MATH 581 ASSIGNMENT 4

DUE THURSDAY MARCH 28

1. (a) Let $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$. Show that
$$\partial^\alpha(u * v) = (\partial^\alpha u) * v = u * (\partial^\alpha v).$$
   (b) Let $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$. Show that
$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v.$$
   (c) Show that
$$1 * (\delta' * \vartheta) \neq (1 * \delta') * \vartheta,$$
   where 1 is the function identically 1 in $\mathbb{R}$, and ϑ is the Heaviside step function.
   (d) Let u and v be the surface measures of the spheres $\{x \in \mathbb{R}^3 : |x| = a\}$ and $\{|x| = b\}$, respectively. Compute $u * v$, and determine its singular support.
2. Prove that $\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{D}'(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open.
3. Show that any distribution satisfying the Laplace equation must be smooth (therefore harmonic in the classical sense).
4. (a) Let $a \in \mathscr{E}(\mathbb{R}^n)$. Prove that the pointwise multiplication $u \mapsto au : \mathscr{S}' \to \mathscr{S}'$ is well-defined and continuous if $a \in \mathscr{O}_M$, that is, for every multi-index α there is a polynomial p such that $|\partial^\alpha a(x)| \leq p(x)$, $x \in \mathbb{R}^n$.
   (b) Prove that if p is a polynomial with no real zeroes, then there are constants $c > 0$ and m such that $|p(\xi)| \geq c(1+|\xi|)^m$ for all $\xi \in \mathbb{R}^n$. Operators $p(D)$ with p satisfying this condition are called *strictly elliptic*.
   (c) Show that if $p(D)$ strictly elliptic, then the equation $p(D)u = f$ has a solution for each $f \in \mathscr{S}'$.
5. Let p be a nonzero polynomial. Show the following.
   a) The equation $p(D)u = f$ has at least one smooth solution for every $f \in \mathscr{D}$.
   b) If all solutions of $p(D)u = 0$ are smooth, then $p(D)$ is hypoelliptic.
   c) If $p(D)$ admits a fundamental solution that is smooth outside some ball of finite radius (centred at the origin), then $p(D)$ is hypoelliptic.
6. Recall Hörmander's theorem that $p(D)$ is hypoelliptic if and only if for any $\eta \in \mathbb{R}^n$ one has $p(\xi + i\eta) \neq 0$ for all sufficiently large $\xi \in \mathbb{R}^n$. Apply this criterion to show the following.
   (a) All elliptic operators are hypoelliptic.
   (b) The wave operator is not hypoelliptic.
   (c) The heat operator is hypoelliptic.

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7. For $s \in \mathbb{R}$, the (Bessel potential) *Sobolev space* $H^s(\mathbb{R}^n)$ is the set of those $u \in \mathscr{S}'(\mathbb{R}^n)$ with $\|u\|_{H^s} := \|\langle D\rangle^s u\|_{L^2} < \infty$, where the *Bessel potential* $\langle D\rangle^s u$ of u is defined by
$$\widehat{\langle D\rangle^s u}(\xi) = \langle \xi\rangle^s \hat{u}(\xi) \equiv (1 + |\xi|^2)^{s/2}\hat{u}(\xi).$$
Prove the following.
(a) $\langle D\rangle^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert space isometry.
(b) For $k \geq 0$ integer, $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.
(c) $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
(d) The (topological) dual of $H^s(\mathbb{R}^n)$ is isometric to $H^{-s}(\mathbb{R}^n)$.
(e) The *trace operator* $\gamma : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^{n-1})$ defined by
$$(\gamma u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0),$$
has a unique extension to a bounded linear operator $\gamma : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.