MATH 581 ASSIGNMENT 5

DUE THURSDAY APRIL 11

1. Consider the Cauchy problem

$$\partial_t u = \sum_{k=1}^{n} A_k \partial_k u, \qquad u|_{t=0} = f,$$

where u is a vector function with m components, each A_k is a (possibly complex) $m \times m$ matrix, and f is a given (vector) function. We say that the problem is *strongly well-posed* if for any $f \in L^2$, there exists a solution $u \in C^0(\overline{\mathbb{R}}_+, L^2)$, which satisfies the estimate

$$\|u(t)\|_{L^2} \leq C e^{\alpha t} \|f\|_{L^2}, \qquad t \geq 0,$$

with some constants α and C, and u is the only solution in $C^0(\overline{\mathbb{R}}_+, L^2)$. In each of the following cases, prove that the corresponding Cauchy problem is strongly well-posed.
(a) Symmetric hyperbolic: All A_k are Hermitian.
(b) Strictly hyperbolic: For all nonzero $\xi \in \mathbb{R}^n$, the eigenvalues of $P(\xi) = \sum_{k=1}^{n} A_k \xi_k$ are real and distinct.

2. The Maxwell equations for 3 dimensional electromagnetism in vacuum are

$$\partial_t E = \nabla \times B, \qquad \partial_t B = -\nabla \times E, \tag{1}$$

and

$$\nabla \cdot E = 0, \qquad \nabla \cdot B = 0, \tag{2}$$

where $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ are the electric and magnetic fields, respectively. Show that the system (1) is symmetric hyperbolic. Then show that the constraints (2) are preserved by the evolution, i.e., that if one starts with initial data satisfying the constraints (2), and if E and B evolve according to (1), then (2) will be satisfied for all time.

3. Prove the strong well-posedness of the Cauchy problem for the system

$$\partial_t u = P(\partial) u + Q(\partial) v,$$
$$\partial_t v = H(\partial) v + M u,$$

where u and v are vector functions, $P(\partial)$ is a second order parabolic operator, $Q(\partial)$ is an arbitrary first order operator, $H(\partial)$ is a first order symmetric hyperbolic operator, and M is simply a matrix (i.e., a zeroth order operator). The operators $P(\partial)$, $Q(\partial)$, and $H(\partial)$ may contain lower order terms, and the spatial dimension is n.

Date: Winter 2019.

4. For isotropic and homogeneous materials, the elastodynamics equations are given by

$$\partial_t^2 u = \mu\Delta u + \lambda\nabla(\nabla\cdot u),$$

where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the displacement field, and μ and λ are real parameters, satisfying $\mu > 0$ and $\mu + \lambda > 0$. In components, the equations read

$$\partial_t^2 u_k = \mu\Delta u_k + \lambda\partial_k(\partial_1 u_1 + \ldots \partial_n u_n), \qquad k = 1, \ldots, n.$$

We consider the corresponding Cauchy problem with the initial data $u|_{t=0} = f$ and $\partial_t u|_{t=0} = g$. Show that the Cauchy problem is well-posed in the following sense: For any initial data $(f, g) \in H^s \times H^{s-1}$ with some $s \in \mathbb{R}$, there exists a unique solution $u \in \mathscr{C}(\mathbb{R}, H^s) \cap \mathscr{C}^1(\mathbb{R}, H^{s-1})$, satisfying

$$\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \leq C(\|f\|_{H^s} + \|g\|_{H^{s-1}}), \qquad t \in \mathbb{R},$$

for some constant $C > 0$.