De Giorgi-Nash-Moser's theorem

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MATH 581 - Partial Differential Equations

April 22, 2019

De Giorgi-Nash-Moser's regularity theorem

Theorem 1. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of*

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \right) = 0 \tag{1}$$

assuming that the measurable and bounded coefficients $a_{i,j}$ satisfies the structural conditions,

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j} \xi_i \xi_j, \quad |a_{i,j}(x)| \leq \Lambda, \tag{2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, with constants $0 < \lambda < \Lambda < \infty$. Then u is Hölder continuous in Ω. More precisely, for any $\omega \subset\subset \Omega$, there exist some $\alpha \in (0,1)$ and a constant C with

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}, \tag{3}$$

for all $x, y \in \omega$. α depends on n, $\frac{\Lambda}{\lambda}$ and ω, C in addition on $\operatorname{Osc}_{\omega}(u) := \sup_{\omega}(u) - \inf_{\omega}(u)$.

Preliminary H^1 bound

Proposition 1. *Let $u \in W^{1,2}(\Omega)$ satisfying to the problem (1) on the ball $B_1 \subset\subset \Omega$, then we have the following gradient estimate*

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C_1(n, \lambda, \Lambda) \|u\|_{L^2(B_1)}, \tag{4}$$

where C_1 is a constant.

Proof. Choose η a cut-off function such that

$$\begin{cases} \eta = 1 & \text{in} \quad B_{1/2}, \\ 0 \leq \eta \leq 1 & \text{in} \quad B_1, \\ \eta = 0 & \text{in} \quad B_1^c. \end{cases} \tag{5}$$

for which $\nabla \eta$ is bounded and $\|\nabla \eta\|_{L^\infty}$ depends only on n. We can write the first intermediate estimate

$$\int_{B_{1/2}} |\nabla u|^2 \leq \int_{B_1} \eta^2 |\nabla u|^2 \leq \frac{1}{\lambda} \int_{B_1} \eta^2 \sum_{i,j=1}^{n} a_{i,j} \partial_i u \partial_j u.$$

Then recalling that u satisfies $Lu = 0$ in B_1, we get from integration by parts that

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq \frac{2}{\lambda} \int_{B_1} |\eta u| \sum_{i,j=1}^{n} |a_{i,j} \partial_i u \partial_j \eta| \leq 2 \frac{\Lambda}{\lambda} \int_{B_1} |u \nabla \eta| \cdot |\eta \nabla u|,$$

from which we can deduce after applying Cauchy-Schwarz inequality that

$$\left(\int_{B_1} \eta^2 |\nabla u|^2 \right)^{\frac{1}{2}} \leq 2 \frac{\Lambda}{\lambda} \left(\int_{B_1} |u \nabla \eta|^2 \right)^{\frac{1}{2}} \leq 2 \frac{\Lambda}{\lambda} \|\nabla \eta\|_{L^\infty} \left(\int_{B_1} |u|^2 \right)^{\frac{1}{2}}.$$

After squaring the last inequality we obtain the gradient estimate

$$\int_{B_{1/2}} |\nabla u|^2 \leq \int_{B_1} \eta^2 |\nabla u|^2 \leq 4 \left(\frac{\Lambda}{\lambda} \right)^2 \|\nabla \eta\|_{L^\infty}^2 \left(\int_{B_1} |u|^2 \right), \tag{6}$$

where the constant C is given by $C(n, \lambda, \Lambda) = 4 \left(\frac{\Lambda}{\lambda} \right)^2 \|\nabla \eta\|_{L^\infty}^2$. □

L^∞ bound and Moser's iterations

Definition 1 (Subsolution and supersolution)**.** A function $u \in W^{1,2}(\Omega)$ is called a weak *subsolution* (resp. *supersolution*) of L, denoted $Lu \geq 0$ (resp. $Lu \leq 0$) if for all positive functions $\phi \in H_0^{1,2}(\Omega)$, we have that

$$\int_\Omega \sum_{i,j} a_{i,j} \partial_i u \partial_j \phi \leq 0, \tag{7}$$

(resp ≥ 0 for supersolution). All the inequality are assumed to hold except possibly on sets of measure zero.

