The Method of Moving Planes and Applications to Elliptic Equations

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Abstract

In this set of notes, we shall investigate the qualitative behaviour of classical positive solutions to a large class of elliptic problems in symmetric domains of $\mathbb{R}^n$. More precisely, we begin by establishing maximum and comparison principles and use these together with the method of moving planes to show that all positive classical solutions of $-\Delta u = f(u)$ in the unit ball are radially symmetric about the origin, provided f satisfies a Lipschitz condition. Furthermore, we treat classical solutions to nonlinear eigenvalue-type problems in $\mathbb{R}^n$. Finally, we will discuss applications of this method to systems and unbounded domains.

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1 Introduction

In this document, we will mainly be concerned with the qualitative properties of classical solutions to a large number of elliptic problems. More specifically, we will be interested in the radial symmetry of *positive* classical solutions to elliptic partial differential equations in symmetric domains of $\mathbb{R}^n$.

Radial symmetry is an extremely important property when considering solutions to various equations and systems in $\mathbb{R}^n$. Indeed, knowing that a particular equation or system admits only radial classical solutions allows one to reduce the given problem to that of an ordinary differential equation (ODE). Of course, these systems and equations are in general much easier to handle and, consequently, one can readily deduce many other properties and sometimes even closed forms of solutions. Furthermore, existence and non-existence questions are much better understood when working in the context of ODEs. Put differently, establishing the radial symmetry of solutions is a useful tool for establishing the existence or non-existence of classical solutions to a given problem. Within this approach, it often proves useful to make use of a priori decay estimates together with symmetry properties.

1.1 Introducing the Method of Moving Planes

Having now partially motivated the symmetry of solutions, we give the main setup of the problem at hand. Our goal in these notes is to introduce the very useful *method of moving planes*, which is a particularly viable tool for establishing the symmetry of solutions. Despite its numerous recent applications, the heart of this method can be understood using only a handful of elementary concepts. More precisely, we will require

1. Hopf's lemma for elliptic operators;
2. strong and weak maximum principles for elliptic operators;
3. various comparison principles.

We will devote the first part of this project to establishing the tools required for the method of moving planes. Once we have these, our next goal will be to illustrate the method for positive solutions of the equation

$$\begin{cases} -\Delta u = f(u) & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0) \end{cases} \tag{1}$$

where we assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous. Before moving on, we point out some important problems that fall into this category.

- Let $k \geq 0$ and fix $\beta, \gamma > 0$. Then, any result established for (1) will also apply to solutions of

  $$-\Delta u + \gamma u = \beta u^{2k+1} \quad \text{in } B_1(0).$$

  In particular, this covers the nonlinear stationary Schrödinger equation

  $$-\Delta u + u = u^3 \quad \text{in } B_1(0)$$

  with homogeneous boundary condition.

- Problem (1) covers the eigenvalue problem in the ball $B_1(0)$ given by

  $$\begin{cases} -\Delta u = \lambda u & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0) \end{cases}$$

  for $\lambda \neq 0$.

In addition to (1), it will be possible to deduce the radial symmetry for a sub-class of positive classical solutions to the following problem:

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{2}$$

for appropriate n and p.

Without further ado, let us now formulate our main result, which is Theorem 5.1 in [CL17].

Theorem 1 (Theorem 5.1 in [CL17])**.** *Let $u \in C^2(B_1(0)) \in C(\bar{B}_1(0))$ be a positive solution of* (1)*. Then, u is radially symmetric and monotone decreasing about the origin.*

As was implied above for Problem (2), the method of moving planes applies even to unbounded domains (e.g. $\mathbb{R}^n$). In fact, it is often enough to have a domain which is symmetric in one of the coordinates. However, the situation becomes much more complex when working in an unbounded domain. We will illustrate this phenomenon when establishing the following analogue of Theorem 1 for a nonlinear eigenvalue-type problem in the whole of $\mathbb{R}^n$:

Theorem 2 (Theorem 5.2 in [CL17]). *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of the equation*

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n, \tag{3}$$

with $n \geq 3$, $p := \frac{n+2}{n-2}$ and such that $u \in O(|x|^{2-n})$. Then, there exists a point $x_0 \in \mathbb{R}^n$ about which u is radially symmetric and monotone decreasing.

Remark 1. It is actually known (see [CL17]) that any $u \in C^2(\mathbb{R}^n)$ satisfying the hypothesis of Theorem 2 must be of the form

$$u(x) = \frac{[n(n-2)\lambda^2]^{(n-2)/4}}{\left(\lambda^2 + |x - x_0|^2\right)^{(n-2)/2}}$$

in all of $\mathbb{R}^n$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. It is also easy to check that the above is always a solution of (3).

Although we will only illustrate the method of moving planes for these simpler cases, it is nonetheless useful when studying much more general equations. For instance, it was proven in [Gid81] that any positive function $u \in C^2(\mathbb{R}^N)$ satisfying

$$-\Delta u + m^2 u = \lambda u^3 \quad \text{in } \mathbb{R}^3$$

for some $\lambda, m > 0$ and

$$\lim_{|x| \to \infty} u(x) = 0,$$

will be radially symmetric and decreasing about some $x_0 \in \mathbb{R}^3$. However, the approach taken in this paper is much less elementary and first requires some heavy a priori asymptotic analysis. Nonetheless, the approach within the method itself does not change when considering these more general equations, systems, and domains. We also point out that the method often continues to apply when treating solutions to both systems of differential and integral equations. For the latter, one typically begins by establishing regularity results and can sometimes use the resulting symmetry to obtain sharp decay estimates for solutions.

This document will be broken down into three main components. First, we will establish important maximum principles for uniformly elliptic operators. In particular, we will be handling Hopf's lemma which plays a critical role in the maximum principles for elliptic operators. Subsequently, we will introduce the comparison, decay, and narrow region principles. Finally, we will piece all of these together in order to prove Theorems 1 and 2.

2 Preliminary Results

In this section, we somewhat extend the well known maximum principles for the Laplacian Δ to more general elliptic operators. These results are standard and can be found in a variety of texts (see Evans, Gilbarg-Trundinger, or Jürgen Jost's book), but are nevertheless critical components for the method of moving planes.

Unlike in the case of harmonic functions, this journey begins with a weak maximum principle. Later, we will strengthen the statement to obtain a general *strong maximum principle.*

Theorem 3 (Weak maximum Principle for Elliptic operators, Theorem 2 in §6.4.1 of [Eva10])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and consider an elliptic operator of the form*

$$L = -\sum_{i,j} a_{ij}\partial_{ij} + \sum_{i} b_i \partial_i + c,$$

where the functions b_1, c are assumed to be bounded and (a_{ij}) are functions satisfying the uniform ellipticity condition with a constant $\lambda > 0$. That is,

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda\left|\xi\right|^2 \qquad \forall \xi \in \mathbb{R}^n.$$

Let us suppose furthermore that $c \geq 0$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ in all of Ω. Then,

$$\min_{\overline{\Omega}} u \geq \min_{\partial\Omega} u^-$$

with $u^- = \min\{0, u\}$.

