MATH 581 ASSIGNMENT 3

DUE TUESDAY MARCH 12

1. Derive the analogue of the Kirchhoff formula for the solution of the Cauchy problem for the homogenous wave equation in $\mathbb{R}^n$, for general spatial dimension n.
2. Consider the variable coefficient wave equation
$$\partial_t^2 u - \sum_{i,k=1}^{n} a_{ik}\partial_i\partial_k u = 0,$$
where $[a_{ik}(x)]$ is a symmetric, positive definite matrix, depending smoothly on $x \in \mathbb{R}^n$. Suppose that $Q \subset \mathbb{R}^{n+1} \subset \mathbb{R}^n \times (0,T)$ is a domain, and introduce the notations
$$S_t = \bar{Q} \cap \big(\mathbb{R}^n \times \{t\}\big), \qquad \partial^* Q = \partial Q \setminus (S_0 \cup S_T).$$
By using the multiplier method, establish a local energy inequality of the form
$$E(S_T) \leq E(S_0),$$
where
$$E(S) = \frac{1}{2}\int_S \Big(|\partial_t u|^2 + \textstyle\sum_{ik} a_{ik}(\partial_i u)(\partial_k u)\Big),$$
under a suitable condition on the lateral boundary $\partial^* Q$. What is the local speed of information propagation?
3. Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain with C^1 boundary, and let $\phi, \psi \in C^\infty(\Omega) \cap C_0(\bar{\Omega})$. Here $C_0(\bar{\Omega})$ indicates the class of functions with homogenous Dirichlet boundary condition, and note that there is no growth restriction on ϕ and ψ at ∞. Show that there exists a unique smooth solution to the initial-boundary value problem
$$\begin{aligned} \Box u + u &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \\ u(\cdot, 0) &= \phi, \\ \partial_t u(\cdot, 0) &= \psi. \end{aligned}$$
4. Consider the Cauchy problem
$$\begin{aligned} \Box u &= 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \\ u(\cdot, 0) &= 0, \\ \partial_t u(\cdot, 0) &= \psi, \end{aligned}$$
and denote the map $\psi \to \partial_t u(\cdot, t)$ by $W'(t) : \mathscr{D}(\mathbb{R}^3) \to \mathscr{D}(\mathbb{R}^3)$. Show that $W'(t)$ cannot be extended to a bounded operator $W'(t) : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ for any $p \neq 2$ and $t > 0$.

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5. Let p be a nontrivial polynomial of n variables, and let f be a real analytic function in a neighbourhood of $0 \in \mathbb{R}^n$.
   a) Prove that the set $\{\xi \in \mathbb{R}^n : p(\xi) = 0\}$ is closed and of measure zero.
   b) Show that there is a neighbourhood of $0 \in \mathbb{R}^n$, on which the equation $p(\partial)u = f$ has a solution. Supposing that $p(\xi) = \sum_\alpha a_\alpha \xi^\alpha$, here the operator $p(\partial)$ is given by
   $$p(\partial) = \textstyle\sum_\alpha a_\alpha \partial^\alpha.$$
   c) Extend this local solvability result to linear operators with analytic coefficients. That is, assuming that $\{a_\alpha\}$ is a finite collection of real analytic functions in a neighbourhood of $0 \in \mathbb{R}^n$, with the property that $p(\xi) = \sum_\alpha a_\alpha(0)\xi^\alpha$ is a nontrivial polynomial, show that the equation
   $$\textstyle\sum_\alpha a_\alpha \partial^\alpha u = f,$$
   has a solution on a neighbourhood of $0 \in \mathbb{R}^n$.
6. Let p be a nontrivial polynomial of n variables, and let $H \subset \mathbb{R}^n$ be a (closed) half-space.
   a) Show that if $u \in C^\infty(\mathbb{R}^n)$ satisfies $p(\partial)u = 0$ in $\mathbb{R}^n$ and $\operatorname{supp} u \subset H$, and if the boundary of H is noncharacteristic for the constant coefficient operator $p(\partial)$, then $u \equiv 0$. Provide a counterexample when ∂H is characteristic and p is a nonconstant homogeneous polynomial.
   b) Show that if we require that u is compactly supported, then the noncharacteristic condition on ∂H can be dropped, i.e., prove that if $u \in C_c^\infty(\mathbb{R}^n)$ satisfies $p(\partial)u = 0$ in $\mathbb{R}^n$ then $u \equiv 0$. Imply that if $u \in C_c^\infty(\mathbb{R}^n)$ then $\operatorname{supp} u$ is contained in the convex hull of $\operatorname{supp} p(\partial)u$.