Theorem 2 (DGNM L^∞ bound)**.** *Let L satisfy* (2) *and $u \in W^{1,2}(\Omega)$ be a positive subsolution of L, i.e. $Lu \geq 0$ and $u > 0$. Then u satisfies*

$$\|u\|_{L^\infty(B_{1/2})} \leq C_2(n, \lambda, \Lambda)\|u\|_{L^2(B_1)}, \tag{8}$$

where C_2 is a constant.

Lemma 1. *Under the hypotheses of theorem 2, and if we let $1/2 \leq r \leq r + w \leq 1$ then u satisfies*

$$\|\nabla u\|_{L^2(B_r)} \leq C_3(n, \lambda, \Lambda) w^{-1} \|u\|_{L^2(B_{r+w})}, \tag{9}$$

where C_3 is a constant.

Proof. Again choose a cut-off function η such that

$$\begin{cases} \eta = 1 & \text{in} \quad B_r, \\ 0 \leq \eta \leq 1 & \text{in} \quad B_{r+w}, \\ \eta = 0 & \text{in} \quad B_{r+w}^c. \end{cases} \tag{10}$$

for which $\nabla \eta$ can be made bounded with $\|\nabla \eta\|_{L^\infty} \leq \frac{1}{w}$. Then the proof follows the exact same steps as the one done for proposition 1. □

Definition 2 (Sobolev conjugate)**.** If $1 \leq p < n$, *the Sobolev conjugate of p* is

$$p^* := \frac{np}{n-p}. \tag{11}$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p. \tag{12}$$

Theorem 3 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{13}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Lemma 2. *Under the hypotheses of theorem 2, and if we let $1/2 \leq r \leq r + w \leq 1$, then u satisfies*

$$\|u\|_{L^{2^*}(B_r)} \lesssim w^{-1} \|u\|_{L^2(B_{r+w})} \tag{14}$$

Proof. Let η be a cut-off function satisfying to the following

$$\begin{cases} \eta = 1 & \text{in} \quad B_r, \\ 0 \leq \eta \leq 1 & \text{in} \quad B_{r+w}, \\ \eta = 0 & \text{in} \quad B_{r+w}^c. \end{cases} \tag{15}$$

and for which $\nabla \eta$ is bounded with $\|\nabla \eta\|_{L^\infty} \leq \frac{1}{2w}$. Then combining the Gagliardo-Nirenberg-Sobolev inequality, proposition [1] and lemma [1] applied to ηu, we prove that

$$\|u\|_{L^{2^*}(B_r)} \leq \|\eta u\|_{L^{2^*}(B_{r+w/2})} \lesssim \|\nabla(\eta u)\|_{L^2(B_{r+w/2})} \lesssim \frac{1}{2w}\|u\|_{L^2(B_{r+w/2})} + \|\nabla u\|_{L^2(B_{r+w/2})} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})}.$$

□

Lemma 3. *If $\beta > 1$ and u is a positive subsolution of equation* (1) *i.e. $Lu \geq 0$ and $u > 0$, then u^β is also a subsolution of equation* (1).

Proof. Using the coercivity of L given in conditions (2) we have that

$$Lu^\beta = \sum_{i,j} \partial_j(a_{i,j}\partial_i(u^\beta)) = \sum_{i,j} \partial_j(a_{i,j}\beta\partial_i u u^{\beta-1}) = \beta u^{\beta-1}\sum_{i,j}\partial_j(a_{i,j}\partial_i u) + \beta(\beta-1)u^{\beta-2}\sum_{i,j} a_{i,j}\partial_i u\partial_j u,$$
$$\geq \beta u^{\beta-1}Lu + \lambda\beta(\beta-1)u^{\beta-2}|\nabla u|^2,$$

and recalling that u is positive and a subsolution of equation (1) we can conclude that $Lu^\beta \geq 0$. □