Proof. We begin by picking $\alpha > 0$ such that

$$\alpha^2\lambda - \alpha\left\|b_1\right\|_{L^\infty} - \left\|c\right\|_\infty > 0. \tag{4}$$

It will soon become clear towards why α was chosen as such. Next, consider an arbitrary $\varepsilon > 0$ and define

$$u_\varepsilon(x) := u(x) - \varepsilon e^{\alpha x_1}$$

A simple computation shows that

$$\begin{aligned} Lu_\varepsilon = Lu - \varepsilon L\left(e^{\alpha x_1}\right) &\geq \varepsilon\left(a_{11}\alpha^2 e^{\alpha x_1} - b_1\alpha e^{\alpha x_1} - ce^{\alpha x_1}\right) \\ &\geq \varepsilon e^{\alpha x_1}\left(\alpha^2\lambda - \alpha\left\|b_1\right\|_\infty - \|c\|_\infty\right) \\ &> 0. \end{aligned} \tag{5}$$

Here, the second inequality arises from the uniform ellipticity condition with $\xi = (1, 0, \ldots, 0)$.

Having completed these preliminary computations, suppose for a contradiction that

$$\min_{\overline{\Omega}} u_\varepsilon < \min_{\partial\Omega} u_\varepsilon^-. \tag{6}$$

Then there must exist a point $x_0 \in \Omega$ such that

$$u_\varepsilon(x_0) = \min_{\overline{\Omega}} u_\varepsilon < \min_{\partial\Omega} u_\varepsilon^- \leq 0.$$

At this point x_0, we have $\nabla u_\varepsilon = 0$ and $D^2 u_\varepsilon(x_0)$ must be positive semi-definite (here, D^2 denotes the Hessian matrix). On the other hand, using that the matrix $(a_{ij}(x_0))$ is positive definite, we see that (after diagonalizing the matrix - we refer the reader to [Eva10] for details)

$$-\sum_{i,j} a_{ij}(x_0)\partial_{ij}u_\varepsilon(x_0) \leq 0$$

Combining the above inequality with $\nabla u_\varepsilon(x_0) = 0$, we observe

$$\begin{aligned} Lu_\varepsilon(x_0) &= -\sum_{i,j} a_{ij}(x_0)\partial_{ij}u_\varepsilon(x_0) + \sum_i b_i(x_0)\partial_i u_\varepsilon(x_0) + c(x_0)u_\varepsilon(x_0) \\ &\leq c(x_0)u(x_0) \leq 0 \end{aligned}$$

because $c \geq 0$ and $u_\varepsilon(x_0) < 0$. Of course, the above contradicts equation (5). This renders assumption (6) absurd, hence we must have

$$\min_{\overline{\Omega}} u_\varepsilon \geq \min_{\partial\Omega} u_\varepsilon^-$$

Finally, since Ω is bounded we notice that

$$\min_{\overline{\Omega}} u \geq \min_{\overline{\Omega}} u_\varepsilon \geq \min_{\partial\Omega} u_\varepsilon^- \geq -C\varepsilon + \min_{\partial\Omega} u^-$$

for some constant $C > 0$ independent of ε. Since the choice of $\varepsilon > 0$ was arbitrary, this establishes the result. $\square$

We now turn our attention towards the famous Hopf lemma.

Theorem 4 (Hopf's Lemma, Lemma 3.1 in [CL17])**.** *Let B be a ball in $\mathbb{R}^n$ and consider the elliptic operator*

$$L = -\sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$$

where the functions b_i, c are bounded in B and $c \geq 0$. Furthermore, we assume that (a_{ij}) satisfy the uniform ellipticity condition with constant $\lambda > 0$, i.e;

$$\sum_{ij} a_{ij}\xi_i\xi_j > \lambda |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n$$

Assume further that $u \in C^2(B) \cap C^1(\bar{B})$ satisfies $Lu \geq 0$ in B. If there exists $x_0 \in \partial B$ such that $u(x_0) \leq 0$ and

$$u(x_0) < u(x) \qquad \forall x \in B,$$

then one has $\nabla u(x_0) \neq 0$ and $\partial_\nu u(x_0) < 0$ for any outward pointing directional derivative ν.

Proof. Without loss of generality, we may assume that B is centered at the origin. Let $r > 0$ denote the radius of B and define $D = B_{r/2}(x_0) \cap B$. Fix $\varepsilon > 0$ and α appropriately (we postpone explaining how these constants our chosen) and define

$$v(x) = u(x) + \varepsilon w(x) \quad \text{on } \bar{D}$$

where $w(x) = e^{-\alpha r^2} - e^{-\alpha |x|^2}$. After a computation, one has for every $x \in D$ that

$$\begin{aligned} Lw(x) &= e^{-\alpha|x|^2}\left(4\alpha^2 \sum_{i,j} a_{ij}(x)x_i x_j + 2\alpha \sum_i b_i(x)x_i - c(x)\right) + c(x)e^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\left(4\alpha^2\lambda |x|^2 - 2\alpha \|b\|_\infty \sum_i |x_i| - \|c\|_\infty\right) \\ &\geq e^{-\alpha|x|^2}\left(4\alpha^2\lambda \frac{r^2}{4} - 2\alpha \|b\|_\infty r\sqrt{n} - \|c\|_\infty\right) \end{aligned}$$

where $\|b\|_\infty = \max_i \|b_i\|_\infty$. We may therefore pick $\alpha > 0$ so large that $Lw \geq 0$ in D. Then (whatever our choice for $\varepsilon > 0$) we have $Lv \geq 0$ inside the region D. By the weak maximum principle for elliptic operators, it follows that

$$\min_{\overline{D}} v \geq \min_{\partial D} v^-.$$

We now evaluate v on the boundary of D;

1. On $\partial D \cap B$, we know that $u(x) > u(x_0)$. Thus, if we pick $\varepsilon > 0$ sufficiently small then $v(x) > u(x_0) = v(x_0)$ on $\partial D \cap B$.

2. On $\partial D \cap \partial B$ we have $v(x) = u(x) \geq u(x_0) = v(x_0)$.

Since $v(x_0) = u(x_0) \leq 0$, we deduce from the above observations that

$$\min_{\overline{D}} v \geq v(x_0).$$

It follows from the above that $\partial_\nu v(x_0) \leq 0$ for any outward directional derivative at x_0. Recalling that $v = u + \varepsilon w$, we conclude that

$$\partial_\nu u = \partial_\nu v - \varepsilon \partial_\nu w \leq -\varepsilon \partial_\nu w.$$

Since w is radially increasing in B, we see that $\partial_\nu w \geq 0$. In fact, and direct computation shows that $\partial_\nu w > 0$. Ergo, we have

$$\partial_\nu u \leq -\varepsilon \partial_\nu w < 0,$$

as desired. □

We introduce now a refinement of Hopf's Lemma which will come in handy in the proof of Theorem 2. This version of the result is of interest since it does not require the function c to be non-negative.

