KPP REACTION-DIFFUSION EQUATIONS

THOMAS TENDRON

Abstract. We consider a class of quasilinear parabolic partial differential equations and study the spreading properties of solutions to these equations. We are specifically interested in the formation of travelling waves. We also consider front-like solutions to the Cauchy problem. We first follow a paper by Nadin and Rossi ([1]) for nonlinearities that depend on time and on the solution. In a second part, we follow a paper by Nadin ([2]) and consider nonlinearities depending on space and on the solution. For both types of nonlinearities, we also study the solutions to the equation in a random environment.

Contents

1. Introduction

In this note, we will be concerned with spreading properties of solutions to certain quasilinear reaction-diffusion equations. Specifically, we will look at equations of the form

$$\partial_t u(x,t) - \Delta u(x,t) = f(x,t,u), \; x \in \mathbb{R}^N, t \in \mathbb{R} \tag{1}$$

with positivity, boundedness and regularity assumptions on the reaction f.

Let us first motivate the discussion by presenting a brief history of the subject and some background material on the homogeneous case, namely when $f = f(u)$. One of the simplest nonlinearities studied is

$$f(u) = u - u^2 \tag{2}$$

which was first derived in 1937 by Fisher ([3]) in the context of population genetics, based on heuristic arguments. Fisher looked for special solutions of (1), with the simple nonlinearity

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(2), having the constant shape of a wave and travelling with some constant velocity. In the $(1+1)$-dimensional case, this translates into the requirement that

$$\partial_t u = -c\partial_x u$$

Now, it is easy to solve $\partial_t u + c\partial_x u = 0$ using the method of characteristics:

$$\begin{cases} \dot{t}(s) = 1 \\ \dot{x}(s) = c \\ \dot{u}(x(s), t(s)) = 0 \end{cases} \Rightarrow \begin{cases} t(s) = s + t_0 \\ x(s) = cs + x_0 \\ u(x(s), t(s)) = u_0 \end{cases}$$

so for $t_0 = 0$ and $u_0 = \phi(x_0)$ for any function ϕ, we obtain

$$\begin{cases} t = s \\ x = cs + x_0 = ct + x_0 \Leftrightarrow x_0 = x - ct \\ u(x,t) = \phi(x_0) = \phi(x - ct) \end{cases}$$

So the notion of transition wave is essentially that of a solution of the form $u(t,x) = \phi(x-ct)$. In the context studied by Fisher, the solution $u(t,x)$ represents the frequency of some mutant gene at time t and at position x. In 1937, Kolomogorov, Petrovskii and Piskounov studied the same model as Fisher, but with a more general nonlinearity $f = f(x,t,u)$ satisfying some boundedness and regularity conditions ([4]). They introduced mathematical rigour in the study of these equations. In particular, they showed by a now standard iteration argument that the Cauchy problem for (1), with $N = 1$, has a unique solution. (Theorem 1, [4]) They also discussed monotonicity properties of solutions with respect to space, time and the nonlinearity. In 1975, Aronson and Weinberger investigated front propagation in the multidimensional case for the *homogeneous* reaction $f = f(u)$. ([5]) They showed the existence of plane wave solutions in any direction $e \in \mathbb{S}^{N-1}$ (Theorem 4.1, [5]) and they discussed the so-called hair-trigger effect, which essentially identifies the constant solution $u \equiv 0$ as an unstable state, and the constant solution $u \equiv 1$ as a stable state by showing that a solution u of the Cauchy problem with any non-trivial initial data $0 \leq u_0 \leq 1$ will blow-up to 1 as $t \to \infty$ (Corollary 3.1, [5]).

In our case, we will focus on reactions satisfying the conditions identified by Kolmogorov, Petrovskii and Piskounov. In section 2, we will show that in the *time-heterogeneous* case $f = f(t,u)$, given any $e \in \mathbb{S}^{N-1}$, there exists a solution u with the profile $u(\cdot, t)$ having the shape of a wave decaying from 1 to 0 in direction e for any t. Such solutions are called *transition waves*. We will then apply this existence result to obtain the existence of random transition waves for the time-heterogeneous case in a random environment. One can then ask whether a general solution to the Cauchy problem will automatically possess some of the defining spreading properties of a transition wave. It is indeed the case, and it will be the purpose of the end of section 3 to make the notion of *front-like* solution precise. This corresponds to a paper by Nadin and Rossi in 2012 ([1]). In section 3, we will continue with the study of the asymptotic behavior of solutions to the Cauchy problem, but this time in the *space-heterogeneous* case $f = f(x,u)$ in a random environment. Following a paper by Nadin in 2015 ([2]), we will study how randomness of the coefficient and reaction affect the *spreading speed* of a solution. Specifically, we will derive an inequality showing that heterogeneity increases the propagation speed, and we will show that refining the environment, that is making the change of variable $x \mapsto \frac{x}{L}$ for $L > 0$ small, slows down the propagation.

We now give the precise assumptions regarding the reaction term in (1). The nonlinearity $f = f(x,t,u)$ in equation (1) is assumed to be *KPP monostable.* Namely, f satisfies the following conditions:

(1) $f(x,t,\cdot)$ is Lipschitz continuous in $[0,1]$ and of class C^1 in a neighborhood of 0, uniformly with respect to $(x,t) \in \mathbb{R}^N \times \mathbb{R}$
(2) for a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $f(x,t,0) = f(x,t,1) = 0$
(3) $\forall u \in (0,1)$, $\operatorname{ess\,inf}_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} f(x,t,u) > 0$
(4) for a.e. $(x,t,u) \in \mathbb{R}^N \times \mathbb{R} \times [0,1]$, $f(x,t,u) \leq \mu(x,t)u$, where $\mu(x,t) \coloneqq \frac{\partial f}{\partial u}\big|_{u=0}(x,t)$

We will make use of the strong maximum principle for quasilinear parabolic equations of second order. ([8], Theorem 2 page 3)

Theorem 1.1. (Strong maximum principle) *Let P be the parabolic operator in divergence form defined by*

$$Pu = -u_t + \operatorname{div} A(x,t,u,\nabla u) + B(x,t,u)$$

Suppose that A is continuously differentiable with respect to its last two arguments and B is Lipschitz continuous with respect to its last argument. Let $\Omega \subset \mathbb{R}^{N+1}$ be a nonempty open connected set. Suppose that u attains its nonnegative maximum on Ω at some point (x_0,t_0) in the interior of Ω, then we have $u(x,t) = u(x_0,t_0)$ for all $(x,t) \in \Omega$.

We also recall the comparison principle for quasilinear parabolic differential equations. It is a special case of Theorem 1 in [6].

Theorem 1.2. (Comparison principle) *Let P be the parabolic operator defined by*

$$Pu = -u_t + \Delta u + f(x,t,u)$$

Suppose that f is Lipschitz continuous with respect to its last argument. Let $Q = \Omega \times (0,T) \subset \mathbb{R}^{N+1}$ be a nonempty open connected set. If $u, v \in C^{2,1}(\overline{Q})$ (two derivatives in space and one in time) are functions such that $Pu \geq Pv$ in Q and $u(x,0) \leq v(x,0)$ on Ω, then $u \leq v$ in $\overline{Q}$.

2. Propagation phenomena in the time-heterogeneous case

We restrict our attention to the time heterogeneous case: $f = f(t,u)$. It is further assumed that f satisfies

(5) $f(\cdot,u) \in L^\infty(\mathbb{R})$, for all $u \in [0,1]$
(6) $\exists C > 0, \exists \gamma, \delta \in (0,1]$ such that for a.e. $(t,u) \in \mathbb{R} \times [0,\delta]$, we have $f(t,u) \geq \mu(t)u - Cu^{1+\gamma}$

This last assumption provides a lower bound for $f(t,u)$ when u is close to 0. It is used in the proof of the existence result below.
The notion of solution considered in the paper is that of a strong L^{N+1}-solution. Namely, define for any compact set $K \subset \mathbb{R}^N \times \mathbb{R}$ the space

$$W^{2,1}_{N+1}(K) \coloneqq \{u \in L^{N+1}(K) : \partial_t u, \partial_{x_i} u, \partial_{x_i x_j} u \in L^{N+1}(K), \forall i,j \in \{1,\ldots,N\}\}$$

Then, we want to find $u \in W^{2,1}_{N+1,loc}(\mathbb{R}^N \times \mathbb{R})$ such that

$$\partial_t u - \Delta u = f(t,u), \text{ for a.e. } x \in \mathbb{R}^N,\ t \in \mathbb{R} \tag{3}$$

The authors define the *least mean* $\underline{g}$ of a function $g \in L^\infty(\mathbb{R})$ by

$$\underline{g} \coloneqq \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} g(s)ds$$

The results use the notion of *almost planar generalized transition wave* defined as follows. An (almost planar) generalized transition wave in the direction $e \in \mathbb{S}^{N-1}$ of equation (1.1) is a solution u which can be written as $u(x,t) = \phi(x \cdot e - \int_0^t c(s)ds, t)$, where $c \in L^\infty(\mathbb{R})$ and $\phi : \mathbb{R} \times \mathbb{R} \to [0,1]$ satisfies $\lim_{z\to-\infty} \phi(z,t) = 1$ and $\lim_{z\to\infty} \phi(z,t) = 0$. The functions ϕ and c are respectively called the *profile* and the *speed* of the generalized transition wave u. If u is a generalized transition wave, then its profile ϕ satisfies

$$\begin{cases} \partial_t\phi - \partial_{zz}\phi - c(t)\partial_z\phi = f(t,\phi), \; z \in \mathbb{R}, t \in \mathbb{R} \\ \lim_{z\to-\infty} \phi(z,t) = 1 \text{ and } \lim_{z\to\infty} \phi(z,t) = 0 \text{ uniformly in } t \in \mathbb{R} \end{cases} \tag{4}$$

The main result is the following.