From where applying lemma 2 to u^β leads us to

$$\|u\|^\beta_{L^{\frac{2\beta n}{n-2}}(B_r)} = \|u^\beta\|_{L^{\frac{2n}{n-2}}(B_r)} \lesssim w^{-1}\|u^\beta\|_{L^2(B_{r+w})} = (Cw)^{-1}\|u\|^\beta_{L^{2\beta}(B_{r+w})}, \tag{16}$$

which, if we let $s = \frac{n}{n-2}$, gives the following result

Lemma 4. *Under the hypotheses of theorem 2, if we let $1/2 \leq r \leq r+w \leq 1$ and $p \geq 2$, then u satisfies*

$$\|u\|_{L^{sp}(B_r)} \leq (Cw^{-1})^{2/p}\|u\|_{L^p(B_{r+w})}. \tag{17}$$

Let $p \in \mathbb{R}$, $R > 0$, $x_0 \in \Omega$ and take $u \in L^p(B_R(x_0))$ positive, we define then the function Φ such that

$$\Phi(p,R) := \left(\fint_{B_R(x_0)} u^p\right)^{\frac{1}{p}}. \tag{18}$$

Lemma 5.

$$\lim_{p\to\infty}\Phi(p,R) = \sup_{B(x_0,R)} u := \Phi(\infty,R), \tag{19}$$

$$\lim_{p\to-\infty}\Phi(p,R) = \inf_{B(x_0,R)} u := \Phi(-\infty,R). \tag{20}$$

Proof. The function $\Phi(\cdot,R)$ is monotonically increasing. Indeed, using Hölder's inequality we have that for any $p < p'$ and $u \in L^{p'}(\Omega)$

$$\left(\fint_\Omega u^p\right)^{\frac{1}{p}} \leq \frac{1}{|\Omega|^{1/p}}\left(\int_\Omega 1^{\frac{p'}{p'-p}}\right)^{\frac{p'-p}{pp'}}\left(\int_\Omega u^{p'}\right)^{\frac{p}{p'p}} = \left(\fint_\Omega u^{p'}\right)^{\frac{1}{p'}}. \tag{21}$$

Moreover, by definition of the essential supremum we know that for any $\varepsilon > 0$ there exists $\delta > 0$ such that,

$$|A_\varepsilon| := \left|\left\{x \in B(x_0,R) : u(x) \geq \sup_{B(x_0,R)} u - \varepsilon.\right\}\right| > \delta \tag{22}$$

Therefore we can bound Φ below as follow

$$\left(\fint_{B(x_0,R)} u^p\right)^{\frac{1}{p}} \geq \frac{1}{|B(x_0,R)|^{1/p}}\left(\int_{A_\varepsilon} u^p\right)^{\frac{1}{p}} \geq \left|\frac{\delta}{B(x_0,R)}\right|^{\frac{1}{p}}(\sup_{B(x_0,R)} u - \varepsilon), \tag{23}$$

hence

$$\lim_{p\to\infty}\Phi(p,R) \geq \sup_{B(x_0,R)} u - \varepsilon. \tag{24}$$

Combining the results (21) and (24), we prove (19), and (20) follows immediately by replacing u with u^{-1}. □

We are ready now to prove the DeGiorgi-Nash-Moser L^∞ bound.

