Invisibility Cloaking for Electrostatics and Electromagnetics

Math 581 - Partial Differential Equations 2

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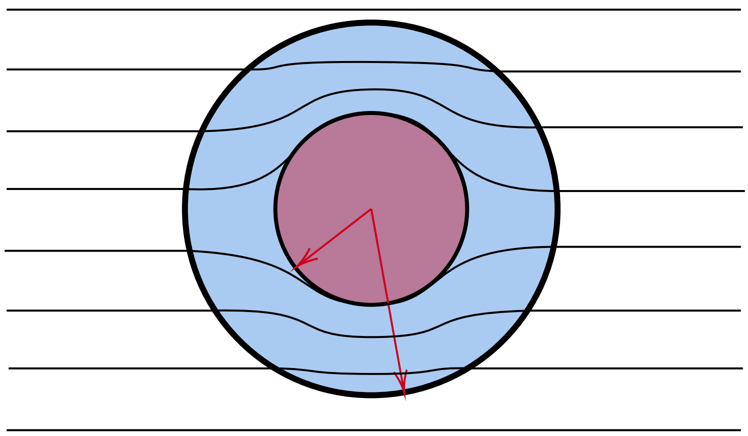

April 22, 2019

1 Introduction

In this report, we examine mathematical invisibility cloaking, in particular for the Electrostatic setting in regions of space of dimension $n \geq 3$, as well as the Electromagnetic setting in three dimensions. The cloaking techniques we examine are not only of importance to the theory of partial differential equations, but have vast practical applications; namely invisibility of certain metamaterials to the human eye.

We say that a region is *cloaked* if its content, as well as its cloak, are indistinguishable from the background space with respect to some exterior wave measurements. So if we have a cloaked region, we could put an object in the region and it would not only be invisible, but we would be oblivious to the fact that it is even being hidden. Thus the notion of invisibility is directly related to classical *inverse problems* (see Appendix); we seek counterexamples in which different parameters lead to the same data.

In general, to obtain an ideal invisibility cloak, one first selects a region Ω in the background space. Then, consider a point $p \in \Omega$ and let F be a diffeomorphism which blows up p to a region D within Ω. The medium around p is "compressed" via the push-forward of the transformation F to form the cloaking medium in $\Omega \setminus \bar{D}$. The region D is then what we refer to as the *cloaked region*. This general construction will be used in both of the settings in this report.

Our first main goal will be able to establish invisibility for a class of transformations in the setting where $\Omega \in \mathbb{R}^n$ and the metric g on Ω corresponds to an anisotropic conductivity. To do so, we shall first explore invisibility for boundary value problems on Riemannian manifolds, with the purpose of introducing the role of Dirichlet to Neumann Maps. Next, we specify the electrostatic setting as a special case, and proceed to introduce the main theorem for Electrostatic Cloaking. Then, we construct an important example in three dimensions to illustrate the main theorem, and finally provide proof of the Electrostatic Cloaking Theorem.

Our other main goal will be to examine invisibility in the Electromagnetic setting; namely for Maxwell's equations. After specifying the setting and introducing the notion of finite energy solutions, we determine to what extent the invisibility constructions we have seen can be used to establish Electromagnetic cloaking.

2

Invisibility for Electrostatics

2.1 Importance of Dirichlet to Neumann Maps

Let (M, g) be a compact two dimensional Riemannian manifold with non-empty boundary. Consider the manifold $\tilde{M} := M \setminus \{x_0\}$ with metric

$$\tilde{g}_{ij}(x) = \frac{1}{\text{dist}_M(x, x_0)^2} g_{ij}(x)$$

where $\text{dist}_M(x, x_0)$ denotes the distance between x and x_0 on (M, g). Note that $(\tilde{M}, \tilde{g})$ is no longer compact, but it is complete (recall that a geodesically complete Riemannian manifold is one for which every maximal geodesic is defined on $\mathbb{R}$) and has boundary $\partial\tilde{M} = \partial M$.