Theorem 2.1. *Assume that f satisfies the conditions described in the introduction and let $e \in \mathbb{S}^{N-1}$.*

(1) *For all $\gamma > 2\sqrt{\underline{\mu}}$, there exists a generalized transition wave u in direction e with a speed c such that $\underline{c} = \gamma$ and a profile ϕ which is decreasing with respect to z.*
(2) *There exists no generalized transition wave u in direction e with a speed c such that $\underline{c} < 2\sqrt{\underline{\mu}}$.*

2.1. Existence of transition waves.

Lemma 2.2. *Under the assumptions of Theorem 1, for all $\gamma > 2\sqrt{\underline{\mu}}$, there exists a function $c \in L^\infty(\mathbb{R})$ with $\underline{c} = \gamma$, such that (2.1) admits some uniformly continuous generalized sub and supersolutions $\underline{\phi}(z,t)$, $\overline{\phi}(z)$ satisfying*

(1) $\overline{\phi}(\infty) = 0$, $\underline{\phi}(-\infty, t) = 1$ *uniformly in $t \in \mathbb{R}$, and $\overline{\phi}$ is nonincreasing in $\mathbb{R}$*
(2) $0 \leq \underline{\phi} < \overline{\phi} \leq 1$ *uniformly in $t \in \mathbb{R}$, $\forall z \in \mathbb{R}$, $\inf_{t\in\mathbb{R}}(\overline{\phi} - \underline{\phi})(z,t) > 0$ and $\forall\tau$, $\lim_{z\to\infty} \frac{\overline{\phi}(z+\tau)}{\underline{\phi}(z,t)} < 1$ uniformly in $t \in \mathbb{R}$*
(3) $\exists\xi \in \mathbb{R}$, $\inf_{t\in\mathbb{R}} \underline{\phi}(\xi,t) > 0$

Condition 1 says that $\overline{\phi}$ and $\underline{\phi}$ behave like transition waves. Condition 2 means that $\overline{\phi}$ and $\underline{\phi}$ are ordered and asymptotically close to each other, but distinct. Lastly, condition 3 ensures that the subsolution is not identically equal to zero.

Proof. Since f is KPP, a solution to the linearization of problem (3) near 0 gives a super solution of (3). The linearized problem is

$$\begin{cases} \partial_t\phi - \partial_{zz}\phi - c(t)\partial_z\phi = \mu(t)\phi, \; z \in \mathbb{R}, t \in \mathbb{R} \\ \lim_{z\to-\infty} \phi(z,t) = 1 \text{ and } \lim_{z\to\infty} \phi(z,t) = 0 \text{ uniformly in } t \in \mathbb{R} \end{cases} \tag{5}$$

Fix $\gamma > 2\sqrt{\underline{\mu}}$. We choose a speed $c = c(t)$ so that there is an exponential solution of the form $\psi(z) = e^{-\kappa z}$ for the linear problem (5). We have by direct computation

$$\begin{aligned} &0 = \partial_t\phi - \partial_{zz}\phi - c(t)\partial_z\phi - \mu(t)\phi = (-\kappa^2 + \kappa c(t) - \mu(t))e^{-\kappa z} \\ &\Rightarrow -\kappa^2 + \kappa c(t) - \mu(t) = 0 \\ &\Rightarrow c(t) = \frac{\mu(t)}{\kappa} + \kappa \end{aligned}$$

We impose the condition $\underline{c} = \gamma$. Namely, by an equivalent definition of the least mean of c and linearity of the integral

$$\gamma = \lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} \left(\frac{\mu(s)}{\kappa} + \kappa\right) ds = \frac{\underline{\mu}}{\kappa} + \kappa$$

We solve for κ in $-\kappa^2 + \kappa\underline{c} - \underline{\mu} = 0$. Since $\gamma > 2\sqrt{\underline{\mu}}$, we obtain two positive real roots:

$$\kappa = \frac{\gamma \pm \sqrt{\gamma^2 - 4\underline{\mu}}}{2}$$

Define κ to be the smallest root, i.e.

$$\kappa := \frac{\gamma - \sqrt{\gamma^2 - 4\underline{\mu}}}{2}$$

By extending $f(t,\cdot)$ linearly outside $[0,1]$, since $\psi \geq 0$, we can assume that ψ is a global supersolution. Set $\overline{\phi}(z) = \min(\psi(z), 1)$. We note in particular that the form $c = \kappa + \kappa^{-1}\mu$ of the speed naturally appears when choosing a speed for which such a ψ exists. The construction of the subsolution $\underline{\phi}$ is a bit harder and relies on an equivalent definition of the least mean, and on assumption (6) on f given in the introduction. By assumption (6) on f, we know that we can find $C > 0$, $\gamma, \delta \in (0,1]$ such that for a.e. $(t,u) \in \mathbb{R}\times[0,\delta]$, we have

$$f(t,u) \geq \mu(t)u - Cu^{1+\gamma} \tag{6}$$

It is therefore enough to find a function $A \in W^{1,\infty}(\mathbb{R})$ and a constant $h > \kappa$ such that the function $\varphi(z) := \psi(z) - e^{A(t)-hz}$ satisfies

$$\partial_t\varphi - \partial_{zz}\varphi - c(t)\partial_z\varphi \leq \mu(t)\varphi - C\varphi^{1+\gamma}, \text{ for a.e. } z > 0, t \in \mathbb{R} \tag{7}$$

The authors prove the following variational characterization of the least mean of a function $B \in L^\infty(\mathbb{R})$:

$$\underline{B} = \sup_{A\in W^{1,\infty}(\mathbb{R})} \operatorname*{ess\,inf}_{t\in\mathbb{R}} (A' + B)(t) \tag{8}$$

Computing the partial derivatives of φ explicitely gives

$$\partial_t\varphi - \partial_{zz}\varphi - c(t)\partial_z\varphi - \mu(t)\varphi(t) = [-A'(t) + h^2 - c(t)h + \mu(t)]e^{A(t)-hz}$$

so it is enough to show that

$$A'(t) + B(t) \geq C\varphi^{1+\gamma}e^{hz-A(t)}, \text{ for a.e. } z > 0, t \in \mathbb{R} \tag{9}$$

where

$$B(t) := -h^2 + c(t)h - \mu(t)$$

Choosing $h \in (\kappa, (1+\gamma)\kappa)$ and using $\varphi \leq \psi$ gives

$$\varphi^{1+\gamma}e^{hz-A(t)} \leq e^{-\kappa(1+\gamma)z+hz-A(t)} = e^{-A(t)+(h-\kappa(1+\gamma))z} \leq e^{-A(t)}, \ \forall z > 0, t \in \mathbb{R} \tag{10}$$

So if we can show $\operatorname{ess\,inf}_{\mathbb{R}}(A' + B) > 0$ for some $A \in W^{1,\infty}(\mathbb{R})$, then up to adding a positive constant to A and using (10), equation (9) holds. By equation (8), it suffices to show that

$\underline{B} > 0$. But

$$\begin{aligned}
\underline{B} &= \lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} \left(-h^2 + c(s)h - \mu(s)\right) ds \\
\text{(as } c(s) = \tfrac{\mu(s)}{\kappa} + \kappa\text{)} \quad &= \lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} \left(-h^2 + c(s)h - \mu(s)\right) ds \\
&= \lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} h\left(\kappa - h + \mu(s)\left(\frac{1}{\kappa} - \frac{1}{h}\right)\right) ds \\
\text{(linearity of the integral)} \quad &= h\left(\kappa - h + \underline{\mu}\left(\frac{1}{\kappa} - \frac{1}{h}\right)\right) \\
\text{(since } \gamma = \underline{c} = \tfrac{\mu}{\kappa} + \kappa\text{)} \quad &= -h^2 + \gamma h - \underline{\mu}
\end{aligned}$$