Proof. Consider a sequence of balls such that

$$B(0,1/2) \subset \cdots \subset B(0,r_{k+1}) \subset B(0,r_k) \subset \cdots \subset B(0,r_0) = B(0,1) \subset\subset \Omega, \tag{25}$$

i.e. $1/2 \leq r_k \leq 1$ for every $k \geq 0$. For instance, one can choose $r_k = \frac{1}{2} + \frac{1}{2(k+1)}$ so that $r_{k+1} - r_k = O(\frac{1}{k^2})$. From here we use Moser's technique which consist of iterating the result of lemma 4 in order to trap higher L^p norms,

$$\|u\|_{L^2(B_1)} \geq A_0\|u\|_{L^{2s}(B_{r_1})} \geq \cdots \geq A_0\cdots A_{k-1}\|u\|_{L^{2s^k}(B_{r_k})}, \tag{26}$$

where $A_k = (C(r_k - r_{k-1})^{-1})^{s^{-k}}$. Nonetheless, we remark that

$$\log(\prod_{k=0}^{N} A_k) = \sum_{k=0}^{N} s^{-k} \log(C(r_k - r_{k-1})), \tag{27}$$

is the partial sum of a convergent series since

$$s^{-k} \log(C(r_k - r_{k-1})) = O\left(\frac{\log(k)}{s^k}\right). \tag{28}$$

Hence, combining lemma 5 and the previous remark, we can take the limit in both sides of equation (26) and prove that there exists a constant C such that

$$\|u\|_{L^\infty(B_{1/2})} \leq C(n, \lambda, \Lambda)\|u\|_{L^2(B_1)}. \tag{29}$$

□

Moser-Harnack's inequality

Theorem 4 (Moser-Harnack's inequality). *Let u be a positive weak solution to Lu=0 in a domain Ω of $\mathbb{R}^n$, and let $\omega \subset\subset \Omega$. Then*

$$\sup_\omega u \leq c \inf_\omega u \tag{30}$$

with c depending on n, ω, Ω and $\frac{\Lambda}{\lambda}$.

Theorem 5 (Weak Moser-Harnack's inequality). *If the elliptic operator L satisfies the conditions* (2)*, u weak solution of Lu=0 such that* $0 < u < 1$ *on* B_1 *and*

$$\left|\left\{x \in B_{1/2} : u(x) > 1/10\right\}\right| \geq \frac{1}{10}\left|B_{1/2}\right|, \tag{31}$$

then,

$$\inf_{B_{1/2}} u \geq \gamma, \tag{32}$$

where γ depends on n, and $\frac{\Lambda}{\lambda}$.

Lemma 6. *If $u \in W^{1,2}(\Omega)$ is a weak solution of L and k is some real number, then the function v defined by*

$$v = \max(u, k)$$

is also a weak subsolution to L.

Corollary 1. *Let u be a weak solution to Lu=0 on Ω and let $r > 0$ and $x \in \Omega$ such that $B_r(x) \subset \Omega$, then*

$$\operatorname*{Osc}_{B_{r/2}(x)} u \leq (1-\gamma) \operatorname*{Osc}_{B_r(x)} u. \tag{33}$$

Proof. The key to this proof rely a scaling argument. Indeed, without lost of generality, since u is bounded, we can assume that

$$\inf_{B_r(x)} u = 0, \quad \sup_{B_r(x)} u = 1, \quad r = 1,$$
$$\left|\left\{x \in B_{1/2} : u(x) \geq t\right\}\right| \geq t\left|B_{1/2}\right|.$$

Then using the weak Moser-Harnack's inequality, we readily verify that

$$\operatorname*{Osc}_{B_{1/2}} u \leq (1-\gamma) = (1-\gamma) \operatorname*{Osc}_{B_1} u. \tag{34}$$

□

Now we are able to prove the Hölder regularity of weak solution to the problem (1).

Proposition 2. *Let $u : B_1 \mapsto \mathbb{R}$ satisfy* (33)*. Then,*

$$\|u\|_{C^\alpha(B_{1/2})} \lesssim \|u\|_{L^\infty(B_1)}, \tag{35}$$

for some $\alpha > 0$ depending on γ

Proof. Let $x, y \in B_{1/2}$, we define $d = |x-y|$ and $a = \frac{1}{2}|x+y|$. Then, in order to establish a link u and d, we can recursively apply the result from corollary 1 to get that

$$|u(x)-u(y)| \leq \operatorname*{Osc}_{B_{d/2}(a)} u \leq (1-\gamma) \operatorname*{Osc}_{B_d(a)} u \leq \cdots \leq (1-\gamma)^k \operatorname*{Osc}_{B_{2^k d}(a)} u. \tag{36}$$