Theorem 5 (Second Version of Hopf's Lemma, §9.5 Lemma 1 in [Eva10])**.** *Let B be a ball in $\mathbb{R}^n$ and consider the elliptic operator*

$$L = -\Delta + c$$

where c is bounded in B. Assume further that $u \in C^2(B) \cap C^1(\bar{B})$ satisfies $Lu \geq 0$ in B. If there exists $x \in \partial B$ such that

$$0 = u(x_0) < u(x) \qquad \forall x \in B,$$

then one has $\partial_\nu u(x_0) < 0$ for any outward pointing directional derivative ν and, in particular, $\nabla u(x_0) \neq 0$.

Proof. We first fix $\alpha \geq \|c\|_\infty^{1/2}$ and notice that the function $v(x) = e^{-\alpha x_1} u(x)$ satisfies

$$-\Delta v - 2\alpha \partial_1 v = e^{-\alpha x_1} \left(\alpha^2 u - \Delta u \right) \geq e^{-\alpha x_1} \left(\alpha^2 u - cu \right) \geq 0$$

where we have used that $-\Delta u \geq -cu$. Since v also satisfies

$$0 = v(x_0) < v(x) \qquad \forall x \in B,$$

it follows from Hopf's Lemma (Theorem 4) that $\partial_\nu v(x_0) < 0$. On the other hand, since $u(x_0) = 0$ we have

$$\partial_\nu v(x_0) = e^{-\alpha x_1} \partial_\nu u(x_0)$$

which, combined with $\partial_\nu v(x_0) < 0$, implies that $\partial_\nu u(x_0) < 0$ as desired. $\square$

Having now established this critical result, we move on to the strong maximum principle.

Theorem 6 (Strong Maximum Principle, Theorem 3.3 in [CL17])**.** *Consider a domain $\Omega \subseteq \mathbb{R}^n$ and define*

$$L = -\Delta + \sum_i b_i \partial_i + c$$

where b_i and c are bounded on Ω. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ and $u \geq 0$ in Ω. If u vanishes at some point in Ω, then $u \equiv 0$ in Ω. In particular, if there exists a point on $\partial\Omega$ where $u > 0$, then $u > 0$ in Ω.

Proof. Suppose ad absurdum that $u \not\equiv 0$ and

$$\Omega_+ := \{x \in \Omega : u(x) > 0\} \subsetneq \Omega.$$

Then, $\Omega \cap \partial\Omega_+$ is non empty. For the sake of readability, we now divide the proof into multiple steps.

Step 1: We begin by finding a ball $B \subseteq \Omega_+$ such that ∂B contains a point in $\Omega \cap \partial\Omega_+$. In order to construct such a ball, we pick an arbitrary point $x_1 \in \Omega \cap \partial\Omega_+$ and let

$$d = \operatorname{dist}(x_1, \partial\Omega) > 0.$$

Now, pick x_2 to be a point in $\Omega^+ \cap B_{d/2}(x_1)$ and define

$$r = \operatorname{dist}(x_2, \partial\Omega_+) \leq \frac{d}{2}.$$

We claim that the ball $B := B_r(x_2)$ is as desired.

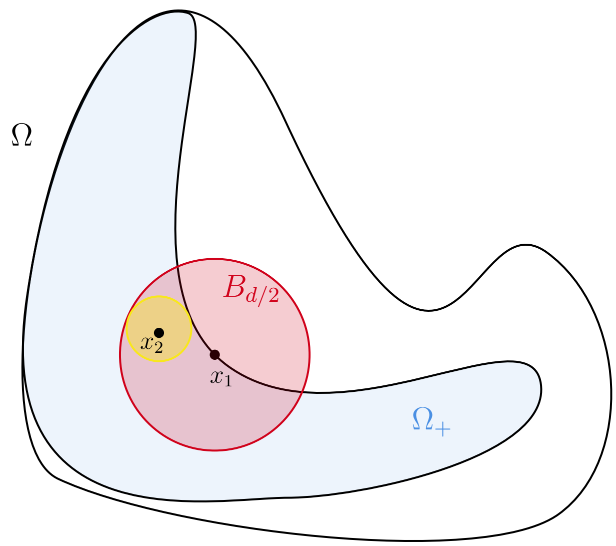

Figure 1: Depiction of the construction used in Step 1.

First, it is clear that $B \subseteq \Omega^+$. Furthermore, since $\partial\Omega_+$ is closed, there exists a point $y_0 \in \partial\Omega_+$ such that $r = |x_2 - y_0|$. That is, we can find $y_0 \in \partial B \cap \partial\Omega_+$. It remains to show that $y_0 \in \Omega$. Indeed, this follows from the triangle inequality

$$|y_0 - x_1| \leq |y_0 - x_2| + |x_2 - x_1| < d.$$

Recalling that $d = \operatorname{dist}(x_1, \partial\Omega)$, we conclude that $y_0 \in \Omega$.

Step 2: For any $\rho > 0$ sufficiently small, there exists a point x_0 such that $B_\rho(y_0) \subseteq \Omega$ and $B_{\rho/4}(x_0) \subseteq \Omega_+$ and y_0 is on the boundary of $B_{\rho/4}(x_0)$. To see this, pick $\rho \in (0, r)$ such that $B_\rho(y_0) \subseteq \Omega$. Now, let consider the point

$$x_0 = y_0 + \frac{\rho}{4}\frac{(x_2 - y_0)}{|x_2 - y_0|} = y_0 + \frac{\rho}{4r}(x_2 - y_0).$$

Clearly, $B_{\rho/4}(x_0) \subseteq B \subseteq \Omega_+$ and $y_0 \in \partial B_{\rho/4}(x_0)$.

STEP 3: There exists a positive function ψ such that

$$\begin{cases} -\Delta\psi = \frac{\lambda_1}{\rho^2}\psi & \text{in } B_\rho(y_0) \\ \psi = 0 & \text{on } \partial B_\rho(y_0). \end{cases} \tag{7}$$

where λ_1 is a constant independent of ρ. Indeed, it is well known that there exists a positive solution to the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } B_1(0) \\ \phi = 0 & \text{on } \partial B_1(0). \end{cases}$$

More specifically, we may let λ_1 be the first positive eigenvalue for the above problem and ϕ the corresponding (positive) solution. Then it is readily seen that

$$\psi(x) = \phi\left(\frac{x - y_0}{\rho}\right)$$

solves equation (7).

