all t in $(0;T)$.

Then by definition of the strong derivatives, it means that we can redefine u on a one dimensional set of measure zero such that $\lim_{n\to\infty} \nabla u_n(t) = \nabla u(t)$ in L^2 for all $0 < t < T$.

Now we can use the same reasoning used to get the previous equalities to obtain that

$$||\nabla u||_2^2(t) = ||\nabla u||_2^2(0) - 2\int_0^t \int_\Omega u_t.\Delta u$$

That is to say that $||\nabla u||_2$ is absolutely continuous and we have the required equality.

From this point we always consider that the functions have been redefined to be able to use the equality (5).

Now we need to recall some embedding result :

Sobolev embedding theorem ([3], [4])
Let $1 \leq p < n$ and $u \in W^1_{p,0}(\Omega)$ (with Dirichlet boundary condition). Ω is a bounded smooth domain in R^n. Then $u \in L^q(\Omega)$ where $q = \frac{np}{n-p}$ and there exists $c > 0$ such that

$$||u||_{L^q} \leq c||Du||_{L^p}$$

If $r < q$ then W^1_p has a compact injection into L^r.

This leads to the next lemma, where $n = 3$ and $p = 2$.

Lemma 2
Let $u \in K_T$. Then there exists $c > 0$ such that

$$\boxed{||\nabla u||_6 \leq c||\Delta u||_2} \tag{11}$$

The other inequalities we are going to use come from a direct application of Sobolev inequalities and were found in [3] or in [5].

Now we are going to make a new energy decrease appear by using the previous lemmas. To do so we first recall the problem introduced by Serrin

(2.1) : $u_t = \Delta u - (u.\nabla)v - (v'.\nabla)u - \nabla p'$ in $\Omega \times \mathbb{R}^+_*$

(2.2) : div $u = 0$ in $\Omega \times \mathbb{R}^+_*$

(2.3) : $u(x,t) = 0$ on $\partial\Omega$, $\forall t > 0$

(2.4) : $u(x,0) = a(x)$ in Ω

Lemma 3

Let $u \in K_T$ be a solution of (2.1-4) with $\nabla v', \nabla v \in L_T^{2,\infty}$ and $\nabla a \in L^2$.

If ${}_T||\nabla v||_{2,\infty}$ and ${}_T||\nabla v'||_{2,\infty}$ are small enough, then there exists $\alpha > 0$ such that

$$\boxed{||\nabla u||_2(t) \leq ||\nabla a||_2 e^{-\alpha t}} \tag{12}$$

Proof of Lemma 3

We first multiply (2.1) by $-\Delta u$. Thanks to the decomposition of L^2, the term with $\nabla p'$ is going to disappear.

Now we integrate the resulting equation on the domain Ω. The first term makes exactly appear the equation (5) of Lemma 1. Therefore we obtain

$$\frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 + ||\Delta u||_2^2 - \int_\Omega (v'.\nabla u + u.\nabla v).\Delta u = 0 \tag{13}$$

Now Sobolev theorem gives $||v'||_4 \leq c||\nabla v'||_2$ and $||\nabla u||_4 \leq c'||\Delta u||_2$ as $4 < 6$.

Therefore using Holder's inequality we obtain

$$|\int_\Omega v'.\nabla u.\Delta u| \leq ||v'||_4||\nabla u||_4||\Delta u||_2$$

and we use the previous remarks to get

$$|\int_\Omega v'.\nabla u.\Delta u| \leq C||\nabla v'||_2||\Delta u||_2^2$$

with $C > 0$.

Then similarly Sobolev theorem gives $||u||_\infty \leq c||\nabla u||_1 \leq c'||\Delta u||_2$.
Therefore using Holder's inequality we obtain

$$\begin{aligned} |\int_\Omega u.\nabla v.\Delta u| &\leq ||\nabla v||_2||u||_\infty||\Delta u||_2 \\ &\leq C'||\nabla v||_2||\Delta u||_2^2 \end{aligned}$$

with $C' > 0$.

Therefore we can take ${}_T||\nabla v||_2^{2,\infty}$ and ${}_T||\nabla v'||_2^{2,\infty}$ small enough so that

$$||\Delta u||_2^2 - \int_\Omega (v'.\nabla u + u.\nabla v).\Delta u \geq \frac{1}{2}||\Delta u||_2^2$$

But by Sobolev theorem we have $||\nabla u||_2 \leq c||\Delta u||_2$.
Therefore we can find $\alpha > 0$ such that

$$\frac{d}{dt}||\nabla u||_2^2 + \alpha||\nabla u||_2^2 \leq 0$$

This lemma is going to lead to the decrease of $||\nabla u||_2$ with u being a solution of the Navier and Stokes equations. The result is given by the next lemma.