We then choose k carefully such that $\frac{1}{4} < 2^k d \leq \frac{1}{2}$. Then $k = \log_2(\frac{1}{d}) + O(1)$ and

$$|u(x)-u(y)| \leq (1-\gamma)^k \operatorname*{Osc}_{B_{1/2}(a)} u \leq (1-\gamma)^k \operatorname*{Osc}_{B_1(a)} u \leq 2(1-\gamma)^k \|u\|_{L^\infty(B_1)}. \tag{37}$$

Also by being more precise in the constant in $O(1)$, we see that we can safely say that $k \leq \log_2(\frac{1}{d}) + 2$ and so

$$(1-\gamma)^k \leq 4(1-\gamma)^{\log_2(\frac{1}{d})} = 4d^{-\log_2(1-\gamma)}. \tag{38}$$

Hence we conclude by letting $\alpha = \alpha(\gamma) = -\log_2(1-\gamma) = \gamma + O(\gamma^2)$. □

Therefore, the proof of DeGiorgi-Nash-Moser's theorem boils down to proving the weak Moser-Harnack's inequality. We will attack the proof using the same approach than the one for differential Harnack's inequality in the case of Laplace operator.

Lemma 7. *Let u be a weak solution to $Lu = 0$ and $u > 0$ on B_1. Then $\|\nabla \log u\|_{L^2(B_{1/2})} \lesssim 1$.*

Proof. Choose η a cut-off function such that

$$\begin{cases} \eta = 1 & \text{in } \; B_{1/2}, \\ 0 \leq \eta \leq 1 & \text{in } \; B_1, \\ \eta = 0 & \text{in } \; B_1^c. \end{cases} \tag{39}$$

Then, using the elliptic condition we have

$$\int_{B_{1/2}} |\nabla \log u|^2 \leq \int_{B_1} \eta^2 |\nabla \log u|^2 = \int_{B_1} \eta^2 |\nabla u|^2 u^{-2} \leq \frac{1}{\lambda} \int_{B_1} \sum_{i,j=1}^n \eta^2 a_{i,j} \frac{\partial_i u}{u} \frac{\partial_j u}{u} = \frac{1}{\lambda} \int_{B_1} \sum_{i,j=1}^n \eta^2 a_{i,j} \partial_i u \partial_j u^{-1},$$

which gives when we integrate by parts that

$$\int_{B_{1/2}} |\nabla \log u|^2 \leq 2\frac{\Lambda}{\lambda} \int_{B_1} \eta |\nabla \eta| |\nabla u| u^{-1} = \int_{B_1} \eta |\nabla \eta| |\nabla \log u|.$$

And again by Cauchy-Schwarz inequality

$$\int_{B_1} \eta^2 |\nabla \log u|^2 \lesssim \int_{B_1} \eta^2 |\nabla \log u| \int_{B_1} |\nabla \eta|^2, \tag{40}$$

which let us conclude that $\|\nabla \log u\|_{L^2(B_{1/2})} \lesssim 1$. □

Let $w = -\log u$ and $v = w - \log(10)$, the following Poincaré inequality will give us a bound on the L^2 norm of w instead of the actual bound on ∇w.