STEP 4: We are now ready to derive a contradiction. Recalling that $y_0 \in \Omega \cap \partial\Omega_+$, it is clear that $u(y_0) = 0$. Let us now define the function

$$v : B_{\rho/4}(x_0) \subseteq B_\rho(y_0) \to \mathbb{R}_+, \quad v = u/\psi.$$

where we pick $\rho > 0$ sufficiently small so that

$$\frac{\lambda_1}{\rho^2} + \frac{1}{\rho}\sum_i b_i(x)\frac{\partial_i\phi\left((x-y_0)/\rho\right)}{\phi\left((x-y_0)/\rho\right)} + c \geq 0$$

in $B_{\rho/4}(x_0)$. By definition, we see that v is non-negative and $v(y_0) = 0$. In particular, v is minimized at y_0 hence $\nabla v(y_0) = 0$. On the other hand, we will show that this contradicts Hopf's Lemma. First, we see that

$$v(y_0) = 0 < v(x) \qquad \forall x \in B_{\rho/4}(x_0)$$

By assumption, we know that $Lu \geq 0$ in Ω. In particular, in $B_\rho(y_0)$ there holds

$$
\begin{aligned}
0 &\leq \frac{1}{\psi} L(v\psi) \\
&= \frac{1}{\psi}\left[-\psi\Delta v - 2\nabla v\nabla\psi - v\Delta\psi + \psi\sum_i b_i\partial_i v + v\sum_i b_i\partial_i\psi + cv\psi\right] \\
&= -\Delta v + \sum_i \left(b_i - 2\frac{\partial_i\psi}{\psi}\right)\partial_i v - \left(\frac{\Delta\psi}{\psi} + \sum_i b_i\frac{\partial_i\psi}{\psi} + c\right) v \\
&= -\Delta v + \sum_i \tilde{b}_i\partial_i v + \tilde{c}v
\end{aligned}
$$

where the functions $\tilde{b}_i$ and $\tilde{c}$ are easily seen to be bounded. Furthermore, we see that

$$
\tilde{c}(x) = \frac{\lambda_1}{\rho^2} + \frac{1}{\rho}\sum_i b_i(x)\frac{\partial_i\phi\left((x-y_0)/\rho\right)}{\phi\left((x-y_0)/\rho\right)} + c \geq 0
$$

in $B_{\rho/4}(x_0)$. The conditions of Hopf's Lemma are therefore satisfied in the ball $B_{\rho/4}(x_0)$ and we conclude that $\nabla v(y_0) \neq 0$. This contradiction concludes the proof. $\square$

Remark 2. The choice of $\rho/4$ for the radius of the ball about x_0 is to guarantee that $B_{\rho/4}(x_0)$ is compactly contained in $B_\rho(y_0)$. Therefore, $1/\psi$ is bounded in $B_{\rho/4}(x_0)$ and we may indeed conclude that the functions $\tilde{b}_i$ and $\tilde{c}$ are bounded. Furthermore, if $x \in B_{\rho/4}(x_0)$ then $(x-y_0)/\rho \in B_{1/2}(0)$ so $\phi^{-1}\left((x-y_0)/\rho\right)$ may be bounded independently of ρ.

The idea of using eigenfunctions to establish stronger maximum principles for elliptic operators is one that we will encounter again. Specifically, the section titled comparison principles is entirely devoted to this idea. Essentially, the idea will be to introduce auxiliary functions satisfying a certain eigenvalue problem and to apply the maximum principles above to products of these solutions. This will be made precise in the section that follows.

2.1 Comparison Principles

As mentioned previously, here we will be concerned with the *comparison principles* required for the method of moving planes. More precisely, we will establish an important maximum principle by comparing our solutions with a positive function ϕ satisfying an eigenvalue problem in the domain Ω. Comparison principles are in general very useful (and attainable) results for the method of moving planes. We remark that these are useful even when considering unbounded domains. For instance, comparison techniques (together with some a priori estimates) are used in [Gid81] when working with nonlinear Schrödinger equations in all of $\mathbb{R}^N$.

Theorem 7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and suppose $\phi, \lambda : \overline{\Omega} \to \mathbb{R}$ satisfy*

$$\begin{cases} \phi > 0 & \text{in } \overline{\Omega} \\ -\Delta\phi + \lambda\phi \geq 0 & \text{in } \Omega. \end{cases}$$

Consider a function $c : \Omega \to \mathbb{R}$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $c(x) > \lambda(x)$ at all points in Ω where u is negative. If u solves

$$\begin{cases} -\Delta u + cu \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

then $u \geq 0$ in Ω. Furthermore, if Ω is unbounded then under the additional assumption that

$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} \frac{u(x)}{\phi(x)} \geq 0,$$

the theorem continues to hold true.

Proof. Suppose for a contradiction that $u \not\geq 0$ in Ω and define $v = u/\phi$. Since $v \geq 0$ on $\partial\Omega$, and

$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} v(x) \geq 0$$

if Ω is unbounded, it is clear that v must achieve it's minimum at some point $x_0 \in \Omega$. At this point, we have

$$\nabla v(x_0) = 0, \qquad -\Delta v(x_0) \leq 0.$$

On the other hand, a straightforward computation yields

$$\begin{aligned} -\Delta v &= \frac{1}{\phi}\left(-\Delta u + u\frac{\Delta\phi}{\phi}\right) + 2\left(\frac{\nabla u}{\phi} - u\frac{\nabla\phi}{\phi^2}\right) \cdot \frac{\nabla\phi}{\phi} \\ &= \frac{1}{\phi}\left(-\Delta u + u\frac{\Delta\phi}{\phi}\right) + 2\nabla v \cdot \frac{\nabla\phi}{\phi}. \end{aligned}$$

Thus, at x_0 we have

$$\frac{1}{\phi}\left(-\Delta u + u\frac{\Delta\phi}{\phi}\right) = -\Delta v(x_0) \leq 0.$$

But the above is impossible since

$$-\Delta u(x_0) + u(x_0)\frac{\Delta\phi(x_0)}{\phi(x_0)} \geq -c(x_0)u(x_0) + \lambda(x_0)u(x_0) > 0.$$

Note that we have used that $u(x_0) < 0$ to derive this inequality. $\square$

From this we may deduce a very noteworthy result. Namely, we have the following:

Corollary 7.1 (Narrow Region Principle)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $c : \Omega \to \mathbb{R}$ be bounded below by some constant m. Suppose furthermore that, for some $a \in \mathbb{R}$,*

$$\Omega \subset \{x \in \mathbb{R}^n : a \leq x_1 \leq a + \ell\}$$

where $\ell > 0$ is such that $m > -1/\ell^2$. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves

$$\begin{cases} -\Delta u + cu \geq 0 & \textit{in } \Omega \\ u \geq 0 & \textit{on } \partial\Omega \end{cases}$$

then $u \geq 0$ in Ω.

In the case that Ω is unbounded, then provided that

$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} u(x) \geq 0,$$

the result still holds.

Proof. In order to apply Theorem 7, it suffices to find a function $\phi : \overline{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} \phi > 0 & \text{in } \overline{\Omega} \\ -\Delta\phi - \dfrac{1}{\ell^2}\phi \geq 0 & \text{in } \Omega. \end{cases}$$

It is easy enough to provide an explicit form for such a function. Indeed we consider

$$\phi(x) = \sin\left(\frac{x_1 - a + \varepsilon}{\ell}\right)$$

for fixed $0 < \varepsilon < \ell$. Clearly, $\phi > 0$ in $\overline{\Omega}$. Furthermore, for any $x \in \Omega$ we have

$$-\Delta\phi(x) - \frac{1}{\ell^2}\phi(x) = 0.$$

Applying the previous theorem with ϕ as above and $\lambda(x) = -1/\ell^2$ concludes the result in the case where Ω is bounded.