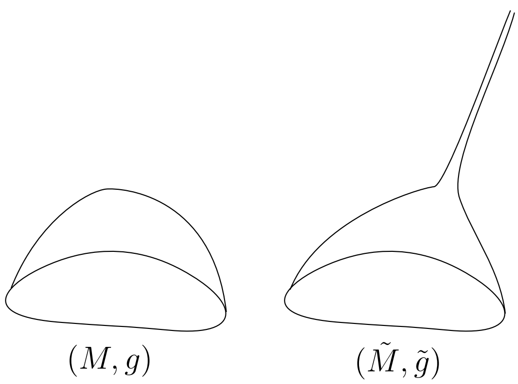

Figure 2.1: The manifolds (M, g) and $(\tilde{M}, \tilde{g})$. Essentially, in $\tilde{M}$ we have "pulled" the point x_0 to infinity. (Figure created using TikZ TeX package)

We consider the boundary value problems (Dirichlet Problems)

$$\begin{cases} \Delta_g u = 0 & \text{in} \quad M \\ u = f & \text{on} \quad \partial M \end{cases} \tag{2.1}$$

and

$$\begin{cases} \Delta_{\tilde{g}} \tilde{u} = 0 & \text{in} \quad \tilde{M} \\ \tilde{u} = f & \text{on} \quad \partial\tilde{M} \end{cases}$$

where Δ_g denotes the *Laplace-Beltrami Operator* associated to g.

Remark. The Laplace operator Δ can be generalized to operate on functions defined on Riemannian manifolds; this is called the Laplace-Beltrami operator. In local coordinates, it is given by

$$\Delta_g u = \frac{1}{\sqrt{det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{det(g)} g^{ij} \frac{\partial u}{\partial x_j} \right)$$

where (g^{ij}) is the matrix inverse of the metric tensor (g_{ij}).

The Dirichlet problems are uniquely solvable and hence define the *Dirichlet to Neumann maps* $\Lambda_{M,g} f = \partial_\nu u|_{\partial M}$, $\Lambda_{\tilde{M},\tilde{g}} f = \partial_\nu \tilde{u}|_{\partial \tilde{M}}$, where ν denotes the outer unit normal. Note that the Dirichlet to Neumann map is a particular *Poincaré-Steklov* operator which maps the value of a harmonic function on the boundary to the normal derivative on the boundary. So, in this setting, if we have $\Lambda_{M,g} = \Lambda_{\tilde{M},\tilde{g}}$, then the boundary measurements for the manifolds (M, g) and $(\tilde{M}, \tilde{g})$ coincide. Hence the media represented by g and $\tilde{g}$ are indistinguishable via boundary measurements, so we have a form of invisibility if the Dirichlet to Neumann maps are equal.

2.2 Electrostatic Setting and Main Theorem

With the previous section in mind, we shall now introduce the setting for which we shall work with for the remainder of this chapter. Let $\Omega \subset \mathbb{R}^n$ (where $n \geq 3$) be a domain with metric $g = g_{ij}$. Here u shall denote the electric potential, f is the prescribed voltage on the boundary $\partial\Omega$, and let σ denote the *anisotropic conductivity* on Ω, which is defined by a symmetric positive semi-definite matrix-valued function $\sigma = (\sigma^{ij}(x))$. The conductivity is related to the metric g by

$$\sigma^{ij} = |\det(g)|^{\frac{1}{2}} g^{ij} \ , \ g^{ij} = \det(\sigma)^{\frac{2}{n-2}} \sigma^{ij} \tag{2.2}$$

The governing partial differential equation of electrostatics is the *Conductivity Equation* :