Since κ is the smallest root of $-x^2 + x\gamma - \underline{\mu} = 0$, we can choose $h \in (\kappa, (1+\gamma)\kappa)$ such that $\underline{B} > 0$. So, there exists $A \in W^{1,\infty}(\mathbb{R})$ such that (9) holds. Moreover, we can add a constant $\alpha < 0$ to $A(t)$ so that $\varphi < \delta$. So, φ is a subsolution of (3) in $(0,\infty)\times\mathbb{R}$. Since $\varphi \geq 0$ if and only if $-\kappa z \geq A(t) - hz$, i.e. if and only if $-\kappa z \geq \frac{A(t)}{h-\kappa}$, α can be chosen so that $\varphi \leq 0$ for $z < 0$. Hence, by the arbitrariness of the extension of $f(t,u)$ for $u < 0$, it follows that $\underline{\phi}(z,t) := \max(\varphi(z,t), 0)$ is a generalized subsolution of (3). The only property of $\underline{\phi}, \overline{\phi}$ that is not immediate from the construction is: for any $\tau > 0$, we obtain

$$\lim_{z\to\infty} \frac{\overline{\phi}(x+\tau)}{\underline{\phi}(z,t)} = \lim_{z\to\infty} \frac{e^{-\kappa\tau-\kappa z}}{e^{-\kappa z} - e^{A(t)-hz}} = \lim_{z\to\infty} \frac{e^{-\kappa\tau}}{1 - e^{A(t)-(h-\kappa)z}} = e^{-\kappa\tau} < 1$$

□

Proof. (of (1) of Theorem 2.1) Fix $\gamma > 2\sqrt{\underline{\mu}}$. We want to construct a generalized transition wave in direction $e \in \mathbb{S}^{N-1}$ with speed c satisfying $\underline{c} = \gamma$ and a profile ϕ decreasing with respect to z. The authors obtain existence of a solution by considering for each $n \in \mathbb{N}$ the solution ϕ_n of the problem

$$\begin{cases} \partial_t\phi - \partial_{zz}\phi - c(t)\partial_z\phi = f(t,\phi) & z \in \mathbb{R},\ t > -n \\ \phi(z,-n) = \overline{\phi}(z) & z \in \mathbb{R} \end{cases}$$

Note that such solutions ϕ_n exists by a paper by Amann ([10]).By the comparison principle, we obtain $\underline{\phi} \leq \phi_n \leq \overline{\phi}$ so in particular ϕ_n is nonincreasing since $\overline{\phi}$ is nonincreasing by lemma 1. Indeed, suppose by contradiction that $\exists z_0 > 0$ such that $\phi_n(z+z_0,t) - \phi_n(z,t) > 0$. Since $(z,t) \mapsto \phi_n(z+z_0,t) - \phi_n(z,t)$ solves the PDE with initial condition $\overline{\phi}(z+z_0) - \overline{\phi}(z) \leq 0$, then the comparison principle implies that $\phi_n(z+z_0,t) - \phi_n(z,t) \leq 0$. Contradiction. Let K be a compact subset of $\mathbb{R}\times\mathbb{R}$. Since

$$\sup_{n\in\mathbb{N}} \|\phi_n\|_{W_p^{2,1}(K)} \leq \|\overline{\phi}\|_{W_p^{2,1}(K)} < \infty$$

and $W_p^{2,1}(K)$ is a reflexive Banach space (i.e. every bounded sequence has a weakly convergent subsequence), it follows that there exists $\phi \in W_p^{2,1}(K)$ such that $\phi_n \rightharpoonup \phi$ in $W_p^{2,1}(K)$. Thus, it follows by continuity of the embedding that $\phi_n \rightharpoonup \phi$ in $C(K)$. Moreover, we have by the Rellich-Kondrachov embedding theorem ([10], Theorem 6.2. page 144) that for any $p \in (N,\infty)$, $W_p^{2,1}(K)$ is compactly embedded in $C(K)$. Now, let $(\phi_{n_k})_{k\in\mathbb{N}}$ be any subsequence of $(\phi_n)_{n\in\mathbb{N}}$. By compactness of the embedding and the equivalence of compactness and sequential compactness, there exists a subsequence $(\phi_{n_{k_l}})_{l\in\mathbb{N}}$ of $(\phi_{n_k})_{k\in\mathbb{N}}$ such that $\phi_{n_{k_l}} \to \phi$ in $C(K)$ where we have used the uniqueness of the weak limit ϕ in $C(K)$. By arbitrariness of $(\phi_{n_k})_{k\in\mathbb{N}}$, it follows that $\phi_n \to \phi$ in $C(K)$, and ϕ is a nonincreasing solution of the problem

$$\partial_t\phi - \partial_{zz}\phi - c(t)\partial_z\phi = f(t,\phi) z \in \mathbb{R},\ t \in \mathbb{R}$$

which statisfies $\underline{\phi} \leq \phi \leq \overline{\phi}$. The rest of the proof consists of showing the two remaining properties of a transition wave: ϕ is decreasing as a function of z, and $\phi(-\infty,t) = 1$ uniformly in t. The first one holds by the parabolic strong maximum principle. Indeed, since ϕ is nonincreasing, we have $\phi(z+z_0,t)-\phi(z,t) \leq 0$ for all $z_0 > 0$, for all $z \in \mathbb{R}$, $t \in \mathbb{R}$. Suppose by contradiction that ϕ is not decreasing. Then, there is $z_0 > 0$ such that $\phi(z+z_0,t)-\phi(z,t) = 0$. But then, by the parabolic strong maximum principle, we infer $\phi(z + z_0, t) - \phi(z, t) = 0$ for all $z \in \mathbb{R}$, $t \in \mathbb{R}$., so ϕ is constant, which is a contradiction. The second property, namely $\phi(-\infty, t) = 1$ uniformly in t, is shown by using a change of coordinate system, similar arguments as above, and direct computation. Define

$$\theta := \lim_{z\to-\infty} \inf_{t\in\mathbb{R}} \phi(z,t)$$

We wish to show that $\theta = 1$. Consider a sequence $(t_n)_{n\geq1} \subset (0,\infty)$ such that $\lim_{n\to\infty} \phi(-n,t_n) = \theta$. Define the functions

$$v_n(z,t) := \phi\left(z - n - \int_{t_n}^{t_n+1} c(s)ds, t + t_n\right)$$

A simple computation shows that

$$\partial_t v_n - \partial_{zz} v_n = f(t + t_n, v_n), \quad z \in \mathbb{R}, t \in \mathbb{R}$$

Moreover, we have

$$\lim_{n\to\infty} v_n(0,0) = \lim_{n\to\infty} \phi\left(-n - \int_{t_n}^{t_n+1} c(s)ds, t_n\right) = \lim_{n\to\infty} \phi(-n,t_n) = \theta$$

and

$$\liminf_{n\to\infty} v_n(z,t) \geq \theta \tag{11}$$

locally uniformly in $(z,t) \in \mathbb{R}^2$. Now, by the exact same method as detailed above when we used the Rellich-Kondrachov embedding theorem and sequential compactness, we obtain that $(v_n)_{n\geq1}$ converges weakly in $W^{2,1}_{p,loc}(\mathbb{R}^2)$ and strongly in $L^\infty_{loc}(\mathbb{R}^2)$ to some function v which satisfies

$$\partial_t v - \partial_{zz} v = g(z,t), \text{ a.e. } z \in \mathbb{R}, t \in \mathbb{R} \tag{12}$$

where g is, up to a subsequence, the weak limit in $L^p_{loc}(\mathbb{R}^2)$ of $f(t + t_n, v_n)$. Moreover, by (11) and construction of v, θ is the minimum value of v and $v(0,0) = \theta$. So, by the strong minimum principle, $v \equiv \theta$ in $\mathbb{R} \times (-\infty, 0]$. So by (12), $g \equiv 0$ in $\mathbb{R} \times (-\infty, 0)$, hence for a.e. $(z,t) \in \mathbb{R} \times (-\infty, 0)$. By Lipschitz continuity of $f(t,\cdot)$, there exists a constant $c > 0$ such that

$$|f(t,\theta) - f(t,v_n)| \leq c|\theta - v_n|$$

So we obtain

$$\operatorname*{ess\,inf}_{s\in\mathbb{R}} f(s,\theta) \leq \operatorname*{ess\,inf}_{s\in\mathbb{R}} f(s,v_n) + c|\theta - v_n| \leq g(z,t) + c|\theta - v_n| \overset{n\to\infty}{\to} g(z,t) \equiv 0$$

for a.e. $z \in \mathbb{R}$, $t \in (-\infty, 0)$. Since, f is KPP, we conclude that $f(t,\theta) = 0$ for all $t \in \mathbb{R}$ so either $\theta = 0$ or $\theta = 1$. Take $\xi \in \mathbb{R}$ from (3) of lemma 2.2. Then, since ϕ is decreasing as in its first argument and $\phi \geq \underline{\phi}$ by construction, we obtain

$$\theta = \lim_{z\to-\infty} \phi(z,t) \geq \inf_{t\in\mathbb{R}} \phi(\xi,t) \geq \inf_{t\in\mathbb{R}} \underline{\phi}(\xi,t) > 0$$

so $\theta = 1$. □

2.2. Nonexistence of transition waves.

Proof. (of (2) *of Theorem 2.1)* To prove the non-existence of transition waves with speed $c = c(t)$ with least mean satisfying $\underline{c} < 2\sqrt{\underline{\mu}}$, we proceed by contradiction. Assume that there exists such a transition wave solution u. We will derive the contradiction that $\underline{c} \geq 2\sqrt{\underline{\mu}}$. To reach the contradiction, we use the definition of least mean, the assumption that $f(t,\cdot)$ is C^1 near 0, and a technical lemma to construct subsolutions u_n that propagate with speed less than $2\sqrt{\underline{\mu}}$ and then compare these subsolutions with the transition wave u.

