Ian Weih-Wadman
Math 581

Notes on Cheeger Estimates and Nodal Sets of the p-Laplacian

The p-Laplacian can be defined as a map $\Delta_p : W^{1,p} \to W^{-1,q}$ where

$$\langle \Delta_p u, v \rangle = \int |\nabla u|^{p-2} \nabla u \cdot \nabla v.$$

The weak formulation of the p-Laplacian equation arises naturally as a variational problem; a solution is a minimizer of the energy functional

$$L(u) = \int_\Omega |\nabla u|^p.$$

Nonlinear eigenvalue problems for differential operators of this type can be formulated as another type of variational problem, where the class of admissible functions to be minimized over is restricted to only those functions in $W_0^{1,p}$ satisfying an integral constraint of the form

$$\int G(u) = 0$$

where G is a fixed smooth function. Then if $g = G'$, there is a real number λ such that

$$-\langle \Delta_p u, v \rangle = \lambda \langle g(u), v \rangle. \tag{1}$$

A more detailed discussion of this approach can be found in chapter 8 of [1].

Unlike the eigenvalues of a linear operator, there is no guarantee in general that scaling an eigenfunction $u \to au$ gives another eigenfunction. For the particular choice $g = |u|^{p-2}u$, however, we retain this property, so we consider this type of homogeneous constraint to be the natural setting for the eigenvalue problem for the p-Laplacian. One of the uses of this scaling property is to show that

$$u(x,t) = e^{-\lambda t} u(x)$$

solves the nonlinear time evolution problem

$$(\partial_t - \Delta_p) u = 0.$$

This nonlinear parabolic PDE generalizes the heat equation, and has been used to model flow in porous media.

The Dirichlet eigenvalue problem consists of finding which values of λ admit a nontrivial solution to (1) in $W_0^{1,p}$. This problem is extremely difficult, in fact for $p \neq 2$ it is not even known whether or not the spectrum is discrete. However, if we restrict attention to the principal eigenvalue, we find many results in common with the linear case $p = 2$. In particular, the principal eigenvalue is simple, positive and isolated, and its associated eigenfunction has constant sign.

The relative ease of studying the principal eigenvalue, when compared to general eigenvalues, is the due to the fact that it can be expressed as the minimum of a Rayleigh quotient

$$\lambda_p = \inf_{u \in W_0^{1,p}} \frac{\int_\Omega |\nabla u|^p}{\int |u|^p} \tag{2}$$

much like the principal eigenvalue of the Laplacian. We prove first that this infimum is actually attained for some eigenfunction u.

Theorem 1. *Let $1 < p < n$ and Ω a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and Ω having C^1 boundary. There is at least one function u which does not change signs and which minimizes the Rayleigh quotient (2). This function is an eigenfunction associated with λ.*

Proof. Let u_k be a minimizing sequence of the Rayleigh quotient. Since (2) is invariant under the scaling $u \to au$, we may assume that

$$\int |u_k|^p = 1, \ \forall n \in \mathbb{N}. \tag{3}$$

Since bounded subsets of $W^{1,p}$ have weakly convergent subsequences, we can pass to a subsequence which converges weakly to some u in $W^{1,p}$. Weakly convergent sequences can be badly behaved in general; however, under our hypotheses we can apply the Rellich-Kondrachov theorem to extract a subsequence which converges strongly in L^p to the limit function $u \in W^{1,p}$. Clearly

$$\int_\Omega |u|^p = 1,$$

and with the sequence u_k normalized as in (3) we find that

$$\left| \lambda_p - \int_\Omega |\nabla u_k|^p \right| \to 0, \text{ as } k \to \infty.$$

By the weak convergence in $W^{1,p}$ we also have

$$\lambda_p = \lim \int_\Omega |\nabla u_k|^p = \int_\Omega |\nabla u|^p.$$

Since $W^{1,p}$ is uniformly convex, the Radon-Riesz theorem finishes the existence proof.

The proof that u is an eigenfunction is standard for variational problems, so I only provide a rough sketch here. Simply observe that for any $v \in W^{1,p}$,

$$\frac{d}{dt}\int_\Omega |u+tv|^p = p\int_\Omega |u+tv|^{p-1}v$$

and

$$\frac{d}{dt}\int_\Omega |\nabla u+\nabla tv|^p = p\int_\Omega |\nabla u+\nabla tv|^{p-2}[\nabla u+\nabla tv]\cdot\nabla v.$$

Comparing these at $t=0$ proves that the equation is satisfied for all such v, since otherwise there would be some small number ϵ such that $u+\epsilon v$ contradicts the minimality of λ_p. If u minimizes the Rayleigh quotient, then so does $|u|$. Hence there is a first eigenfunction which does not change signs. □

The following theorem states an important uniqueness property of the principal eigenfunction.

Theorem 2. *If u is an eigenfunction with eigenvalue λ_p, and v is any eigenfunction of constant sign, then $u = kv$ for some k.*

The proof is omitted since it is quite technical and not particularly instructive, being based on direct computations and the convexity of the map $x \to |x|^p$. It can be found in [4].