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \ \Omega \\ u = f & \text{on } \ \partial\Omega \end{cases} \tag{2.3}$$

where $f \in H^{\frac{1}{2}}(\partial\Omega)$. Note that this Conductivity Equation is a specific case of the Dirichlet Problem (2.1). We shall consider solutions $u \in L^\infty(\Omega)$ of the Conductivity Equation (2.3) in the sense of distributions. Namely, $u \in H^1(\Omega)$ and $\sigma\nabla u \in H(\Omega; \nabla\cdot) = \{v \in L^2(\Omega) : \nabla \cdot v \in L^2(\Omega)\}$. If such a unique solution exists, then we have the resulting Dirichlet to Neumann map; rather a "Voltage to Current" map:

$$\Lambda_\sigma f = n \cdot \sigma \nabla u|_{\partial\Omega}$$

We are now ready to state the main theorem:

Theorem 2.2.1 (Perfect Cloaking for Electrostatics)**.** *Let $\Omega \subset \mathbb{R}^n$ where $n \geq 3$ and let $g = g_{ij}$ be a metric on Ω. Let $D \subset \Omega$ and $y \in D$ be such that there exists a C^∞-diffeomorphism*

$$F : \Omega \setminus \{y\} \to \Omega \setminus \bar{D}$$

satisfying $F|_{\partial\Omega} = Id$, $dF(x) \geq c_0 I$, and $det(dF(x)) \geq c_1 dist_{\mathbb{R}^n}(x, y)^{-1}$ where dF is the Jacobian matrix in Euclidean coordinates of $\mathbb{R}^n$ and $c_0, c_1 > 0$. Let $\tilde{g} = F_* g$ on $\Omega \setminus \bar{D}$ and $\hat{g}$ be an extension of $\tilde{g}$ into D such that it is positive definite on the interior of D. Finally, let γ and $\hat{\sigma}$ be the anisotropic conductivities related to metrics g and $\hat{g}$ by (2.2). Then, the Conductivity equation (2.3) with conductivity $\hat{\sigma}$ is uniquely solvable, and

$$\Lambda_{\hat{\sigma}} = \Lambda_\gamma$$

2.3 Concrete Example in Three Dimensions

In this section, we would like to give an explicit construction to use the main theorem 2.2.1. Let $\Omega := B(0,2) \subset \mathbb{R}^3$ and $D := B(0,1) \subset \Omega$. Define the map $F : B(0,2) \setminus \{0\} \to B(0,2) \setminus \bar{B}(0,1)$ given by

$$F : x \mapsto \left(\frac{|x|}{2} + 1\right) \frac{x}{|x|} \tag{2.4}$$

and note that $F|_{\partial B(0,2)} = \mathrm{Id}$.

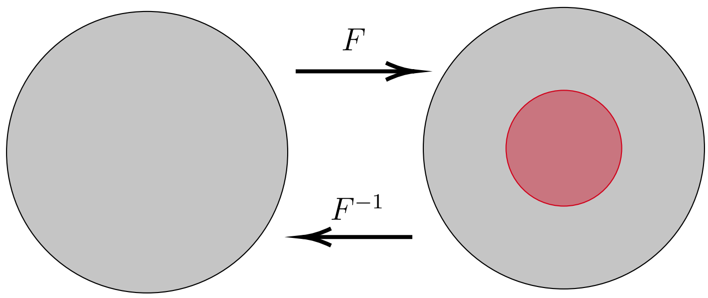

Figure 2.2: The map $F : B(0,2) \setminus \{0\} \to B(0,2) \setminus \bar{B}(0,1)$. F stretches the origin to the ball $\bar{B}(0,1)$. (Figure created using TikZ)

Let $\gamma = 1$ be the homogenous conductivity in Ω and define $\sigma = F_*\gamma$. Let g be the metric corresponding to γ and $\tilde{g}$ be the metric corresponding to σ (see (2.2)). We have the Jacobian

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{2.5}$$

and hence with respect to standard spherical coordinates on $\Omega \setminus \{0\}$,

$$g = J^T J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

Thus, $\det(g) = r^4 \sin^2\theta$ and so by (2.2),

$$\gamma = \begin{bmatrix} r^2\sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \frac{1}{\sin\theta} \end{bmatrix}$$