Specifically, based on an auxiliary result from the paper [12], and on very similar techniques as in the proof of our lemma 2.2, the authors show the existence of a family of uniformly continuous subsolutions $(v_n)_{n\geq1}$ of

$$\partial_t v - \Delta v = (\mu(t+t_n) - \epsilon)v$$

for $\epsilon > 0$ small enough, such that the subsolutions satisfy

$$\begin{cases} \inf_{0\leq t<\eta T, |x|\leq\gamma t} v_n(x,t) > C \\ v_n(x,0) = 0 & |x| > R \\ 0 \leq v_n \leq 1 & \text{some } C > 0 \end{cases} \tag{13}$$

where $T > 0$, $\eta \in \mathbb{N} \cup \{\infty\}$, and $\gamma > 0$ are chosen in terms of t_n, and t_n is such that

$$\frac{1}{nT}\int_{t_n}^{t_n+nT} c(s)ds < \underline{c} + 2\epsilon \tag{14}$$

which is possible by definition of the least mean. let $\sigma > 0$ be obtained by the regularity assumption on f near 0: $f(t,\omega) \geq (\mu(t) - \epsilon)\omega$ for all $\omega \in [0,\sigma]$. So, we obtain a sequence $(u_n)_{n\geq1}$ of uniformly continuous subsolutions of the problem

$$\partial_t u - \Delta u = f(t,u)$$

of the form $u_n(x,t) = \sigma v_n(x, t - t_n)$. Since u is assumed to be a transition wave, for any $R > 0$, there is $L > 0$ large enough so that

$$\inf_{|x|<R} u(x - Le + e\int_0^t c(s)ds, t) > \sigma$$

for all $t \in \mathbb{R}$ so we may assume that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$u(x - Le + e\int_0^t c(s)ds, t) \geq u_n(x,t)$$

An application of the comparison principle and taking the lim inf show that

$$\liminf_{n\to\infty} u(\gamma^n nTe + e\int_0^{t_n} c(s)ds, t_n + nT) \geq \liminf_{n\to\infty} u_n(\gamma^n nT, t_n + nT) \geq \sigma C > 0$$

by (13). So by properties of the profile of a transition wave, the argument must be finite, with

$$\infty > \limsup_{n\to\infty}\left(\gamma^n nT + \int_0^{t_n} c(s)ds\right)$$

$$\text{(as } c \geq 0\text{)} \qquad > \limsup_{n\to\infty}\left(\gamma^n nT + \int_{t_n}^{t_n+nT} c(s)ds\right)$$

$$\text{(by (14))} \qquad > \limsup_{n\to\infty}\left(2\sqrt{\underline{\mu}-2\epsilon} - \underline{c} - 3\epsilon\right) nT$$

which implies that $2\sqrt{\underline{\mu}-2\epsilon} - \underline{c} - 3\epsilon \leq 0$ after rearranging and letting $\epsilon \to 0$ that $\underline{c} \geq 2\sqrt{\underline{\mu}}$. Contradiction. □

2.3. **Application to transition waves in random environment.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and consider the reaction-diffusion equation with random nonlinear term

$$\partial_t u - \Delta u = f(t,\omega,u),\ x \in \mathbb{R}, t \in \mathbb{R}, \omega \in \Omega \tag{15}$$

Assume that f satisfies all the conditions listed in the introduction. Moreover, suppose that $\forall t \in \mathbb{R}$, $u \mapsto \frac{f(t,\omega,u)}{u}$ is nonincreasing in $[1-\delta, 1]$ for some $\delta = \delta(\omega) \in (0,1)$, and f is a stationary ergodic random function with respect to t. This last condition means that there exists a group $(\pi_t)_{t\in\mathbb{R}}$ of transformations of Ω such that the following hold:

- $(\pi_t)_{t\in\mathbb{R}}$ are measure-preserving: $\forall t \in \mathbb{R}$, $\forall A \in \mathcal{F}$, we have $\mathbb{P}(A) = \mathbb{P}(\pi_t^{-1}(A))$ (even when $\pi_t(A) \neq A$)
- stationarity: $\forall (t,s,\omega,u) \in \mathbb{R}\times\mathbb{R}\times\Omega\times[0,1], f(t+s,\omega,u) = f(t,\pi_s\omega,u)$
- ergodicity: for all $A \in \mathcal{F}$, if $\pi_t A = A$ for all $t \in \mathbb{R}$, then $\mathbb{P}(A) = 0$ or 1.

Intuitively, stationarity of f means that f will have the same statistical properties no matter when we look:

$$\mathbb{E}[f(t,\cdot)] = \int_\Omega f(t,\omega)d\mathbb{P}(\omega) = \int_\Omega f(0,\pi_t\omega)d\mathbb{P}(\omega) = \mathbb{E}[f(0,\cdot)]$$

We think of ergodicity in terms of the ergodic theorems, namely averages taken in t equal averages taken in ω. The next theorem relies on the definition of *random transition wave* (introduced by Shen in [9]) in the direction $e \in \mathbb{S}^{N-1}$ of equation (15). Such a solution is a function $u : \mathbb{R}^N \times \mathbb{R} \times \Omega \to [0,1]$ which satisfies:

- $\exists \tilde{c} : \Omega \to \mathbb{R}, \exists \tilde{\phi} : \mathbb{R}\times\Omega \to [0,1]$ bounded measurable functions such that
$$u(x,t,\omega) = \tilde{\phi}(x.e - \int_0^t \tilde{c}(\pi_s\omega)ds, \pi_t\omega) \text{ for all } (x,t,\omega) \in \mathbb{R}^N\times\mathbb{R}\times\Omega$$
- For almost every $\omega \in \Omega$, $(x,t) \mapsto u(x,t,\omega)$ is a solution of (15)
- For almost every $\omega \in \Omega$, $\lim_{z\to-\infty}\tilde{\phi}(z,\omega) = 1$ and $\lim_{z\to\infty}\tilde{\phi}(z,\omega) = 0$

Theorem 2.3. *Let $e \in \mathbb{S}^{N-1}$. Under the previous hypotheses, for all $\gamma > 2\sqrt{\underline{\mu}}$, there exists a random transition wave u in direction e with random transition speed $\tilde{c}$ such that $c(t,\omega) := \tilde{c}(\pi_t\omega)$ has least mean γ, and a random profile $\tilde{\phi}$ which is decreasing with respect to z. Moreover for all $\gamma < 2\sqrt{\underline{\mu}}$, there doesn't exist any random transition wave u in direction e with random transition speed $\tilde{c}$ such that $c(t,\omega) := \tilde{c}(\pi_t\omega)$ has least mean γ.*

We note first that for this statement to make sense, we would like that $\underline{\mu}$ is $\mathbb{P}$-almost surely constant. It is indeed the case, as shown in the following lemma.

Lemma 2.4. *Let $\tilde{g} \in L^\infty(\Omega, \mathbb{R})$ and define*

$$G(\omega) = \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} \tilde{g}(\pi_s\omega)ds$$

Then G is constant on a set of probability 1. We call this constant value the least mean of the random stationary ergodic function g defined by $g(t,\omega) := \tilde{g}(\pi_t\omega)$ and we denote it $\underline{g}$.