Lemma 8 (Poincaré inequality)**.** *Let $H = \{v \leq 0\} \cap B_r$. For all $v \in W^{1,1}(B_r)$, we have*

$$\int_{B_r} v_+^2 \leq \frac{Cr^2 |B_r|}{|H|} \int_{B_r} |\nabla v_+|^2. \tag{41}$$

Proof. Let $u = v_+$, then by the usual Poincaré inequality we have

$$\int_{B_r} |\nabla u|^2 \geq \frac{C}{r^2} \int_{B_r} |u - \bar{u}|^2 \geq \frac{C}{r^2} \int_H |u - \bar{u}|^2 = \frac{C|H|}{r^2 |B_r|} \int_{B_r} |\bar{u}|^2. \tag{42}$$

Moreover we also have by Poincaré inequality that

$$\int_{B_r} |\nabla u|^2 \geq \frac{C|H|}{r^2 |B_r|} \int_{B_r} |u - \bar{u}|^2, \tag{43}$$

and by adding the two previous inequalities we get

$$\int_{B_r} |\nabla u|^2 \geq \frac{C|H|}{2r^2 |B_r|} \left(\int_{B_r} |u - \bar{u}|^2 + \int_{B_r} |\bar{u}|^2 \right) \geq \frac{C|H|}{2r^2 |B_r|} \int_{B_r} |u|^2 \tag{44}$$

□

Lemma 9. *Let u be a weak solution to $Lu = 0$ and $u > 0$ on B_1. Moreover, if u satisfies*

$$|A| := \big|\{x \in B_{1/2} : u(x) > 1/10\}\big| \geq \frac{1}{10}\big|B_{1/2}\big|, \tag{45}$$

then, $\|w\|_{L^2(B_{1/2})} \lesssim 1$.

Proof. The proof is a straight forward application of Poincaré inequality. Indeed we have

$$\left(\int_{B_{1/2}} |w|^2\right)^{\frac{1}{2}} - \frac{1}{|B_{1/2}|^{1/2}} \int_{B_{1/2}} w \leq \left(\int_{B_{1/2}} |w - \bar{w}|^2\right)^{\frac{1}{2}} \lesssim \left(\int_{B_{1/2}} |\nabla w|^2\right)^2, \tag{46}$$

and by hypotheses

$$|A| := \big|\{x \in B_{1/2} : w(x) \leq \log(10)\}\big| \geq \frac{1}{10}\big|B_{1/2}\big|, \tag{47}$$

Therefore,

$$\begin{aligned}
\left(\int_{B_{1/2}} |w|^2\right)^{\frac{1}{2}} &\lesssim \left(\int_{B_{1/2}} |\nabla w|^2\right)^2 + \frac{1}{|B_{1/2}|^{1/2}} \int_{B_{1/2}-A} w + \frac{1}{|B_{1/2}|} \int_A w, \\
&\lesssim \left(\int_{B_{1/2}} |\nabla w|^2\right)^{\frac{1}{2}} + \frac{1}{|B_{1/2}|^{1/2}} \int_{B_{1/2}} w_+ + 1, \\
&\lesssim \left(\int_{B_{1/2}} |\nabla w|^2\right)^{\frac{1}{2}} + \left(\int_{B_{1/2}} w_+^2\right)^{\frac{1}{2}} + 1,
\end{aligned}$$

and using Poincaré's inequality we prove that

$$\left(\int_{B_{1/2}} |w|^2\right)^{\frac{1}{2}} \lesssim \left(\int_{B_{1/2}} |\nabla w|^2\right)^{\frac{1}{2}} + 1. \tag{48}$$

□

Hence, we now have a L^2 bound on w and we can conclude witth the following lemme

Lemma 10. *Let $w = -\log u$, then w is a weak subsolution and satisfy $Lw \geq 0$.*

Proof. The proof follows with a straight forward computation

$$-\sum \partial_j(a_{ij}\partial_i \log u) = -\sum \partial_j(a_{ij}\partial_i u^{-1}) = Lu \cdot u^{-1} + \sum a_{ij}(\partial_i u)(\partial_j u)u^{-2} \geq 0. \tag{49}$$

□

Since, $w = -\log u > 0$ because $u < 1$, using previous results we have the upper bound

$$\|w\|_{L^\infty(B_{1/2})} \lesssim \|w\|_{L^2(B_{1/2})} \lesssim 1, \tag{50}$$

and the proof of the weak Harnack inequality follows by exponentiating the previous inequality.