Now, in the annulus $\{r : 1 < r < 2\}$, since $\sigma = F_*\gamma$,

$$\sigma = \begin{bmatrix} 2(r-1)^2\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & \frac{2}{\sin\theta} \end{bmatrix}$$

So, using (2.2), we lastly have that

$$\tilde{g} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4(r-1)^2 & 0 \\ 0 & 0 & 4(r-1)^2\sin^2\theta \end{bmatrix}$$

Thus, letting $\hat{\sigma}$ be the continuation of σ that is C^∞ in D, all the conditions in the main theorem 2.2.1 are met. Hence, $\Lambda_{\hat{\sigma}} = \Lambda_\gamma$ and in particular, boundary measurements for the homogenous conductivity $\gamma = 1$ and the degenerated conductivity $\hat{\sigma}$ are the same. So the map (2.4) is a singular cloaking transformation and the "hole" $D = B(0,1)$ is the cloaked region. We can place some object in $B(0,1)$ and it would be undetectable and unaffected by exterior currents; the object is indistinguishable from a homogenous material. This is precisely our notion of mathematical invisibility.

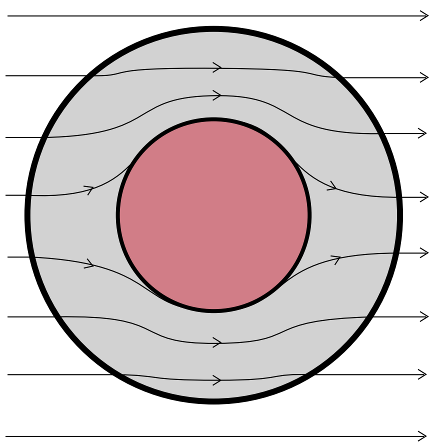

Figure 2.3: Analytic solution for the currents corresponding to conductivity σ. (Figure created using TikZ)

2.4 Proof of the Main Theorem

Here, we provide a proof of theorem 2.2.1. We will first need the following proposition, whose proof can be found in [5]. The proof uses a Brownian motion argument and we will need Hunt's Theorem as well as Kakutani's Formula (see Appendix). Hence the proof of the proposition is not of particular interest to us in the setting of this course.

Proposition 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ where $n \geq 3$ and let $g = g_{ij}$ be a metric on Ω. Let u satisfy*

$$\begin{cases} \Delta_g u = 0 \quad in \quad \Omega \\ u|_{\partial\Omega} = f_0 \in C^\infty(\partial\Omega) \end{cases}$$

Let $D \subset \Omega$ and $y \in D$ such that there exists a diffeomorphism $F : \Omega \setminus \{y\} \to \Omega \setminus \bar{D}$ satisfying $F|_{\partial\Omega} = Id$. Let $\tilde{g} = F_* g$ and v be a function satisfying

$$\begin{cases} \Delta_{\tilde{g}} v = 0 \quad in \quad \Omega \setminus \bar{D} \\ v|_{\partial\Omega} = f_0 \\ v \in L^\infty(\Omega \setminus \bar{D}) \end{cases}$$

Then, u and $F^* v$ coincide and have the same Cauchy data on $\partial\Omega$,

$$\partial_\nu u|_{\partial\Omega} = \partial_{\tilde{\nu}} F^* v|_{\partial\Omega}$$

where ν is the unit normal in metric g and $\tilde{\nu}$ is the unit normal in metric $\tilde{g}$. Moreover, for $c_0 := u(y)$, we have that $\lim_{x \to \partial D} v(x) = c_0$.

