MATH 581 ASSIGNMENT 1

DUE THURSDAY JANUARY 31

1. Consider the function $v(x,t) = \frac{x}{t}E(x,t)$ for $x \in \mathbb{R}$ and $t > 0$, where
$$E(x,t) = \frac{\theta(t)e^{-x^2/4t}}{2\sqrt{\pi t}},$$
is the heat kernel of $\mathbb{R}$. Show that $\partial_t v = \Delta v$ in $\mathbb{R} \times (0,\infty)$, and that $v(x,t) \to 0$ as $t \to 0^+$ for each fixed $x \in \mathbb{R}$. How do we reconcile this with Tychonov's uniqueness theorem?
2. By using the mean value property, formulate and prove a strong maximum principle for caloric functions in general spacetime domains.
3. Formulate and prove a weak maximum principle for functions satisfying
$$\partial_t u = \Delta u + \sum_{i=1}^{n} b_i \partial_i u + cu,$$
on (spacetime) cylindrical domains, where b_i and c are bounded continuous functions. Can you relax the continuity assumptions on b_i and c?
4. With $\Omega \subset \mathbb{R}^n$ a bounded smooth domain, consider the initial-boundary value problem
$$\begin{cases} \partial_t u = \Delta u + au & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = g & \text{on} \quad \Omega \times \{0\}, \end{cases}$$
where $g \in C(\bar{\Omega})$ and $a \in C^\infty(\bar{\Omega})$ are given functions.
   (a) Show the existence of a solution u in the class $C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$.
   (b) Show that the solution is unique in the same class.
   (c) Assuming $a \equiv 0$, show that
   $$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq (4\pi t)^{-\frac{n}{2}} \|g\|_{L^1(\Omega)}, \qquad \text{for all} \quad t > 0.$$
   (d) Show that there exists $c > 0$ with the property that if $\|a\|_\infty \leq c$ then the L^2-norm of $u(\cdot,t)$ decays exponentially in time.
   (e) Under some smallness condition on a, can you establish an exponential decay in stronger norms, such as H^k or L^∞?

Date: Winter 2019.

5. (*Duhamel's principle*) Recall the Green representation formula

$$u(y,s) = \int_Q E^*_{y,s} Hu + \int_{\partial^- Q} u E^*_{y,s} + \int_{\partial^* Q} (E^*_{y,s}\partial_\nu u - u\partial_\nu E^*_{y,s}), \qquad (y,s) \in Q,$$

for $u \in C^2(\bar{Q})$, where

$$H = \partial_t - \Delta$$

is the heat operator,

$$E^*_{y,s}(x,t) = E^*(x-y,t-s) = E(x-y,s-t) = \frac{\theta(s-t)}{(4\pi(s-t))^{n/2}} \exp\big(-\frac{|x-y|^2}{4(s-t)}\big)$$

is the backward heat kernel centred at (y,s), and

$$Q = \Omega \times (0,T), \qquad \partial^* Q = \partial\Omega \times (0,T), \qquad \partial^- Q = \Omega \times \{0\},$$

with $\Omega \subset \mathbb{R}^n$ a C^1 domain, and $T > 0$. The vector ν is the outward unit normal to $\partial\Omega$. If we formally take $Q = \mathbb{R}^n$ in the formula, we get

$$u(y,s) = \int_{\mathbb{R}^n} E(x-y,s)g(x)d^n x + \int_0^s \int_{\mathbb{R}^n} E(x-y,s-t)f(x,t)d^n x\, dt$$

for $y \in \mathbb{R}^n$ and $s \in (0,T)$, where $f = Hu$ and $g = u(\cdot,0)$. This can be rewritten as

$$u(\cdot,s) = e^{s\Delta} g + \int_0^s [e^{(s-t)\Delta} f(\cdot,t)]\, dt, \qquad (0 < s < T), \qquad (*)$$

where the heat propagator $e^{t\Delta}$ is defined by

$$[e^{t\Delta} g](y) = \int_{\mathbb{R}^n} E(x-y,t)g(x)d^n x.$$

(a) Suppose that $u \in C^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$, and for each $0 < a < T$, the derivatives $\partial_t u$, $\partial_i u$, and $\partial_i\partial_k u$ are bounded in $\mathbb{R}^n \times [a,T]$. Show that u satisfies $(*)$ with $f = Hu$ and $g = u(\cdot,0)$.

(b) Let $g \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$, with $\partial_i f$ bounded and continuous in $\mathbb{R}^n \times [a,T]$ for each $0 < a < T$. Show that u given by $(*)$ satisfies

$$\begin{cases} \partial_t u = \Delta u + f & \text{for} \quad 0 < t < T, \\ u = g & \text{for} \quad t = 0. \end{cases}$$

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.

Hand in your work in class, before the lecture on the due date. Email submissions are accepted only if you type your solutions in LaTeX. Please do not use any other means without discussing it with the instructor (In particular, we no not have a homework box). Please try not to use extensions but if you need, you can get individual extensions by sending me an email preferably well in advance of the due date stating your proposed new deadline.