On the other hand, if Ω is unbounded then notice that ϕ is in fact bounded below in Ω by some positive number. Therefore,

$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} u(x) \geq 0 \implies \liminf_{\substack{|x|\to\infty \\ x\in\Omega}} \frac{u(x)}{\phi(x)} \geq 0$$

and the result once again follows from Theorem 7. □

Although it appears to be innocuous at a first glance, the narrow region principle is fundamental to the method of moving planes. As will become apparent in the proof of Theorem 1, Corollary 7.1 is what will make it possible for us to "start" the moving plane.

Next, we have a "decay principle" analogous to the conclusions drawn from the previous corollary. We give this result below.

Corollary 7.2 (Decay Principle)**.** *Let $c : \mathbb{R}^n \to \mathbb{R}$ and for every $R > 0$ define the set $\Omega_R = \{x \in \mathbb{R}^n : |x| > R\}$. Suppose there exists $R > 0$, $\Omega \subseteq \Omega_R$ and a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$c(x) > -\frac{q(n-2-q)}{|x|^2} \qquad \text{in } \{x \in \Omega : u(x) < 0\}$$

where $0 < q < n-2$ is some constant. If u solves

$$\begin{cases} -\Delta u + cu \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} u(x)\,|x|^q \geq 0$$

then $u \geq 0$ in Ω.

Proof. As for the previous corollary, the proof amounts to verifying that the assumption in Theorem 7 are satisfied with the function u, c as in the statement of this Corollary, $\phi(x) = 1/|x|^q$ and $\lambda(x) = -q(n-2-q)/|x|^2$. □

3 Proofs of Main Results

We are now prepared to prove our main results. Namely, we are ready to illustrate two elegant applications of the method of moving planes. Of course, we begin with Theorem 1 which holds for open balls in $\mathbb{R}^n$. This first theorem truly exemplifies the elegance of the method of moving planes.

3.1 The Proof of Theorem 1

Let $u \in C^2(B_1(0)) \cap C(\bar{B}_1(0))$ be a solution to problem (1).

For every unit vector ν, there exists unique hyper-plane H_ν which passes through the origin and is perpendicular to ν. Our goal is to show that for every unit vector ν, the function u is symmetric about H_ν in the set $B_1(0)$. Since this problem is symmetric, it suffices to prove the statement for the plane perpendicular to $(1, 0, \ldots, 0)$. That is, we show that u is symmetric about the hyper-plane $L_0 = \{x \in \mathbb{R}^n : x_1 = 0\}$.

To prove this statement, we begin with some setup. For each non-negative number λ we denote

$$L_\lambda = \{x \in \mathbb{R}^n : x_1 = \lambda\}.$$

Furthermore, given $x \in \mathbb{R}^n$ we denote by x_λ the refection of x about the line L_λ. More specifically, we have

$$x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n).$$

Now, given $\lambda \geq 0$ denote by $\Sigma_\lambda = \{x \in B_1(0) : x_1 > \lambda\}$ the part of the unit ball with is to the right of L_λ.

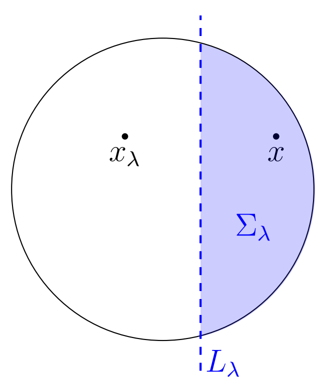

Figure 2: We begin by moving the plane to the left of the region.

Furthermore, define

$$w_\lambda : \overline{\Sigma}_\lambda \to \mathbb{R}, \qquad w_\lambda(x) = u(x_\lambda) - u(x)$$

and notice that w_λ is uniformly continuous in λ as a map taking each λ to $C(\overline{\Sigma}_\lambda)$. We delegate the remainder of the paragraph to verifying this fact. Fix $\varepsilon > 0$ and notice that since u is continuous in a compact set, it is uniformly continuous. Therefore, we may pick $\delta > 0$ such that $|u(x) - u(y)| < \varepsilon$ whenever $x, y \in \bar{B}_1(0)$ satisfy $|x - y| < \delta$. Then given $\eta \leq \lambda$ such that $|\lambda - \eta| < \delta/2$, there holds

$$|w_\lambda(x) - w_\eta(x)| = |u(x_\lambda) - u(x_\eta)| < \varepsilon$$

for every $x \in \overline{\Sigma}_\lambda$.

Our main goal is to show that $w_0 \equiv 0$ in Σ_0 which clearly establishes symmetry about L_0. The basic idea will be to show that

$$\bar{\lambda} := \inf \{\lambda \in [0,1] : w_\lambda \geq 0 \text{ in } \Sigma_\lambda\} = 0.$$

by "sweeping the plane towards the left". By the continuity properties of w_λ, we see that the infimum $\bar{\lambda}$ is achieved. Therefore, if $\bar{\lambda} = 0$ then $w_0 \geq 0$ in Σ_0 and a symmetric argument will allow us to conclude that $w_0 \equiv 0$. So, the bulk of the proof amounts to establishing the equality $\bar{\lambda} = 0$. For the sake of clarity, we divide the proof into two steps.

Step 1: The holds $\bar{\lambda} < 1$ *(i.e. we begin moving the plane L_1 towards the left of the region).*

Proof of Step 1. We begin with a few observation. For any $\lambda \in [0,1]$ we may compute

$$\begin{aligned} -\Delta w_\lambda &= -\Delta u(x_\lambda) + \Delta u(x) \\ &= f(u(x_\lambda)) - f(u(x)) \\ &= c_\lambda(x) w_\lambda(x) \end{aligned}$$

in Σ_λ where

$$c_\lambda(x) = \begin{cases} 0 & w_\lambda(x) = 0 \\ \dfrac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} & \text{otherwise.} \end{cases}$$

We claim that $c_\lambda(x)$ is bounded. Since, by assumption, u is continuous on the closed unit ball, the range of u is a compact set. Since f is locally Lipschitz, it follows that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C\,|x - y| \qquad \forall x, y \text{ in the range of } u.$$

It is clear from the above that $|c_\lambda(x)| \leq C$ for all $\lambda \in [0, 1]$ and $x \in \Sigma_\lambda$.

It follows that, for any $\lambda \geq 0$, the value $-C$ is a lower bound for c_λ. With the hopes of applying the *narrow region principle* (Corollary 7.1), we fix $0 < \ell < 1$ small such that

$$-\frac{1}{\ell^2} < -C$$

Setting $\lambda = 1 - \ell$, we notice that Σ_λ is a bounded domain of width ℓ. Furthermore, $w_\lambda \geq 0$ on $\partial\Sigma_\lambda$. Indeed, on $\partial\Sigma_\lambda \cap \partial B_1(0)$ we have

$$w_\lambda(x) = u(x_\lambda) - u(x) = u(x_\lambda) \geq 0$$

and on $\partial\Sigma_\lambda \cap L_\lambda$ we have

$$w_\lambda(x) = u(x_\lambda) - u(x) = u(x) - u(x) = 0.$$

We conclude that the *narrow region principle* (Corollary 7.1) applies to w_λ whence $w_\lambda \geq 0$ in Σ_λ.