We also shall use the following standard result (for example, see [7]):

Proposition 2.4.2. *The normal trace $\vec{W} \mapsto n \cdot \vec{W}|_{\partial D}$ is a continuous map*

$$H(\Omega \setminus \bar{D}; \nabla\cdot) \cap C(\bar{\Omega} \setminus int(D)) \to H^{-1/2}(\partial D)$$

With the above propositions in hand, we proceed with the proof of the main theorem. First, extend the function v (from proposition 2.4.1) inside D:

$$h(x) := \begin{cases} v(x) & \text{for } \ x \in \Omega \setminus \bar{D} \\ c_0 & \text{for } \ x \in \bar{D} \end{cases}$$

where $c_0 = u(y) = \lim_{x \to \partial D} v(x)$. We need only to show that h is a solution of the Conductivity Equation (2.3).

To that end, first we must show that $h \in H^1(\Omega)$. Well, when $y(x) = F^{-1}(x)$, we have that $v(x) = u(F^{-1}(x))$ and

$$\frac{\partial v}{\partial x^j}(x) = \frac{\partial u}{\partial y^k}(y(x)) \frac{\partial y^k}{\partial x^j}(x)$$

So, since $u \in H^1(\Omega)$ and $\frac{\partial y^k}{\partial x^j} \in L^\infty(\Omega \setminus \bar{D})$, we conclude that $v \in H^1(\Omega \setminus \bar{D})$. Now, $h \in C(\Omega \setminus \text{int}(D))$ and the trace $v \mapsto v|_{\partial D}$ is a continuous map $C(\Omega \setminus \text{int}(D)) \cap H^1(\Omega \setminus \bar{D}) \to L^2(\partial D)$. Hence, $v|_{\partial D}$ is well defined and is the constant function of value c_0. We have that $h|_D = c_0 \in H^1(D)$ and $h|_{\Omega \setminus \bar{D}} = v \in H^1(\Omega \setminus \bar{D})$, and the trace from both sides of ∂D coincide, so we conclude that $h \in H^1(\Omega)$.

Now, define the functions

$$V_j(x) = |\text{det}\tilde{g}(x)|^{\frac{1}{2}} \delta_{jk} \tilde{g}^{ki} \partial_i v$$

and let $\vec{V} := (V_1(x), ..., V_n(x))$. It remains to show that $\nabla \cdot \vec{V} = 0$ in Ω and $\vec{V} \in H(\Omega; \nabla\cdot)$. Well, in $\Omega \setminus \bar{D}$, by the conditions on F (namely $dF(x) \geq c_0 I$ and $\det(dF(x)) \geq c_1 \text{dist}_{\mathbb{R}^n}(x, y)^{-1}$), we have that $|\text{det}\tilde{g}(x)| \leq c_2 \text{dist}_{\mathbb{R}^n}(F^{-1}(x), y)^2$. Hence (in $\Omega \setminus \bar{D}$),

$$|V_j(x)| \leq c_3 \text{dist}_{\mathbb{R}^n}(F^{-1}(x), y)$$

Therefore, $\vec{V} \in H(\Omega \setminus \bar{D}; \nabla\cdot) \cap C(\bar{\Omega} \setminus int(D))$. However, by proposition 2.4.2 we have $n \cdot \vec{V}|_{\partial D} = 0$, and since the normal traces of $\vec{V}$ coincide from both sides of ∂D, we conclude that $\vec{V} \in H(\Omega; \nabla\cdot)$. Moreover,

$$\nabla \cdot \vec{V} = \sum_{j,k=1}^{n} \partial_k \left(|\text{det}\tilde{g}(x)|^{\frac{1}{2}} \tilde{g}^{ki} \partial_i h \right) = 0 \ \text{ in } \ \Omega$$

In conclusion, we have shown that h is a solution to the Conductivity Equation in the sense of distributions, and the proof of the main theorem 2.2.1 is complete.