Proof. Fix any $\epsilon > 0$. Define $A_\epsilon = \{\omega \in \Omega : G(\omega) < \operatorname{ess\,inf}_\Omega G + \epsilon\} \in \mathcal{F}$. Then, $\mathbb{P}(A_\epsilon) > 0$ since we take the essential infimum with respect to $\mathbb{P}$, and we have a strict inequality. We easily compute that $G(\pi_r\omega) = G(\omega)$ for all $r \in \mathbb{R}$. Indeed, we have

$$\begin{aligned} G(\pi_r\omega) &= \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} \tilde{g}(\pi_s\pi_r\omega)ds \\ \text{(group property)} \quad &= \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} \tilde{g}(\pi_{s+r}\omega)ds \\ &= \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_{t+r}^{t+r+T} \tilde{g}(\pi_s\omega)ds \\ &= \sup_{T>0} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} \tilde{g}(\pi_s\omega)ds \\ &= G(\omega) \end{aligned}$$

So $\pi_r A_\epsilon = A_\epsilon$ for all $r \in \mathbb{R}$. So, by ergodicity $\mathbb{P}(A_\epsilon) = 0$ or 1. But we already have that $\mathbb{P}(A_\epsilon) > 0$, so $\mathbb{P}(A_\epsilon) = 1$. Letting $\epsilon \to 0$, we obtain $G \equiv \operatorname{ess\,inf}_\Omega G$ $\mathbb{P}$-a.s.. □

Proof. (of Theorem 2.3) First note that as a direct corollary of Theorem 2.1, we obtain the nonexistence for $\gamma < 2\sqrt{\underline{\mu}}$ in the random environment. So, it remains to prove the existence of random transition waves in the case $\gamma > 2\sqrt{\underline{\mu}}$. The proof is divided into three steps. We already know by Theorem 2.1. that we can find a deterministic transition wave for each $\omega \in A_0$, where A_0 is the set of full probability obtained in Lemma 2.4.. So, in some sens we would like to concatenate all these transition waves to obtain a random transition wave. In order to achieve this, we will derive a general (deterministic) uniqueness result for the profile of transition waves.

Step 1: Let $c \in L^\infty(\mathbb{R})$ and f be KPP. Let I be an open interval and φ, ψ be respectively a sub and a supersolution of

$$\partial_t\phi - \partial_{zz}\phi - c(t)\partial_z\phi = f(t,\phi), z \in I, t \in \mathbb{R} \tag{16}$$

which are uniformly continuous and satisfy $0 \le \varphi \le \psi \le 1$ in $I \in \mathbb{R}$. In this context, we have

$$J := \{z \in I : \inf_{t\in\mathbb{R}} (\varphi - \psi)(z,t) = 0\}$$

is either empty or coincides with I.

As usual, it is enough to show that J is both open and closed in the topology of I. Let $(z_n)_n \subset J$ be such that $z_n \to z$ in I as $n \to \infty$. Then, we have by uniform continuity of ϕ and ψ

$$\inf_{t\in\mathbb{R}} (\varphi - \psi)(z,t) = \lim_{n\to\infty} \inf_{t\in\mathbb{R}} (\varphi - \psi)(z_n,t) = 0$$

so $z \in J$. This proves that J is closed. We show that J is open. Let $z_0 \in J$. So, we can take a minimizing sequence $(t_n)_n$ with $\lim_{n\to\infty}(\varphi - \psi)(z_0,t_n) = 0$. For each

n, take $\Phi_n(z,t) \coloneqq (\psi - \varphi)(z, t+t_n)$. By direct computation, since ψ and φ are respectively a sub and a supersolution of (16), we obtain

$$\partial_t \Phi_n - \partial_{zz}\Phi_n - c(t+t_n)\partial_z \Phi_n - \zeta(z, t+t_n)\Phi_n \geq 0$$

for a.e. $z \in I$, $t \in \mathbb{R}$, where

$$\zeta(z,t) \coloneqq \frac{f(t,\psi) - f(t,\varphi)}{\psi - \varphi} \leq \|f(t,\cdot)\|_{\mathrm{Lip}} < \infty$$

since $f(t,\cdot)$ is Lipschitz continuous. So $\zeta \in L^\infty(I \times \mathbb{R})$. Take $\delta > 0$ such that $[z_0 - \delta, z_0 + \delta] \subset I$. By the parabolic weak Harnack inequality (Theorem 7.22, [11]), there are constants $p, C > 0$ such that for all n

$$\|\Phi_n\|_{L^p((z_0-\delta, z_0+\delta)\times(-2,-1))} \leq C \inf_{(z_0-\delta,z_0+\delta)\times(-1,0)} \Phi_n \leq C\Phi_n(z_0, 0)$$

But the right-hand side goes to 0 as $n \to \infty$ by choice of $(t_n)_n$ so we obtain that $\Phi_n \to 0$ in $L^p((z_0-\delta, z_0+\delta)\times(-2,-1))$ as $n \to \infty$. Now, by uniform continuity of ψ and φ and the definition of Φ_n, we immediately obtain equicontinuity of $(\Phi_n)_n$, and we have uniform boundedness by uniform continuity on a bounded domain, so by the Arzela-Ascoli theorem, it follows that, up to subsequences, $\Phi_n \to 0$ uniformly in $(z_0 - \delta, z_0 + \delta) \times (-2, -1)$ as $n \to \infty$. So, $(z_0 - \delta, z_0 + \delta) \subset J$. Hence, J is open.

Step 2: The next step is to use the strong maximum principle-type result from step 1 to derive a uniqueness result. We have seen this kind of argument before so we just state the result:
Assume that $c \in L^\infty(\mathbb{R})$ and that f is KPP with the extra assumption listed at the beggining of this section. Let ϕ, ψ be a subsolution and a positive supersolution of the equation (4) for the profile which are uniformly continuous and satisfy $0 \leq \psi, \varphi \leq 1$, $\psi(\cdot, t)$ is nonincreasing, and $\forall \tau > 0$, $\lim_{R\to\infty} \sup_{z>r, t\in R} \frac{\varphi(z,t)}{\psi(z-\tau,t)} < 1$. Then, $\varphi \leq \psi$ in $\mathbb{R}^2$.

Step 3: We can now conclude by using step 2. Fix $\omega \in A_0$. Then, by Theorem 2.1, i.e. the existence of deterministic generalized transition waves, we obtain a solution $\phi(z,t,\omega)$ with speed $c(t,\omega)$ satisfying $\underline{c}(\cdot, \omega) = \gamma > 2\sqrt{\underline{\mu}}$. It is easily seen that the conditions of the uniqueness result from step 2 are satisfied for $\phi^s(z,t,\omega) \coloneqq \phi(z, t-s, \pi_s\omega)$ and $\phi(z,t,\omega)$. So, we obtain

$$\forall (z,t,s,\omega) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Omega,\ \phi(z, s+t, \omega) = \phi(z,t,\pi_s\omega)$$

namely $\phi(z,t,\omega)$ is stationary ergodic with respect to t, so $\tilde{\phi}(z,\omega) \coloneqq \phi(z,0,\omega)$ and $\tilde{c}(\omega) \coloneqq c(0,\omega)$ are the profile and the speed of a random transition wave.

□

2.4. **Front-like solutions.** Finally, we have the following result which approximately describes the level set $\{x \in \mathbb{R}^N : u(x,t) = \frac{1}{2}\}$ as $t \to \infty$.

Theorem 2.5. (Spreading properties) *Let f satisfy the hypotheses listed in the introduction and let $u_0 \in C(\mathbb{R}^N)$ be such that $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$. Then, the solution u of the Cauchy problem*

$$\begin{cases} \partial_t u - \Delta u = f(t,u) & x \in \mathbb{R}^N, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N \end{cases}$$

satisfies

$$\forall \gamma < 2\sqrt{\underline{\mu}_+},\ \lim_{t\to\infty} \inf_{|x|<\gamma t} u(x,t) = 1$$

where $\underline{\mu}_+ := \sup_{T>0} \inf_{t>0} \frac{1}{T}\int_t^{t+T} g(s)ds$ *and if in addition* u_0 *is compactly supported, then* $\forall \sigma > 0$,

$$\lim_{t\to\infty} \sup_{|x|\geq 2\sqrt{t\int_0^t \mu(s)ds}+\sigma t} u(x,t) = 0$$

Proof. To illustrate the proof technique, we prove only the second case. The proof of the first case is similar in spirit, and uses techniques already shown in the proof of Theorem 2.1 above. Since u_0 is assumed to be compactly supported, we can find $R > 0$ such that $\mathrm{supp}(u_0) \subset B_R$. For every $\kappa > 0$ and $e \in \mathbb{S}^{N-1}$, we define

$$v_{\kappa,e}(x,t) := \exp\left(-\kappa(x\cdot e - R - \kappa t) + \int_0^t \mu(s)ds\right)$$