Cheeger Estimates

For a domain $\Omega \in \mathbb{R}^n$, there is a result due to Cheeger which relates a certain geometric quantity to the lowest eigenvalue of the Laplacian. In particular, given a subdomain $D \subset \Omega$ which does not touch $\partial\Omega$, define its Cheeger quotient to be

$$Q(D) = \frac{H_{n-1}(\partial D)}{H_n(D))}$$

H_n is the n-dimensional Hausdorff measure of the set. Define the Cheeger constant of Ω to be

$$h(\Omega) = \inf_{D\subset\Omega} Q(D).$$

Cheeger proved the lower bound

$$\lambda_2 \geq \frac{h(\Omega)^2}{4}.$$

for the first eigenvalue of the ordinary Laplacian. It turns out that this estimate generalizes very easily to arbitrary p, which is the content of the following theorem. This theorem, and the rest of the results from this section, can be found in [2]. The proof makes use of a special case of the coarea formula, namely that for smooth functions u

$$\int_\Omega |\nabla u| = \int_{\mathbb{R}}\int H_{n-1}(u^{-1}(t))dt.$$

Theorem 3. *The first eigenvalue λ_p of the p-Laplacian on the domain Ω satisfies*

$$\lambda_p \geq \left(\frac{h(\Omega)}{p}\right)^p.$$

Proof. We begin by approximating by smooth functions. Let $w \in C^\infty(\Omega)$ and let $A_t = \{x \in \Omega : w(x) \leq t\}$. Apply the coarea formula to obtain

$$\int_\Omega |\nabla w| = \int_{\mathbb{R}} \frac{H_{n-1}(\partial A_t)}{H_n(A_t)} H_n(A_t)dt = \int_{\mathbb{R}} Q(A_t)H_n(A_t) \geq h(\Omega) \int_\Omega |w|. \tag{4}$$

By approximation (4) holds on $W^{1,1}$. To relate this $W^{1,1}$ estimate to general p, choose $v \in W^{1,p}$ and let $u = |v|^{p-1}v$. Chain rule and Hölder's inequality lead to

$$||\nabla u||_{L^1} \leq p||v||_{L^p}^{p-1}||\nabla v||_{L^p}$$

and in particular $u \in W^{1,1}$. Then (4) applies and gives

$$h(\Omega) \int_\Omega |v|^p \leq \int_\Omega |\nabla u| \leq p||v||_{L^p}^{p-1}||\nabla v||_{L^p}$$

from which the result follows by a simple rearrangement. □

With this theorem proved, we can obtain a lower bound on $\lambda_p(\Omega)$ provided we can compute the Cheeger constant, though this is a non-trivial task. The best results are found in the case of simply connected planar domains, such as in [3]. A common approach to the computation of this constant is to look for Cheeger domains, which are defined to be $D \subset \Omega$ such that $Q(D) = h(\Omega)$. Note that such minimal domains are not required to be compactly contained, and may touch $\partial\Omega$. There are many results attempting to characterize such subdomains. I present a few such results here without proof.

Theorem 4. *(i) A Cheeger domain is C^1 if Ω is C^1, and analytic except on a set of H_{n-1}-measure zero*

(ii) The boundary of a Cheeger domain has constant mean curvature $h(\Omega)$ wherever it does not touch $\partial\Omega$

(iii) The union of all balls of radius $1/h(\Omega)$ is a Cheeger domain for any convex planar domain.

It is reasonable to wonder how good of an estimate the Cheeger bound is. As p becomes large, this relationship becomes more difficult to understand. On the other hand, when p is very close to 1, the estimate is very good, and we will make this fact rigorous with the next theorem. Observe that

$$\left(\frac{h(\Omega)}{p}\right)^p \to h(\Omega),\ p \searrow 1.$$

The Cheeger lower bound is sharp in this limit.

Theorem 5. *For any domain Ω we have* $\lim_{p\to 1} \lambda_p(\Omega) = h(\Omega)$.

Proof. Let D be compactly contained in Ω, and let w be a function satisfying $w = 1$ on D_k, $w = 0$ outside an ϵ-neighbourhood of D, and $|\nabla w| = 1/\epsilon$ in between. For sufficiently small ϵ, $w \in W_0^{1,\infty}(\Omega)$ and is thus an admissible function for the variational problem. Returning once more to the Rayleigh quotient,

$$\lambda_p \leq \frac{\int_\Omega |\nabla w_k|^p}{\int_\Omega |w_k|^p} \leq \frac{H_{n-1}(\partial D_k)}{H_n(D_k)}\epsilon^{1-p}.$$

Sending $p \to 1$ gives

$$\lim_{p\to 1} \lambda_p \leq Q(D)$$

and since this holds for every D we have the result. □

Therefore we should expect the existence of an eigenvalue very close to $(h(\Omega)/p)^p$ when p is close to 1.