3

Invisibility for Maxwell's Equations

3.1 Electromagnetic Setting

Let (M, g) be a smooth compact oriented connected Riemannian 3-manifold with non-empty boundary. The governing differential equations for classical electromagnetism are *Maxwell's equations*; in time-harmonic form they are:

$$\begin{cases} \text{curl } E(x) = ikB(x) \\ \text{curl } H(x) = J - ikD(x) \end{cases} \tag{3.1}$$

where E is the electric field (a 1-form), H is the magnetic field (a 1-form), D is the electric flux (a 2-form), B is the magnetic flux (a 2-form), and J is the electric current. Recall the constitutive relations:

$$\begin{cases} D(x) = \epsilon(x)E(x) \\ B(x) = \mu(x)H(x) \end{cases} \tag{3.2}$$

where ϵ is the electric permittivity and μ is the magnetic permeability. If we denote (in local coordinates on M)

$$\begin{aligned} E &= E_j(x)dx^j \\ H &= H_j(x)dx^j \\ D &= D^1(x)dx^2 \wedge dx^3 + D^2(x)dx^3 \wedge dx^1 + D^3(x)dx^1 \wedge dx^2 \\ B &= B^1(x)dx^2 \wedge dx^3 + B^2(x)dx^3 \wedge dx^1 + B^3(x)dx^1 \wedge dx^2 \end{aligned}$$

then the constitutive relations (3.2) become

$$\begin{cases} D^j = \epsilon^{jk} E_k \\ B^j = \mu^{jk} H_k \end{cases}$$

and in local coordinates on M we have

$$\epsilon^{jk} = \mu^{jk} = |g|^{\frac{1}{2}} g^{jk}$$

Now, we need to introduce singular material parameters $\tilde{\epsilon}$ and $\tilde{\mu}$ to make cloaking possible. We shall work in the ball $B(0,2) \subset \mathbb{R}^3$; so define the map $F : B(0,2) \setminus \{0\} \to B(0,2) \setminus \bar{B}(0,1)$ given by (2.4). Let the metric g on $B(0,1)$ be the Euclidean metric and define the metric on

$B(0,2) \setminus \bar{B}(0,1)$ by $\tilde{g} = F_* g$. Then, we define our singular permittivity and permeability:

$$\tilde{\epsilon}^{jk} = \tilde{\mu}^{jk} = \begin{cases} |\tilde{g}|^{\frac{1}{2}} \tilde{g}^{jk} & \text{in} \quad B(0,2) \setminus \bar{B}(0,1) \\ \delta^{jk} & \text{in} \quad B(0,1) \end{cases}$$

which are singular on $\partial B(0,1)$. Note the similarity to the relation (2.2).

3.2 Notion of Solutions and Main Theorem

Now, for the remainder of this chapter, let $\epsilon = 1$ and $\mu = 1$.

Definition 3.2.1. We say that $(\tilde{E}, \tilde{H})$ is a *finite energy solution* to Maxwell's equations (3.1) on $B(0,2)$ if $\tilde{E}$ and $\tilde{H}$ are 1-forms and $\tilde{D} := \tilde{\epsilon}\tilde{E}$, $\tilde{B} := \tilde{\mu}\tilde{H}$ are 2-forms in $B(0,2)$ with $L^1(B(0,2), dx)$ -coefficients satisfying:

$$||\tilde{E}||^2_{L^2(B(0,2),|\tilde{g}|^{1/2} dV_0(x))} = \int_{B(0,2)} \tilde{\epsilon}^{jk} \tilde{E}_j \bar{\tilde{E}}_k dV_0(x) < \infty$$
$$||\tilde{H}||^2_{L^2(B(0,2),|\tilde{g}|^{1/2} dV_0(x))} = \int_{B(0,2)} \tilde{\mu}^{jk} \tilde{H}_j \bar{\tilde{H}}_k dV_0(x) < \infty$$

where dV_0 is the standard Euclidean volume and

$$\int_{B(0,2)} \left((\nabla \times \tilde{h}) \cdot \tilde{E} - ik\tilde{h} \cdot \tilde{\mu}(x)\tilde{H} \right) dV_0(x) = 0$$
$$\int_{B(0,2)} \left((\nabla \times \tilde{e}) \cdot \tilde{H} + \tilde{e} \cdot (ik\tilde{\epsilon}(x)\tilde{E} - \tilde{J}) \right) dV_0(x) = 0$$

for all 1-forms $\tilde{e}, \tilde{h}$ on $B(0,2)$ having in the Euclidean coordinates components in $C_0^\infty(B(0,2))$.