Lemma 4
Let $u \in K_T$ be a solution of the Navier and Stokes equations with $\nabla a \in L^2$ and $f \in L_T^{2,\infty}$. If $||\nabla a||_2$ and ${}_T||f||_{2,\infty}$ are small enough, then $||\nabla u||_2$ is uniformly bounded in time.

Proof of Lemma 4
The proof is almost the same as the one of the Lemma 3. We begin by multiplying by Δu and integrating over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 + ||\Delta u||_2^2 - \int_\Omega (u.\nabla u - f).\Delta u = 0 \tag{14}$$

Now we take the same inequalities found in the proof to get that we can find some constant $c > 0$ such that

$$\begin{aligned} ||u||_\infty &\leq c||\Delta u||_2 \\ ||u||_\infty &\leq c||\nabla u||_2 \\ ||\nabla u||_2 &\leq c||\Delta u||_2 \end{aligned}$$

Then we choose $||\nabla a||_2 \leq \frac{1}{4c^2}$ and ${}_T||f||_2^{2,\infty} \leq \frac{1}{16c^3}$. We use the same type of inequalities (Holder and Sobolev) to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 &\leq -||\Delta u||_2^2 + ||u||_\infty||\nabla u||_2||\Delta u||_2 + ||f||_2||\Delta u||_2 \\ &\leq -||\Delta u||_2^2 + c||\nabla u||_2^2||\Delta u||_2 + \frac{1}{16c^3}||\Delta u||_2 \end{aligned}$$

Now we suppose that $||\nabla u||_2$ reaches $\frac{1}{4c^2}$, it yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 &\leq -||\Delta u||_2^2 + \frac{1}{4c}||\nabla u||_2||\Delta u||_2^2 + \frac{1}{4c}||\nabla u||_2||\Delta u||_2 \\ &\leq -||\Delta u||_2^2 + \frac{1}{4}||\Delta u||_2^2 + \frac{1}{4}||\Delta u||_2^2 \\ &\leq 0\end{aligned}$$

Therefore when $||\nabla u||_2$ reaches $\frac{1}{4c^2}$, it is decreasing afterwards. This means that $||\nabla u||_2$ is uniformly bounded in time. We can see that we could be less restrictive and take only $\frac{1}{2c}$ instead of $\frac{1}{4c^2}$, but we want to use this argument again in a future proof and we would need this sharpness.

Existence and uniqueness in X

Now we want to use the Leray-Schauder theorem to obtain our existence result. We just first recall the theorem.

Leray-Schauder theorem
Let A be be a continuous and compact mapping of a Banach space X into itself such that the set $\{x \in X/\lambda Ax = x \text{ for some } \lambda \in [0;1]\}$ is bounded.
Then A has a fixed point.

We set $0 < T < \infty$ and we also define a new set of functions :
$B_{r,T} = \{u \in L_T^{2,\infty} /\ _T||u||_{2,\infty} \leq r\}$ and we choose $R = R(T) > 0$ such that the hypotheses of the previous Lemmas are satisfied whenever $\nabla u \in B_{R,T}$ (u is the function of the next problem) and $f \in B_{R,T}$.

Now we introduce the operator A defined on K_T and for all $u \in K_T$ we have $Au = v$ with v being a solution of the following problem :

(3.1) : $v_t = \Delta v - u.\nabla u + f - \nabla p$ in $\Omega \times \mathbb{R}_*^+$

(3.2) : div $v = 0$ in $\Omega \times \mathbb{R}_*^+$

(3.3) : $v(x,t) = 0$ on $\partial\Omega$, $\forall t > 0$

(3.4) : $v(x,0) = u(x,0) = a(x)$ in Ω

v is a solution of an nonhomogeneous heat problem with a the second term being $-u.\nabla u + f - \nabla p$. Therefore we know that v exists and is unique.

The Banach space we will be using for the definition of A is
$X = \{u \in K_T/\nabla u \in B_{R,T}\}$.

Mapping into X

Let $u \in X$ and we define $v = Au$, following the proof of Lemma 3 we can find

$$\frac{1}{2}\frac{d}{dt}||\nabla v||_2^2 + ||\Delta v||_2^2 - \int_\Omega (u.\nabla u - f).\Delta v = 0 \tag{15}$$

Then using the same inequalities we can find that

$$|\int_\Omega u.\nabla u.\Delta v| \leq c||\nabla u||_2^2||\Delta v||_2$$
$$|\int_\Omega f.\Delta v| \leq ||f||_{2,\infty}||\Delta v||_2$$

Therefore by the definition of Lemma 4 we can use the same proof as Lemma 4 to find that $||\nabla v||_2^2$ is uniformly bounded and that $\nabla v \in B_{R,T}$.

