MATH 581 ASSIGNMENT 2

DUE THURSDAY FEBRUARY 14

1. Let $\varphi \in \mathscr{D}(\mathbb{R})$, $\varphi \neq 0$, and $\varphi(0) = 0$. In each of the following cases, decide if $\varphi_j \to 0$ as $j \to \infty$ in $\mathscr{D}(\mathbb{R})$. Does it hold $\varphi_j \to 0$ pointwise or uniformly?
   (a) $\varphi_j(x) = j^{-1}\varphi(x - j)$;
   (b) $\varphi_j(x) = j^{-n}\varphi(jx)$, where $n > 0$ is an integer.
2. In each case, show that f defines a distribution on $\mathbb{R}^2$, and find its order.
   (a) $f(\varphi) = \int_{\mathbb{R}^2} |x|^{-1}e^{|x|^2}\varphi(x)\mathrm{d}x$;
   (b) $f(\varphi) = \int_{\mathbb{R}} \varphi(s, 0)\mathrm{d}s$;
   (c) $f(\varphi) = \int_0^1 \partial_1\varphi(0, s)\mathrm{d}s$.
3. Compute the following derivatives in the sense of distributions.
   (a) $\partial_x|x|$;
   (b) $\partial_x \log|x|$;
   (c) $\partial_2 f$, where $f \in \mathscr{D}'(\mathbb{R}^2)$ is the distribution from (b) of the previous exercise.
4. Prove the following.
   (a) $\partial_j(au) = (\partial_j a)u + a(\partial_j u)$ for $a \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$.
   (b) There is no distribution on $\mathbb{R}$ such that its restriction to $\mathbb{R} \setminus \{0\}$ is $e^{1/x}$.
5. If u is the characteristic function of the unit ball in $\mathbb{R}^n$, compute $x \cdot \nabla u$.
6. Let $\omega \subset \mathbb{R}$ be an open interval, and $u \in \mathscr{D}'(\omega)$. Let $0 \leq k \leq \infty$.
   (a) Show that if $u' = 0$ then u is a constant function.
   (b) Show that if $u' = f$ with $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
   (c) Show that if $u' + au = f$ with $a \in C^\infty(\omega)$ and $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
7. Find the limits $n \to \infty$ of the following sequences in $\mathscr{D}'(\mathbb{R})$.
   a) $n\phi(nx)$, where ϕ is a nonnegative continuous function whose integral over $\mathbb{R}$ is finite.
   b) $n^k \sin nx$, where $k > 0$ is a constant.
   c) $x^{-1}\sin nx$.
8. For each of the following functions, determine if it is a tempered distribution, and if so compute its Fourier transform.
   (a) $x \sin x$,
   (b) $\frac{1}{x}\sin x$,
   (c) $e^{i|x|^2}$,
   (d) $x\vartheta(x)$, where ϑ is the Heaviside step function,
9. Give an example of $u \in \mathscr{C}(\mathbb{R}^n)$ such that $\varphi \mapsto \int u\varphi$ is a tempered distribution and that there is no polynomial p satisfying $|u(x)| \leq |p(x)|$ for all $x \in \mathbb{R}^n$.

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