Denoting $M \setminus \{0\} := (M_1 \setminus \{0\}) \cup M_2$, we are now ready to state the main theorem:

Theorem 3.2.1 (Cloaking for Electromagnetics)**.** *Let E and H be 1-forms with measurable coefficients on $M \setminus \{0\}$ and let $\tilde{E}$ and $\tilde{H}$ be 1-forms with measurable coefficients on $B(0,2) \setminus \partial B(0,1)$ such that $\tilde{E} = F_* E$, $\tilde{H} = F_* H$. Let J and $\tilde{J}$ be 2-forms with smooth coefficients on $M \setminus \{0\}$ and $B(0,2) \setminus \partial B(0,1)$, that are supported away from $\{0\}$ and $\partial B(0,1)$ such that $\tilde{J} = F_* J$. Then, the following are equivalent:*

1. *$\tilde{E}$ and $\tilde{H}$ satisfy Maxwell's equations*

$$\begin{cases} \nabla \times \tilde{E} = ik\tilde{\mu}(x)\tilde{H} \ , \ \nabla \times \tilde{H} = -ik\tilde{\epsilon}(x)\tilde{E} + \tilde{J} \quad \textit{on} \quad B(0,2) \\ \nu \times \tilde{E}|_{\partial B(0,2)} = f \end{cases}$$

   in the sense of definition 3.2.1.

2. *E and H satisfy Maxwell's equations*

$$\begin{cases} \nabla \times E = ik\mu(x)H \ , \ \nabla \times H = -ik\epsilon(x)E + J \quad \textit{on} \quad M_1 \\ \nu \times E|_{\partial M_1} = f \end{cases} \tag{3.3}$$

   and

$$\nabla \times E = ik\mu(x)H \ , \ \nabla \times H = -ik\epsilon(x)E + J \quad \textit{on} \quad M_2 \tag{3.4}$$

with Cauchy data

$$\nu \times E|_{\partial M_2} = b^e, \ \nu \times H|_{\partial M_2} = b^h \tag{3.5}$$

that satisfies $b^e = b^h = 0$.

Moreover, if E *and* H *solve equations* (3.3) *through* (3.5) *with* $b^e \neq 0$ *or* $b^h \neq 0$*, then* $\tilde{E}, \tilde{H}$ *are not solutions of Maxwell's equations on* $B(0,2)$ *in the sense of definition 3.2.1.*

In other words, theorem 3.2.1 tells us that finite energy solutions to Maxwell's equations must satisfy the hidden boundary conditions

$$\nu \times \tilde{E} = 0 \ , \ \nu \times \tilde{H} = 0 \ \text{ on } \ \partial B(0,1)$$

So, with the cloaking techniques we have so far, we only have invisibility in the Electromagnetic setting when there are no internal currents in the cloaked region (since theorem 3.2.1 implies that a nonzero current in the cloaked region gives non-existence of finite energy solutions). To get invisibility for more general settings (in other words for "active" objects), we would need a new cloaking technique entirely. This has been examined for example in [4].

4 Conclusion and Open Problems

In this paper, we have used a diffeomorphism which blows up a point, as well as the invariance properties of the governing partial differential equations, to establish invisibility for anisotropic conductivities as well as electromagnetic parameters. This "blow up a point" construction is one of the primary methods used in cloaking, and is moreover equivalent to some of the other established methods (such as blowing up a small region instead of just a point). However, this method does have its limitations, which leads to some interesting open problems.