Then if we use the previous inequalities and then integrate in time we obtain that Δv is clearly in $L_T^{2,2}$ and then from the equation (3.1) we obtain the same thing for v_t.

This yields $v \in K_T$ and then $v \in X$.

Continuity

Let $u, u' \in X$ and $v = Au, v' = Au'$ the associated solutions. Now $w = v - v'$ is solution of a problem similar to (3.1-4) without any forcing term. Then we just notice that

$$\begin{aligned} |\int_\Omega (u.\nabla u - u'.\nabla u').\Delta w| &= |\int_\Omega (w.\nabla u - u'.\nabla w).\Delta w| \\ &\leq \frac{1}{2c}||\Delta w||_2^2 \end{aligned}$$

therefore we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\nabla w||_2^2 &\leq (\frac{1}{2c} - 1)||\Delta w||_2^2 \\ &\leq (\frac{1}{2} - c)||\nabla w||_2^2 \end{aligned}$$

and we can choose c as big as we want so we can take it strictly bigger than $\frac{1}{2}$ such that $\alpha = -(\frac{1}{2} - c) > 0$ and
$||\nabla w||_2(t) \leq ||\nabla(u - u')(\bullet, 0)||_2 e^{-\alpha t}$.

Then we can integrate the previous inequality in time to show that $||\Delta w||_2$ is also bounded by $||\nabla(u - u')(\bullet, 0)||$ and we use equation (3.1) to obtain the same result on w_t.

Therefore we can similarly prove that $w \in X$ and that there exists some $c > 0$ such that
$||v - v'||_{K_T} \leq c||\nabla(v(\bullet, 0) - v'(\bullet, 0))||_2 \leq c||u - u'||_{K_T}$
which gives the continuity of A.

Compactness

Let $(u_n)_n$ be a sequence in X that weakly converges to u. Therefore we can create a sequence $(v_n)_n$ such that $v_n = Au_n$ and we want to show that $(v_n)_n$ converges in X.

If we call $w = v_n - v_m$ for some $n, m \in \mathbb{N}$, then w, u_n and u_m satisfy the hypotheses of the Lemma 3 and we obtain that $||\nabla w||_2(t) \leq ||\nabla(u_n(\bullet, 0) - u_m(\bullet, 0))||_2$.
And as this is true for all time under T and $||\nabla w||_2$ is continuous in time, it means that $(\nabla v_n)_n$ is Cauchy in $L_T^{2,2}$. Then we can use the same arguments as the proof of

the continuity to show that $(v_n)_n$ is Cauchy in X which implies the convergence of the sequence in X.

Now let $V = \{u \in X/\lambda Au = u \text{ for some } \lambda \in [0;1]\}$. If $\lambda = 0$ then directly $u = 0$. Let $u \in X$ non trivial and suppose that we can find $\lambda \in (0;1]$ such that $Au = \frac{u}{\lambda}$.

Then clearly as $\lambda \in (0;1]$ we can use the proof of Lemma 4 to show that u is bounded in X. As a consequence V is bounded.

Finally we can apply Leray-Schauder theorem to obtain that there exists a fixed point of A in X which means that our problem has a unique solution in X.

Proof of the reproductive property

At this point we can prove the reproductive property of the system. We define the operator M of X into itself as :

$Ma = u(\bullet, T)$ with u being the solution associated to the initial function a.

From Lemma 1 and 4, $\nabla u(\bullet, t) \in B_{R,T}$ for all $0 \leq t \leq T$ and the function associated to the semi norm ($||\nabla u||_2$) is continuous in time.

Then let $a, b \in X$ and u, v the associated solutions

$$||\nabla(Ma - Mb)||_2 = ||\nabla(u - v)(\bullet, T)||_2 \leq e^{-\alpha T}||\nabla(a - b)||_2$$

from Lemma 3.

Then if we define U as the closure of $L_\sigma \cap D(\Omega)$ with respect to the semi norm $||\nabla \bullet ||_2$, U is clearly a Banach space and $B_R \subset U$ as well.

Therefore the previous inequality shows that M is a contraction and the Banach theorem gives that M has a fixed point in X. This gives exactly the definition of the reproductive property.

Conclusion

This paper shows how the ideas of the authors developed themselves around some energy decrease. Then we have seen how we can end up with an existence and uniqueness result if the gradient of the initial condition and the forcing term are small enough. This idea led to the introduction of Bessel spaces from which we can derive more properties of the solution. Moreover we do not need restrictions on the forcing term or in the spatial-space, in two dimensions, to prove the uniqueness and existence result

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