Direct computation gives

$$\partial_t v_{\kappa,e} - \Delta v_{\kappa,e} - \mu(t)v_{\kappa,e} = 0, x \in \mathbb{R}^N, t > 0$$

and for all $x \in B_R$,

$$x \cdot e - R < 0$$

so

$$-\kappa(x \cdot e - R - \kappa t) > 0$$

Therefore, since also $\mu(t) \geq 0$ for all $t > 0$, we obtain

$$v_{\kappa,e}(x,t) > 1$$

Since f is KPP, the functions $v_{\kappa,e}$ are supersolutions of $\partial_t u - \Delta u = f(t,u)$. Since, $v_{\kappa,e}(x,t) > 1$ for all $x \in B_R$ and $\mathrm{supp}(u_0) \subset B_R$, then $v_{\kappa,e} \geq u_0$, hence we obtain by the comparison principle that $v_{\kappa,e} \geq u$ everywhere. Let $\sigma > 0$ be arbitrary. Let $x \in \mathbb{R}^N$ and $t > 0$ be such that $|x| \geq 2\sqrt{t\int_0^t \mu(s)ds} + \sigma t$. Take $e = \frac{x}{|x|}$ and $\kappa = \frac{|x|-R}{2t}$. Then, $u(x,t) \leq v_{\kappa,e}(x,t)$ implies

$$u(x,t) \leq \exp\left(-\frac{(|x|-R)^2}{4t} + \int_0^t \mu(s)ds\right) \tag{17}$$

Take $t > \frac{R}{\sigma}$. Then,

$$|x| - R \geq 2\sqrt{t\int_0^t \mu(s)ds} + \sigma t - R > 0$$

so (17) becomes

$$\begin{aligned} u(x,t) &\le \exp\left(-\frac{\left(2\sqrt{t\int_0^t \mu(s)ds} + \sigma t - R\right)^2}{4t} + \int_0^t \mu(s)ds\right) \\ &\le \exp\left(-\frac{\sigma t - R}{t}\sqrt{t\int_0^t \mu(s)ds} - \frac{(\sigma t - R)^2}{4t}\right) \\ &= \exp\left(-\frac{\sigma^2 t}{4} - \frac{R^2}{4t} - \sigma\sqrt{t\int_0^t \mu(s)ds} + R\sqrt{t^{-1}\int_0^t \mu(s)ds} + \frac{\sigma R}{2t}\right) \end{aligned}$$

where the right-hand tends to 0 as $t \to \infty$. So,

$$\lim_{t\to\infty} \sup_{|x|\ge 2\sqrt{t\int_0^t \mu(s)ds + \sigma t}} u(x,t) = 0$$

Since $\sigma > 0$ was arbitrary, we have obtained the result. □

Theorem 2.5 gives us bounds on the location of the level set $\{x \in \mathbb{R}^N : u(x,t) = \frac{1}{2}\}$ as $t \to \infty$. It is natural to ask how these bounds vary with respect to the nonlinearity f and with respect to a coefficients matrix A if we consider the more general operator $\mathrm{div}(A\nabla u)$ instead of simply Δ. This has not been studied for the time-heterogeneous equation yet, so we consider instead the space-heterogeneous case.

3. Relation between spreading speed and coefficient and reaction in the space heterogeneous case in random environment

Since less is known about the time-heterogeneous case, we consider instead the space heterogeneous case.

$$\begin{cases} \partial_t u - \partial_x(a(x,\omega)\partial_x u) = f(x,\omega,u) & \text{in } (0,\infty)\times\mathbb{R}\times\Omega \\ u(0,x,\omega) = u_0(x) & \text{on } \{0\}\times\mathbb{R}\times\Omega \end{cases}$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and we assume that the derivative of the reaction rate $(x,\omega) \in \mathbb{R}\times\Omega \mapsto f_s(x,\omega,0)$ and the diffusion coefficient $a : \mathbb{R}\times\Omega \to (0,\infty)$ are random stationary ergodic functions, $\inf_{x\in\mathbb{R}} a(x,\omega) > 0$ almost surely, and a, a', f and $\mu(x,\omega) := f_s(x,\omega,0)$ are $\mathbb{P}$-almost surely uniformly continuous and bounded with respect to x uniformly in s. Moreover, f is assumed to be KPP $\mathbb{P}$-almost surely. The problem is to find a speed $w^* = w^*(a,f) > 0$ such that the solution $u = u(t,x,\omega)$ of our problem satisfies for a.e. $\omega \in \Omega$:

$$\begin{cases} \lim_{t\to\infty} \inf_{x\in[0,wt)} |u(t,x,\omega) - 1| = 0 & w \in (0,w^*) \\ \lim_{t\to\infty} \sup_{x\ge wt} u(t,x,\omega) = 0 & w > w^* \end{cases}$$

This corresponds to studying the level sets of $u(t,\cdot,\omega)$ as $t \to \infty$. It is known that such a speed w^* exists ([13]). We are interested in the relation between w^* and a and f. Note that this section relies heavily on results from a previous paper by Berestycki and Nadin ([13]).

3.1. **A variational formula.** As before, let $\mu(x,\omega) := f_s(x,\omega,0)$. Assume that u has the form $u(x,t,\omega;\gamma) = e^{\gamma t}\phi_\gamma(x,\omega)$. Computing the linearized version of the PDE gives the new problem:

$$(a(x,\omega)\phi_\gamma')' + \mu(x,\omega)\phi_\gamma = \gamma\phi_\gamma$$

So we consider the linear second order elliptic operator

$$L_0^\omega \phi = (a(x,\omega)\phi')' + \mu(x,\omega)\phi$$

Define the more general operator $L_p^\omega \phi := e^{-px} L_0^\omega(e^{px}\phi)$ for $p \in \mathbb{R}$ and $\phi \in C^2(\mathbb{R})$. After a basic computation, we see

$$L_p^\omega \phi = (a(x,\omega)\phi')' - 2pa(x,\omega)\phi' + (p^2 a(x,\omega)\phi' - pa(x,\omega') + \mu(x,\omega))\phi$$

Define the generalized principle eigenvalues of the operator L_p^ω as follows

$$\overline{k_p}^\omega(a,\mu) := \inf\{\lambda \in \mathbb{R} : \exists \phi \in \mathcal{A} \text{ s.t. } L_p^\omega \phi \le \lambda\phi\}$$

$$\underline{k_p}^\omega(a,\mu) := \sup\{\lambda \in \mathbb{R} : \exists \phi \in \mathcal{A} \text{ s.t. } L_p^\omega \phi \ge \lambda\phi\}$$

where

$$\mathcal{A} := \{f \in C(\mathbb{R}) : f > 0 \text{ in } \mathbb{R},\ \frac{f'}{f} \in L^\infty(\mathbb{R}),\ \lim_{|x|\to\infty} \frac{\ln f(x)}{|x|} = 0\}$$

We quote a useful lemma from [13].

Lemma 3.1. (1) *$k_p^\omega(a,\mu) \ge k_0^\omega(a,\mu)$ for all $p \in \mathbb{R}$ and $\omega \in \Omega$*

(2) *$\overline{k_p^\omega}(a,\mu) = \underline{k_p^\omega}(a,\mu)$ $\mathbb{P}$-a.s.*

(3) *if $\underline{k_p^\omega}(a,\mu) > \Lambda_1^\omega(a,\mu)$, then $k_p(a,\mu) := \underline{k_p^\omega}(a,\mu) = \overline{k_p^\omega}(a,\mu)$ is a classical principle eigenvalue (i.e. there exists $\phi \in \mathcal{A}$ such that $L_p^\omega \phi = k_p^\omega(a,\mu)\phi$ in $\mathbb{R}$) in which case we have the variational formula*

$$w^*(a,f) = \min_{p>0} \frac{k_p(a,\mu)}{p}$$

where

$$\Lambda_1^\omega(a,\mu) = \sup_{\alpha \in H^1(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} (-a(x,\omega)\alpha'(x)^2 + \mu(x,\omega)\alpha(x)^2)\,dx}{\int_{\mathbb{R}} \alpha(x)^2 dx}$$

Proposition 3.2. *We show that*

$$k_p(a,\mu) = \inf_{\theta\in\mathcal{B}} \underline{k_0}(a, \mu + a(\theta+p)^2), \quad \mathbb{P}-\text{a.s.}$$

where $\mathcal{B} := \{\theta : \mathbb{R}\times\Omega \to \mathbb{R} \text{ measurable in } \omega \in \Omega,\ \theta(\cdot,\omega) \in L^\infty(\mathbb{R}), \theta, \text{ stationary },\mathbb{E}[\theta] = 0\}$.