Step 2: There holds $\bar{\lambda} = 0$. *(In other terms, we sweep with the plane until it reaches it's left limit).*

Proof of Step 2. Suppose for a contradiction that $\bar{\lambda} \neq 0$. Notice that $w_{\bar{\lambda}}$ satisfies

$$\begin{cases} w_{\bar{\lambda}} \geq 0 & \text{in } \Sigma_\lambda \\ -\Delta w_{\bar{\lambda}} - c_{\bar{\lambda}}(x) w_{\bar{\lambda}}(x) \geq 0 & \text{in } \Sigma_\lambda. \end{cases}$$

Furthermore, $w_{\bar{\lambda}} > 0$ on all but two points of $\partial\Sigma_{\bar{\lambda}} \cap \partial B_1(0)$. Therefore, the *strong maximum principle* (Theorem 6) implies that $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}}$. In particular, we see that there exists $\alpha > 0$ such that

$$w_{\bar{\lambda}} \geq \alpha \qquad \text{in } \Sigma_{\bar{\lambda}+\ell/2}$$

where $\ell > 0$ is a in Step 1. Using that w_λ is uniformly continuous in the variable λ, we see that given $\delta \in (0, \ell/2)$ sufficiently small, one has

$$w_{\bar{\lambda}-\delta} \geq 0 \qquad \text{in } \Sigma_{\bar{\lambda}+\ell/2}$$

and $\bar{\lambda} - \delta > 0$. Define $\Lambda = \Sigma_{\bar{\lambda}-\delta} \cap \left\{x \in \mathbb{R}^n : x_1 < \bar{\lambda} + \ell/2\right\}$.

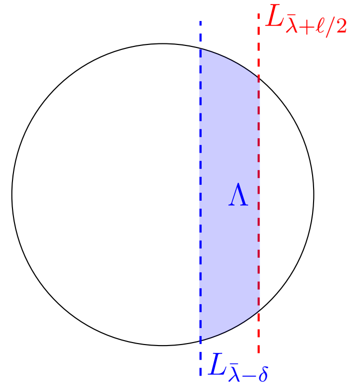

Figure 3: An application of the *narrow region principle* (Corollary 7.1) to the ball.

As in Step 1, it is easy to see that $w_{\bar{\lambda}-\delta} \geq 0$ on $\partial\Lambda$. Once again, we may apply the *narrow region principle* (Corollary 7.1) to see that $w_{\bar{\lambda}-\delta} \geq 0$ in Λ. This contradicts the choice of $\bar{\lambda}$ whence we can infer that $\bar{\lambda} = 0$.

Remark 3. The argument in the above step shows that $w_\lambda \geq 0$ for each $0 \leq \lambda < 1$. This observation is the reason we may conclude that u is monotone decreasing about the origin at the end of the proof.

We can now conclude the proof. From Step 2 it follows that $w_0 \geq 0$ in Σ_0. Sweeping from the right instead of the left, we may similarly deduce that $w_0 \geq 0$ in $-\Sigma_0$. Since w_0 is anti-symmetric about L_0, it follows that $w_0 \equiv 0$ which concludes the proof of Theorem 1.

□

The above application of the method of moving planes heavily relies on the boundedness of our domain. Surprisingly enough, it is still possible to use a similar method to deduce such results for systems defined in all of $\mathbb{R}^n$ by relying on the Decay principle (Corollary 7.2) instead of the Narrow region principle (Corollary 7.1). The proof of Theorem 2 is an example of this.

3.2 The Proof of Theorem 2

We let $u \in C^2(\mathbb{R}^n)$ be as in the statement of the theorem and begin with some setup. As in the proof of Theorem 1, we define

$$L_\lambda = \{x \in \mathbb{R}^n : x_1 = \lambda\}$$

for every $\lambda \in \mathbb{R}$. We also set

$$\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 > \lambda\}$$

to be the half-space to the right of L_λ. Furthermore, for each $x \in \mathbb{R}^n$, we define x_λ to be the reflection of x about L_λ. Finally, define $w_\lambda : \mathbb{R}^n \to \mathbb{R}$ by $w_\lambda(x) = u(x_\lambda) - u(x)$.

Step 1: There exists $\lambda > 0$ large so that $w_\lambda \geq 0$ in Σ_λ *(we begin placing the plane L_λ).*

Proof of Step 1. Fix $0 < q < n-2$ and notice that for $R \gg 1$ be sufficiently large there holds

$$u(x) < \frac{(q(n-2-q)/p)^{1/(p-1)}}{|x|^{(n-2)/2}} \qquad \forall\, |x| \geq R \tag{8}$$

We may indeed find such a R since we have assumed that $u \in O\left(|x|^{2-n}\right)$. Now, for $\lambda > R$ we will show that there exists a function $c_\lambda : \Sigma_\lambda \to \mathbb{R}$ such that

$$c_\lambda(x) > -\frac{q(n-2-q)}{|x|^2} \qquad \text{in } \{x \in \Sigma_\lambda : w_\lambda(x) < 0\}$$

and w_λ solves

$$\begin{cases} -\Delta w_\lambda + c_\lambda w_\lambda \geq 0 & \text{in } \Sigma_\lambda \\ w_\lambda \geq 0 & \text{on } \partial\Sigma_\lambda. \end{cases}$$

Assuming temporarily that the above holds, we also make the following observation;

$$\liminf_{\substack{x \to \infty \\ x \in \Sigma_\lambda}} w_\lambda(x)\, |x|^q = \liminf_{\substack{x \to \infty \\ x \in \Sigma_\lambda}} \left(u(x_\lambda) - u(x)\right) |x|^q = 0$$

since $u \in O(|x|^{n-2})$ and $q < n-2$. Therefore, we are able to conclude from Corollary 7.2 that $w_\lambda \geq 0$ in Σ_λ.

We now prove the aforementioned statements. First of all, it is immediate from the definition of w_λ that $w_\lambda = 0$ on $L_\lambda = \partial\Sigma_\lambda$. To define the function c_λ, we begin by considering

$$f : \mathbb{R}_+ \to \mathbb{R}, \qquad f(a) = a^p.$$

For each $x \in \mathbb{R}^n$, the Mean Value Theorem implies the existence of $\phi_\lambda(x)$ between $u(x_\lambda)$ and $u(x)$ such that

$$\begin{aligned} u^p(x^\lambda) - u^p(x) &= f(u(x^\lambda)) - f(u(x)) \\ &= f'(\phi_\lambda(x))\,(u(x_\lambda) - u(x)) \\ &= p\phi_\lambda^{p-1}(x) w_\lambda(x). \end{aligned}$$

Equivalently, there holds

$$-\Delta w_\lambda(x) + \left(-p\phi_\lambda^{p-1}(x)\right) w_\lambda(x) = 0.$$

It therefore makes sense to define

$$c_\lambda(x) = -p\phi_\lambda^{p-1}(x).$$

Now, let $x \in \Sigma_\lambda$ be an arbitrary point such that $w_\lambda(x) < 0$. Then we have $|x| > \lambda \geq R$ and $u(x_\lambda) < u(x)$. Since $\phi_\lambda(x)$ is between $u(x_\lambda)$ and $u(x)$, we see that

$$0 \leq u(x_\lambda) \leq \phi_\lambda(x) \leq u(x) < \frac{(n/p)^{1/(p-1)}}{|x|^{(n-2)/2}}$$

where the last inequality holds by equation (8). It follows that

$$c_\lambda(x) = -p\phi_\lambda^{p-1}(x) > -p\frac{q(n-2-q)/p}{|x|^{(n-2)(p-1)/2}} = -\frac{q(n-2-q)}{|x|^2}.$$

This concludes the proof of Step 1.