Nodal Sets

The remainder of this note will be dedicated to proving a theorem on the size of nodal sets of eigenfunctions in a planar domain. Given an eigenvalue λ, let Z_λ be the set of zeroes of the associated eigenfunction. For the classical Laplacian $p = 2$, it is known that the nodal set always has empty interior. Furthermore, Yau's conjecture states that

$$\exists c, C : c\lambda^{\frac{1}{2}} \leq H_{n-1}(Z_\lambda) \leq C\lambda^{\frac{1}{2}}.$$

This conjecture has not been proven in the general case, but has been shown to hold for many special cases. For simply connected domains in the plane $\mathbb{R}^2$, we will prove one side of a generalization, namely

$$\exists c : H_1(Z_\lambda) \geq c\lambda^{\frac{1}{p}}. \tag{5}$$

Note that the upper bound is not proven to hold, in fact it is not known that the nodal set of an arbitrary eigenfunction has empty interior for general p. All the arguments that follow will be based on the assumption that Z_λ is a union of curves, though this is not currently known to be true. The theorem will still hold, however, since the H_1 measure of an set of nonempty interior is infinite.

The proof of the generalized Yau Conjecture follows [5]. First, we need a theorem which says roughly that for sufficiently large λ, nodes appear frequently. In particular,

Theorem 6. *Let $R = \left(\frac{C}{\lambda}\right)^{1/p}$ for some $C > \lambda'_p$ where λ'_p is the first eigenvalue of Δ_p on the unit disc. If $B_R \subset \Omega$, then any eigenfunction with eigenvalue λ vanishes in B_R.*

Proof. Observe that the first eigenvalue on a domain Ω is not greater than the first eigenvalue on another $\Omega' \subset \Omega$. This can be seen from the fact that the first eigenfunction on Ω' is admissible in the Rayleigh quotient for Ω, after smooth approximation if necessary. Now let A be any nodal domain of u_λ which contains a ball of radius B_R. It follows that u_λ is the first eigenfunction on A, since the first eigenfunction is the only one of constant sign. Then since

$$\lambda = \lambda_{1,p}(A) \leq \lambda_{1,p}(B_R) \geq \frac{\lambda_{1,p}(B_1)}{R^p}$$

which contradicts our assumption on the radius R. Thus there is no nodal domain containing any B_R, or equivalently the eigenfunction vanishes in every $B_R \subset \Omega$. □

As a side note, if we want to make $\lambda_{1,p}(B_1)$ more explicit, the exact Cheeger estimate for B_1 is quite easy to derive using the isoperimetric inequality. For the next step, we will need a version of the Harnack inequality for quasilinear elliptic equations.

Theorem 7. *If $K(\rho)$ is a square of side length ρ compactly contained in Ω and u_λ is an eigenvalue of the p-Laplacian which is nonnegative on K, then*

$$\sup_{x \in K(\rho)} u_\lambda \leq C \inf_{x \in K(\rho)} u_\lambda \tag{6}$$

for some C which depends only on p, λ, and K.

To avoid this note becoming extremely long, the proof is omitted, but it is due to Trudinger and can be found in [6].

With this quasilinear Harnack inequality, we can finally prove the generalized Yau conjecture for planar domains.

Proof. We can cover Ω with squares of side length proportional to $\lambda^{-\frac{1}{p}}$ each of which contains a zero of u_λ. Assume that λ is so large that we can make the centre of each square a zero of the eigenfunction. Furthermore, each zero lies on a curve of nodes which partitions the square in two, otherwise the Harnack inequality would show that the eigenfunction is zero on the entire square, in which case $H_1(Z_\lambda) = \infty$.

There are two possible types of curve which bisect a square and also intersect the centre. One is a curve that touches two boundaries of the square and the centre of the square. The other is a curve which is closed and completely contained in the square. A completely enclosed curve is a contradiction by the same type of domain monotonicity argument used to prove that u_λ vanishes in B_R. Thus all nodal curves are of the first type, and in particular are at least of length equal to $\lambda^{-\frac{1}{p}}$, the side length of the square. Combining this with the fact that it takes aymptotically at least $Area(\Omega)\lambda^{2/p}$ squares to cover Ω, we obtain the desired result. $\square$

References

[1] L Evans. *Partial Differential Equations.* American Mathematical Society.

[2] B Kawohl and V Fridman. Isoperimetric estimates for the first eigenvalue of the p-laplace operator and the cheeger constant. *Commentations Mathematicae Universitatis Carolinae*, 2003.

[3] Gian Paolo Leonardi, Robin Neumayer, and Giorgio Saracco. The cheeger constant of a jordan domain without necks. *Calculus of Variations and Partial Differential Equations*, 56(6), 2017.

[4] Peter Lindqvist. On the equation $div(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u$. *Proceedings of the American Mathematical Society*, 109(1):157, 1990.

[5] Guillaume Poliquin. Bounds on the principal frequency of the p-laplacian. *Geometric and Spectral Analysis Contemporary Mathematics*, page 349366, 2014.

[6] Neil S. Trudinger. On harnack type inequalities and their application to quasilinear elliptic equations. *Communications on Pure and Applied Mathematics*, 20(4):721747, 1967.