Proof. It is easily shown (by contradiction) that functions $f \in \mathcal{A}$ satisfy in particular

$$\liminf_{R\to\infty} \frac{f(R)}{\int_0^R f(x)dx} = 0 \tag{18}$$

Let $\epsilon > 0$. By definition of $\underline{k_p^\omega}(a,\mu)$, we can choose $p \in \mathbb{R}$, $\omega \in \Omega$ and $\phi \in \mathcal{A}$ such that

$$L_p^\omega \phi \ge (k_p(a,\mu) - \epsilon)\phi \ \text{ in } \mathbb{R}$$

We show that if $\theta \in \mathcal{B}$, then $\psi(x) := \phi(x)e^{\int_0^x \theta(y,\omega)dy}$ is in $\mathcal{A}$:

- We have
$$\frac{\ln \psi(x)}{|x|} = \frac{\ln \phi(x)}{|x|} + \frac{\int_0^x \theta(y,\omega)dy}{|x|} \to \mathbb{E}[\theta(x,\cdot)] = 0 \text{ as } |x| \to \infty$$
by the Birkhoff ergodic theorem for continuous processes and since $\phi \in \mathcal{A}$.
- We have by direct computation: $\frac{\psi'(x)}{\psi(x)} = \frac{\phi'(x)}{\phi(x)} + \theta(x,\omega) \in L^\infty(\mathbb{R})$ and $\psi > 0$ in $\mathbb{R}$ since $\phi > 0$ in $\mathbb{R}$

Therefore, by multiplying $L_p^\omega \phi \geq (k_p(a,\mu) - \epsilon)\phi$ by ψ, integrating over $x \in (-R,R)$, using (18) and integration by parts, we obtain
$$(k_p(a,\mu) - \epsilon) \int_{-R}^{R} \psi^2 dx \leq (o(1) + \Lambda_1^\omega(a, \mu + a(p+\theta)^2)) \int_{-R}^{R} \psi^2 dx$$
so we have almost surely
$$k_p(a,\mu) - \epsilon \leq o(1) + \Lambda_1^\omega(a, \mu + a(p+\theta)^2)$$
Using a result from the previous paper [13], we know that
$$k_0(a, \mu + a(p+\theta)^2) = \Lambda_1^\omega(a, \mu + a(p+\theta)^2)$$
so taking the infimum over $\theta \in \mathcal{B}$, we conclude that
$$k_p(a,\mu) \leq \inf_{\theta \in \mathcal{B}} k_0(a, \mu + a(\theta+p)^2)$$
almost surely.

We can obtain the converse inequality
$$k_p(a,\mu) \geq \inf_{\theta \in \mathcal{B}} \underline{k_0}(a, \mu + a(\theta+p)^2)$$
using similar techniques. More precisely, take $p \in \mathbb{R}$ and a set $\Omega_1 \subset \Omega$ of probability 1 as in pat (3) of lemma 3.1. and such that $k_0(a,\mu) = \Lambda_1^\omega(a,\mu)$. Then, assuming $k_p(a,\mu) > k_0(a,\mu)$ and using that $k_p(a,\mu) = k_{-p}(a,\mu)$ by lemma 3.2. in [2], we can take $\phi, \psi \in \mathcal{A}$ such that
$$L_p^\omega \phi = k_p(a,\mu)\phi \text{ in } \mathbb{R}$$
and
$$L_{-p}^\omega \psi = k_p(a,\mu)\psi \text{ in } \mathbb{R}$$
Let $\alpha :=\sqrt{\phi\psi} \in \mathcal{A}$ and
$$\theta := -\frac{\phi'}{2\phi} + \frac{\psi'}{2\psi}$$
By Corollary 3.4. in [2], we obtain
$$\theta(x+y,\omega) = \theta(x, \pi_y \omega),\ \forall (x,y,\omega) \in \mathbb{R} \times \mathbb{R} \times \Omega_1$$
so θ is a random stationary ergodic functions and we can apply the Birkhoff ergodic theorem to conclude that for all $\omega \in \Omega_1$,
$$\mathbb{E}[\theta] = \lim_{x\to\infty} \frac{1}{x} \int_0^x \theta(y,\omega)dy = \lim_{x\to\infty} \frac{1}{x} \left[-\frac{1}{2}\ln(\phi(y,\omega)) + \frac{1}{2}\ln(\psi(y,\omega)) \right]_0^x = 0$$
since functions in $\mathcal{A}$ are stricly positive in $\mathbb{R}$. This proves that $\theta \in \mathcal{B}$. We have by direct computation
$$(a\alpha')' = -(\mu + |p+\theta|^2 a - k_p(a,\mu))\alpha$$

So, we have constructed a principal eigenfunction $\alpha \in \mathcal{A}$. By proposition 2.2. from [13], We obtain

$$k_p(a,\mu) = \underline{k}_0^\omega(a, \mu + |p+\theta|^2 a) = \overline{k}_0^\omega(a, \mu + |p+\theta|^2 a)$$

But by the first part of the proof

$$k_p(a,\mu) \le \inf_{\tilde{\theta}\in\mathcal{B}} k_0(a, \mu + a(\tilde{\theta}+p)^2)$$

so θ minimizes the infimum on the right-hand side. By convexity of $p \mapsto k_p(a,\mu)$ ([13]), there exists $p_+ \ge 0$ such that $k_p(a,\mu) = k_0(a,\mu)$ for $p \in [0,p_+]$ and $k_p(a,\mu) > k_0(a,\mu)$ for $p > p_+$. Take $p > p_+$ and let $t = \frac{p_+}{p} \in [0,1)$ and $\theta \in \mathcal{B}$ such that $k_0(a, \mu + |p+\theta|^2 a) = k_p(a,\mu)$. Since $\mu \mapsto k_p(a,\mu)$ is convex ([13]), we obtain

$$\begin{aligned} k_0(a, \mu + |p_+ + t\theta|^2 a) &= k_0(a, \mu + (1-t^2)0 + t^2|p+\theta|^2 a) \\ &\le (1-t^2)k_0(a, \mu+0) + t^2 k_0(a, \mu + |p+\theta|^2 a) \\ &= (1-t^2)k_0(a,\mu) + t^2 k_p(a,\mu) \end{aligned}$$

Taking the infimum, we obtain

$$\inf_{\theta\in\mathcal{B}} k_0(a, \mu + |p_+ + t\theta|^2 a) \le (1-t^2)k_0(a,\mu) + t^2 k_p(a,\mu)$$

Letting $p \to p_+$, so $t \to 1$, we obtain

$$\inf_{\theta\in\mathcal{B}} k_0(a, \mu + |p_+ + t\theta|^2 a) \le k_{p_+}(a,\mu) = k_0(a,\mu)$$

It remains to consider the case $p \in (0,p_+)$. Take $t = \frac{p}{p_+} \in [0,1)$. Exactly as above,

$$\inf_{\theta\in\mathcal{B}} k_0(a, \mu + |p_+ + t\theta|^2 a) \le k_0(a,\mu) = k_p(a,\mu)$$

almost surely. □

3.2. **Refining the environment slows down the propagation.** Define the *rescaled coefficient and reaction* $a_L(x,\omega) = a(\frac{x}{L},\omega)$ and $f_L(x,\omega,s) = f(\frac{x}{L},\omega,s)$ for all $(x,\omega,s) \in \mathbb{R}\times\Omega\times[0,1]$. We will show that:

Theorem 3.3. (Dependence with respect to the scaling of the coefficient and reaction) *The function*

$$L > 0 \mapsto w^*(a_L, f_L)$$

is nondecreasing.

We start by interpreting the result. Write $L = \frac{1}{M}$ for $M > 0$. We imagine putting a grid on the spacetime domain. Taking $L > 0$ smaller and smaller (hence $M > 0$ larger and larger) is equivalent to refining the grid. So the fact that function $M > 0 \mapsto w^*(a_{\frac{1}{M}}, f_{\frac{1}{M}})$ is nonincreasing means that *fragmentation of the environment slows down the propagation.*

Proof. For all $L > 0$, a straightforward computation gives $k_p(a_L,\mu_L) = \frac{1}{L^2}k_{pL}(a, L^2\mu)$ for all $p \in \mathbb{R}$. Let $L > 1$. By a Proposition 3.2, we have

$$k_{Lp}(a, L^2\mu) = \inf_{\theta\in\mathcal{B}} \underline{k_0^\omega}(a, a(\theta+Lp)^2 + L^2\mu), \quad \mathbb{P}-\text{a.s.}$$

Since $L \mapsto k_0(a, L^2 d)$ is concave for all bounded uniformly continuous function d (Proposition 3.6 in [14]), we obtain

$$\begin{aligned} k_{Lp}(a, L^2\mu) &\geq \inf_{\frac{\theta}{L}\in\mathcal{B}} \underline{k}_0^\omega(a, L^2(a\left(\frac{\theta}{L}+p\right)^2+\mu)) \\ &\geq L^2 \inf_{\frac{\theta}{L}\in\mathcal{B}} \underline{k}_0^\omega(a, a\left(\frac{\theta}{L}+p\right)^2+\mu) \\ &\geq L^2 k_p(a,\mu) \end{aligned}$$

Hence, by the computations above, we obtain

$$\begin{aligned} \omega^*(a_L, f_L) &= \min_{p>0} \frac{k_p(a_L, \mu_L)}{p} \\ &= \min_{p>0} \frac{k_{Lp}(a, L^2\mu)}{L^2 p} \\ &\geq \min_{p>0} \frac{k_p(a,\mu)}{p} \\ &= \omega^*(a, f) \end{aligned}$$

But the calculations leading to

$$\omega^*(a_L, f_L) \geq \omega^*(a, f)$$

work for any a and f. Therefore, take any $L_1 \geq L_2 > 0$ and define $L := \frac{L_1}{L_2} > 1$, $\tilde{a} := a_{L_2}$ and $\tilde{f} = f_{L_2}$. We obtain

$$\omega^*(a_{L_1}, f_{L_1}) = w^*(\tilde{a}_L, \tilde{f}_L) \geq \omega^*(\tilde{a}, \tilde{f}) = w^*(a_{L_2}, f_{L_2})$$

Hence $L > 0 \mapsto w^*(a_L, f_L)$ is nondecreasing. □

3.3. **Heterogeneity increases the spreading speed.** We want to compare heterogeneous and homogenized coefficients and reactions. We show that heterogeneity speeds up the propagation.