Step 2: For any $\lambda \in \mathbb{R}$, the negative local minimums of $v_\lambda := w_\lambda/\phi$, where $\phi(x) = |x|^{-q}$, occur in the ball $B_R(0)$.

Proof of Step 2. Suppose for a contradiction that there exists a negative local minimum $x_0 \notin B_R(0)$ of v_λ. As in the proof of Theorem 7, a straightforward computation shows that

$$-\Delta v_\lambda = \frac{1}{\phi}\left(-\Delta w_\lambda + w_\lambda \frac{\Delta\phi}{\phi}\right) + 2\nabla v_\lambda \cdot \frac{\nabla\phi}{\phi}.$$

Therefore, at x_0 we have

$$0 \le -\Delta v_\lambda = \frac{1}{\phi}\left(-\Delta w_\lambda + w_\lambda \frac{\Delta \phi}{\phi}\right).$$

On the other hand, as in Step 1 we see that

$$\begin{aligned} -\Delta w_\lambda + w_\lambda \frac{\Delta \phi}{\phi} &= -\Delta w_\lambda - w_\lambda \frac{q(n-2-q)}{|x|^2} \\ &< -\Delta w_\lambda + c_\lambda w_\lambda = 0 \end{aligned}$$

at x_0 where we have used that $w_\lambda(x_0) = v_\lambda(x_0)\phi(x_0) < 0$. This yields a contradiction and thus concludes step 2.

Step 3: If $\bar{\lambda} = \inf\{\lambda \ge 0 : w_\lambda \ge 0 \text{ in } \Sigma_\lambda\} > 0$ then $w_{\bar{\lambda}} \equiv 0$ in $\Sigma_{\bar{\lambda}}$ *(we start sweeping the plane towards the left).*

Proof of Step 3. First, we note that since u is continuous the infimum $\bar{\lambda}$ is achieved. Now, suppose that $\bar{\lambda} > 0$ and notice that, by the strong maximum principle (Theorem 6), we either have $w_{\bar{\lambda}} \equiv 0$ or

$$w_{\bar{\lambda}} > 0 \qquad \text{in } \Sigma_{\bar{\lambda}} \tag{9}$$

It therefore suffices to assume the latter and derive a contradiction. We claim that if equation (9) holds, then there exists $\delta > 0$ small such that

$$w_{\bar{\lambda}-\delta} \ge 0 \qquad \text{in } \Sigma_{\bar{\lambda}-\delta} \tag{10}$$

If this were not the case, then we may find a sequence of positive numbers $\delta_j \to 0$ such that the above fails for each $j \in \mathbb{N}$. Let q be as in step 1 and set

$$v_\lambda = \frac{w_\lambda}{\phi} = w_\lambda |x|^q$$

for each $\lambda > 0$. Notice that for any $\lambda \in \mathbb{R}$, the function $v_\lambda(x)$ tends to 0 as $|x| \to \infty$ since $u \in O(|x|^{2-n})$. It follows that the function $v_{\bar{\lambda}-\delta_j}$ attains it's negative minimum at some point $x_j \in \Sigma_{\bar{\lambda}-\delta_j}$ for each $j \in \mathbb{N}$. By step 2, $|x_j| \le R$ for each $j \in \mathbb{N}$. In particular, passing to a subsequence if necessary, we may suppose that $x_j \to x_0$. Now, we have

$$\begin{aligned} 0 \le v_{\bar{\lambda}}(x_0) &= \lim_{j\to\infty} v_{\bar{\lambda}-\delta_j}(x_j) \le 0 \\ \nabla v_{\bar{\lambda}}(x_0) &= \lim_{j\to\infty} \nabla v_{\bar{\lambda}-\delta_j}(x_j) = 0. \end{aligned}$$

That is, $v_{\bar{\lambda}}(x_0) = 0$ and $\nabla v_{\bar{\lambda}}(x_0) = 0$. Now, we compute

$$\begin{aligned} w_{\bar{\lambda}}(x_0) &= v_{\bar{\lambda}}(x_0)\phi(x_0) = 0 \\ \nabla w_{\bar{\lambda}}(x_0) &= \phi(x_0)\nabla v_{\bar{\lambda}}(x_0) + v_{\bar{\lambda}}(x_0)\nabla\phi(x_0) = 0. \end{aligned}$$

Recalling equation (9), we see that $x_0 \in \partial\Sigma_{\bar{\lambda}}$. However, we assert the second version of Hopf's Lemma (Theorem 5) applies, which yields the contradiction $\nabla w_{\bar{\lambda}}(x_0) \neq 0$. From this, we conclude that equation (10) holds which contradicts our choice of $\bar{\lambda}$ and thus establishes the result of Step 3.

To see that Hopf's Lemma indeed applies, it suffices to find a ball $B \subseteq \Sigma_{\bar{\lambda}}$ with $x_0 \in \partial B$ such that $c_{\bar{\lambda}}$ is bounded in B. To see this recall that $\phi_{\bar{\lambda}}(x)$ was chosen to be between $u(x_{\bar{\lambda}})$ and $u(x)$ where u is continuous and thus bounded on any compact set. It readily follows that $c_{\bar{\lambda}}$ is bounded on any such ball B.

Step 4: We conclude that u is radially symmetric and monotone decreasing about a point. Furthermore, the monotonicity of u is also clear by the same reasoning as in the proof of Theorem 1.

Proof of Step 4. If $\bar{\lambda} > 0$ then by step 3, $w_{\bar{\lambda}} \equiv 0$ whence u is symmetric about $L_{\bar{\lambda}}$. Otherwise, we must have $\bar{\lambda} = 0$ then we may carry out the same procedure from the left instead with

$$\lambda' = \sup\left\{\lambda \leq 0 : w_\lambda \geq 0 \text{ in } -\Sigma_\lambda\right\}.$$

If $\lambda' = 0$ then we conclude that u is symmetric about L_0. Otherwise, we see that u is symmetric about $L_{\lambda'}$ for some $\lambda' < 0$.
In any case, we find that u is symmetric and monotone decreasing in the x_1 direction about the plane $\{x \in \mathbb{R}^n : x_1 = \lambda_1\}$ for some λ_1. Similarly, for each index $i = 1, \ldots, n$ we see that u is symmetric in the x_i direction about the plane $\{x \in \mathbb{R}^n : x_1 = \lambda_i\}$ for some λ_i. We will show that u is in fact radially symmetric about the point

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n).$$

Since u is monotone decreasing about $\boldsymbol{\lambda}$ in each direction x_i, it is clear that u achieves it's maximum at $\boldsymbol{\lambda}$. Now, let ν be an arbitrary unit vector and for each constant α consider the plane

$$H_\alpha = \{x \in \mathbb{R}^n : x \cdot \nu = \alpha\}.$$

By the argument used in the proof of this theorem, we see that the function u is symmetric and monotone decreasing about H_α for some $\alpha \in \mathbb{R}$. In particular, u achieves it's maximum on H_α. If H_α does not intersect the $\boldsymbol{\lambda}$, then there must exist a neighbourhood $U \subseteq \mathbb{R}^n$ where u is maximized. In particular, u is constant in this U so

$$u^p = -\Delta u = 0 \qquad \text{in } U$$

But since the above is where u is maximized, we conclude that u is a positive function whose maximum is 0, i.e. $u \equiv 0$. Ergo, H_α must intersect $\boldsymbol{\lambda}$. Since the unit vector ν was arbitrary, we conclude that u is radially symmetric and monotone decreasing about $\boldsymbol{\lambda}$.