For example, one problem of immense practical importance that still remains unsolved is the construction of a one-way cloak. Recalling Figure 2.3, we see that the current cannot penetrate the inner cloaked region and hence the region is invisible from outside observation. Conversely, if we are to make observations from inside the cloaked region, the exterior space is then invisible. Formally, we refer to this as a *two-way cloak*. Obviously, from a practical standpoint, we would want a cloak that can "see" the exterior space from observations made within the cloaked region. This would be called a *one-way cloak*. Perhaps one way to approach this problem would be to work in a more general geometry framework; maybe the Finsler geometry.

Appendix

Inverse Problems

Here we introduce the basic notion of a mathematical inverse problem.

Definition 4.0.1. A measurement operator $\mathcal{M}$ is a map from the parameters $x \in \mathbb{D}(\mathcal{M}) \subset \mathcal{X}$ to the data $y \in \mathcal{D}$, where $\mathbb{D}(\mathcal{M})$ is the domain of definition of $\mathcal{M}$, $\mathcal{X}$ is some functional space (typically a Banach or Hilbert space), and $\mathcal{D}$ is the space of data.

So, assuming that $\mathcal{M}$ takes values in $\mathcal{D}$, we shall write $y = \mathcal{M}(x)$. Then, the inverse problem is if we can uniquely reconstruct parameters from the knowledge of the measurement operator. For each specific inverse problem, we need to define exactly what the measurement operator is. For example, in our Electrostatic invisibility setting, the Dirichlet to Neumann maps function as the measurement operator.

Brownian Motion

Here we recall some results that are used in the proof of proposition 2.4.1.

We have the result due to G. Hunt (for a proof, see [7]):

Theorem 4.0.1 (Hunt's Theorem). *Let B_t^x be the Brownian motion on manifold M. Let $\Omega \subset M$ be a non-empty open set with smooth boundary and let $\mathcal{F} := M \setminus \Omega$. Define*

$$e_{\mathcal{F}}(x) := \mathbb{P}[\{\exists t > 0 : B_t^x \in \mathcal{F}\}]$$

and let $s_\Omega(x)$ be the super-harmonic potential of Ω. Then, for any $x \in M$ we have that:

$$e_{\mathcal{F}}(x) = s_\Omega(x)$$

Also, recall Kakutani's solution for the Dirichlet Problem:

Theorem 4.0.2 (Kakutani's Formula). *Let Ω be a domain in $\mathbb{R}^n$ and let f_0 be a bounded function on $\partial\Omega$. Consider the problem*

$$\begin{cases} \Delta u(x) = 0 \ , \ \ x \in \Omega \\ \lim_{y \to x} u(y) = f_0(x) \ , \ \ x \in \partial\Omega \end{cases}$$

If there exists a solution u, then it is the expected value of $f_0(x)$ at the random first exit point from Ω for a canonical Brownian motion starting at x.

Bibliography

[1] Hongyu Liu and Gunther Uhlmann (2014), Regularized Transformation-Optics Cloaking in Acoustic and Electromagnetic Scattering, *Inverse Problems and Imaging* (44), 111 – 136.

[2] Allan Greenleaf, Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann (2008), Invisibility and Inverse Problems, *Bull. Amer. Math. Soc.* (46), 55-97

[3] R V Kohn, H Shen, M S Vogelius, and M I Weinstein (2008), Cloaking via Change of Variables in EIT, *Inverse Problems* (24)

[4] Allan Greenleaf, Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann (2007), Full-wave invisibility of active devices at all frequencies, *Commun. Math. Phys.* (275), 749

[5] Allan Greenleaf, Matti Lassas, and Gunther Uhlmann (2003), On Nonuniqueness for Calderon's Inverse Problem , *Math. Res. Lett.* (10), 685–693

[6] Allan Greenleaf, Matti Lassas, and Gunther Uhlmann (2003), Anisotropic Conductivities that cannot be detected by EIT, *Physiol. Meas.* (24), 413-9

[7] Alexander Grigor'Yan (1999), Analytic and Geometric Background of Recurrence and Non-exposion of the Brownian motion on Riemannian Manifolds, *Bull. Amer. Math. Soc.* (36), 135-249