Theorem 3.4.

$$w^*(a, f) \geq w^*(\mathbb{E}\left[\frac{1}{a}\right]^{-1}, \mathbb{E}[f])$$

Proof. We first show that $k_0(a,\mu) \geq \mathbb{E}[\mu]$. We take $\Omega_1 \subset \Omega$ as in lemma 3.1 and λ $\overline{k}_0^\omega(a,\mu)$. By definition of $\overline{k}_0^\omega(a,\mu)$, we can find $\phi \in \mathcal{A}$ such that $L_0^\omega \phi \leq \lambda\phi$ in $\mathbb{R}$, i.e.

$$(a(x,\omega)\phi')' + \mu(x,\omega)\phi \leq \lambda\phi \text{ in } \mathbb{R}$$

Dividing by $\phi > 0$ since $\phi \in \mathcal{A}$ and integrating by parts over $(-R, R)$ with $h_1 = \frac{1}{\phi}$ and $dh_2 = (a(x,\omega)\phi')'dx$ gives

$$\begin{aligned} 2R\lambda &\geq \int_{-R}^{R} \frac{(a(x,\omega)\phi')'}{\phi} dx + \int_{-R}^{R} \mu(x,\omega) dx \\ \text{(IBP)} \quad &\geq a(R,\omega)\frac{\phi'(R)}{\phi(R)} - a(-R,\omega)\frac{\phi'(-R)}{\phi(-R)} + \int_{-R}^{R} \frac{a(x,\omega)(\phi'(x))^2}{(\phi(x))^2} dx + \int_{-R}^{R} \mu(x,\omega) dx \\ \text{(as } a>0) \quad &\geq a(R,\omega)\frac{\phi'(R)}{\phi(R)} - a(-R,\omega)\frac{\phi'(-R)}{\phi(-R)} + \int_{-R}^{R} \mu(x,\omega) dx \end{aligned}$$

Dividing by $2R$, letting $R \to \infty$ and using the Birkhoff ergodic theorem, i.e. that

$$\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \mu(x,\omega)dx = \mathbb{E}[\mu]$$

almost surely, we obtain, as $\frac{\phi'}{\phi} \in L^\infty(\mathbb{R})$ since $\phi \in \mathcal{A}$, $\lambda \geq \mathbb{E}[\mu]$. By arbitrariness of $\lambda \geq \overline{k}_0^\omega(a,\mu)$ and as $k_0(a,\mu) = \overline{k}_0^\omega(a,\mu)$ by choice of $\omega \in \Omega_1$, it follows that

$$k_0(a,\mu) \geq \mathbb{E}[\mu] \tag{19}$$

By Proposition 3.2, we can choose for almost every $\omega \in \Omega$ as sequence $(\theta_n)_{n\geq 1} \subset \mathcal{B}$ such that

$$\underline{k}_0^\omega(a, a|p+\theta_n|^2 + \mu) \leq \underline{k}_p^\omega(a,\mu) + \frac{1}{n}$$

On the other hand, by equttion (19), for each $n \geq 1$, there exists $\Omega_n \subset \Omega$ with full probability such that

$$\underline{k}_0^\omega(a, a|p+\theta_n|^2 + \mu) \geq \mathbb{E}[a|p+\theta_n|^2 + \mu] = \mathbb{E}[\tilde{a}|p+\theta_n(x,\cdot)|^2 + \tilde{\mu}]$$

where $a(x,\omega) = \tilde{a}(\pi_x\omega)$ and $\mu(x,\omega) = \tilde{\mu}(\pi_x\omega)$, where we are using that a and μ are random stationary ergodic. So, for all $\omega \in \bigcap_{n\geq 1}\Omega_n$, i.e. for a.e. $\omega \in \Omega$ and for all $n \geq 1$, we have obtained

$$\underline{k}_p^\omega(a,\mu) + \frac{1}{n} \geq \mathbb{E}[\tilde{a}|p+\theta_n(x,\cdot)|^2 + \tilde{\mu}] \geq \inf_{\theta\in\mathcal{B}} \mathbb{E}[\tilde{a}|p+\theta(x,\cdot)|^2] + \mathbb{E}[\tilde{\mu}]$$

so letting $n \to \infty$,

$$\underline{k}_p^\omega(a,\mu) \geq \inf_{\theta\in\mathcal{B}} \mathbb{E}[\tilde{a}|p+\theta(x,\cdot)|^2] + \mathbb{E}[\tilde{\mu}] \tag{20}$$

Consider the set $\tilde{\mathcal{B}} = \{\tilde{\theta} \in L^2(\Omega) : \mathbb{E}[\tilde{\theta}] = 0\}$. There is a one-to-one correspondence between the sets $\{\theta(x,\cdot) : \theta \in \mathcal{B}\}$ and $\{\tilde{\theta} \in L^\infty(\Omega : \mathbb{E}[\tilde{\theta}] = 0)\}$ given by $\theta(x,\omega) = \tilde{\theta}(\pi_x\omega)$. Moreover, this set is clearly dense in $\tilde{\mathcal{B}}$ so for a.e. $\omega \in \Omega$, we obtain from (20)

$$\underline{k}_p^\omega(a,\mu) \geq \inf_{\tilde{\theta}\in\tilde{\mathcal{B}}} \mathbb{E}[\tilde{a}|p+\theta(x,\cdot)|^2] + \mathbb{E}[\tilde{\mu}] \tag{21}$$

We minimize the function $\tilde{\mathcal{B}} \ni \tilde{\theta} \mapsto \mathbb{E}[\tilde{a}|p+\tilde{\theta}|^2]$ and find a unique minimizer $\tilde{\theta}_0 = p\left(\frac{1}{\mathbb{E}[1/\tilde{a}]\tilde{a}} - 1\right)$. Therefore by linearity of expectation

$$\inf_{\tilde{\theta}\in\tilde{\mathcal{B}}} \mathbb{E}[\tilde{a}|p+\tilde{\theta}|^2] = \mathbb{E}[\tilde{a}|p+\tilde{\theta}_0|^2] = \mathbb{E}\left[\frac{\tilde{a}p^2}{\tilde{a}^2\mathbb{E}[1/\tilde{a}]^2}\right] = \frac{p^2}{\mathbb{E}[1/\tilde{a}]^2}\mathbb{E}[1/\tilde{a}] = \frac{p^2}{\mathbb{E}[1/\tilde{a}]}$$

so by equation (21), we obtain

$$\underline{k}_p^\omega(a,\mu) \geq \inf_{\tilde{\theta}\in\tilde{\mathcal{B}}} \mathbb{E}[\tilde{a}|p+\theta(x,\cdot)|^2] + \mathbb{E}[\tilde{\mu}] = \frac{p^2}{\mathbb{E}[1/\tilde{a}]} + \mathbb{E}[\tilde{\mu}]$$

almost surely, and therefore by lemma 3.1.,

$$w^*(a,\mu) = \min_{p>0} \frac{k_p(a,\mu)}{p} \geq \min_{p>0}\left(\frac{\mathbb{E}[\tilde{\mu}]}{p} + \frac{p}{\mathbb{E}[1/\tilde{a}]}\right) = 2\sqrt{\frac{\mathbb{E}[\tilde{\mu}]}{\mathbb{E}[1/\tilde{a}]}} = w^*(\mathbb{E}[1/a]^{-1}, \mathbb{E}[f])$$

where we have used that a and μ are stationary ergodic. □

3.4. Statement of further results.

- Dependence with respect to the diffusion: if $\mu(x,\omega)$ is constant in (x,ω), then
$$\kappa \mapsto w^*(\kappa a, f)$$
is increasing.
- Dependence with respect to the reaction: if $f_s(x,\omega,0) \leq g_s(x,\omega,0)$ for all $x \in \mathbb{R}$ for a.e. $\omega \in \Omega$, then
$$w^*(a,f) \leq w^*(a,g)$$

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