□

As it turns out, this conclusion of Theorem 2 is not-optimal. Namely, one does not need to assume a priori that $u \in O(|x|^{2-n})$. This does not affect the argument too much, but instead makes use of the same proof together with a comparison function.

4 Further Comments

Having now seen two applications of the method of moving planes, it is clear that the general flavour of the procedure does not vary much. One always uses the same "moving plane" argument together with decay estimates/assumptions and maximum principles. However, it is precisely these additional required tools that makes it hard to apply the method to any given equation, let alone any system of equations. Ergo, before one can hope to apply the method effectively, a detailed a priori analysis of solutions must be conducted.

For instance, using the method of moving planes, one can show that any solution to

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2$$

such that e^u is Lebesgue-integrable in $\mathbb{R}^2$ is radially symmetric about some point. In order to prove this, it will be useful to first establish any such solution u satisfies

$$\frac{u(x)}{\ln|x|} \to \frac{-1}{2\pi} \int_{\mathbb{R}^2} e^{u(x)} \, \mathrm{d}x$$

as $|x| \to \infty$. For more detail, we refer the reader to §5.3. in [CL17].

We now turn our attention back to Theorem 2. As mentioned above, the conclusions drawn in Theorem 2 can be significantly improved. That is, the asymptotic assumption made on the positive regular solutions is not necessary. More precisely, the following improvement of Theorem 2 holds true:

Theorem 8 (Theorem 5.3 in [CL91]). *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of the equation*

$$-\Delta u = u^p \quad in\ \mathbb{R}^n, \tag{11}$$

with $n \geq 3$ and $p := \frac{n+2}{n-2}$. Then, there exists a point $x_0 \in \mathbb{R}^n$ about which u is radially symmetric. Furthermore, if $p < \frac{n+2}{n-2}$ then no positive solution exists.

Proof Sketch. Suppose that u is as in the above theorem. In order to prove this result, one will introduce the Kelvin transform of u

$$v(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right)$$

which satisfies

$$-\Delta v(x) = \frac{1}{|x|^{n+2-p(n-2)}} v^p(x) \qquad \text{in } \mathbb{R}^n \setminus \{0\}\,.$$

In particular, when $p = \frac{n+2}{n-2}$ we find $-\Delta v = v^p$. Since v clearly satisfies the decay assumption $v \in O(|x|^{n-2})$, a similar argument as the one presented in the proof of Theorem 2 shows that v is radially symmetric and monotone decreasing about a point. That is, there exists a point x_0 and a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $v(x) = \phi(|x - x_0|)$ for all x in the domain of v. In order to make this same conclusion about u, we consider 2 distinct cases;

1. If the function v is radially symmetric about the origin then we "move" the origin and find ourselves in the second case. By moving the origin, we mean that we re-start the proof with $\tilde{u}(x) = u(x - x_0)$ for some x_0.

2. If v is radially symmetric about some point which is not the origin, then v is bounded near the origin whence

$$u(x) = \frac{1}{|x|^{n-2}} v\left(\frac{x}{|x|^2}\right)$$

satisfies the decay assumption $u \in O(|x|^{n-2})$ and Theorem 2 applies.

Finally, if $p < \frac{n+2}{n-2}$ then v must be radially symmetric about the origin. Otherwise, v must satisfy

$$-\Delta v(x) = \frac{1}{|x|^{n+2-p(n-2)}} v^p(x) \qquad \text{in } \mathbb{R}^n.$$

In particular, in order to deal with the singularity at the origin we must have $v(x) = 0$. But since $v \geq 0$ we also have $-\Delta v \geq 0$ and it follows from the strong maximum principle (Theorem 6) that $v \equiv 0$. We conclude that v is indeed symmetric about the origin - but moving the origin we once again conclude that $v \equiv 0$. $\square$

For a complete proof of Theorem 8, we refer the reader to Theorem 2.1 in [CL91].

This is by no means the end of the road concerning such results. In fact, it is known that the following stronger result holds true:

Theorem 9 (Theorem 2.1 in [CLO06]). *Fix a dimension $n \geq 3$ and let $0 < \alpha < n$. Suppose $u \in L_{loc}^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ is a positive of*

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{(n+\alpha)/(n-\alpha)}}{|x-y|^{n-\alpha}} \, \mathrm{d}y. \tag{12}$$

Then u is radially symmetric and monotone decreasing about a point.

It was shown in *Classification of Solutions for an Integral Equation* [CLO06] that all positive solutions to equation (12), after multiplication by a constant, solve

$$(-\Delta)^{\alpha/2} u = u^{\frac{n+\alpha}{n-\alpha}}$$

and vice-versa. In particular, the PDE considered in Theorems 2 and 8 corresponds to equation (12) in the special case $\alpha = 2$. We also point out that equations of the form

$$(-\Delta)^{\alpha/2} u = u^p$$

are of particular importance in physics as they appear in the theory of non-linear optics. Furthermore, there is currently active research looking into the fractional Lane-Emden system

$$\begin{cases} (-\Delta)^{\alpha/2} u = v^p \\ (-\Delta)^{\alpha/2} v = u^q. \end{cases} \tag{13}$$

Using the method of moving planes, Miaomiao Cai and Linfeng Mei were able to prove similar results for the above system. Before stating their result, we define, $C^{1,1}_{loc}$ to be those functions that continuously differentiable with locally Lipschitz derivatives. Furthermore, let $L_\alpha(\mathbb{R}^n)$ denote the collection of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+\alpha}} \,\mathrm{d}x < \infty.$$

Theorem 10 (Theorem 2 in [CM17]). *Suppose* $1 \le p, q \le \frac{n+\alpha}{n-\alpha}$ *and suppose* $u, v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n)$ *are positive solution to system* (13). *If* $p = q = \frac{n+\alpha}{n-\alpha}$ *then* u, v *are radially symmetric about some point. Otherwise, no positive solutions in* $C^{1,1}_{loc}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n)$ *exist.*

The method of moving planes even extends to determine results for a weighted version of the aforementioned system. Specifically, let $n \ge 3$ be the dimension, $p, q > 0$ with $pq > 1$, $0 < \alpha < n$ and $0 \le \sigma_1, \sigma_2 < \alpha$. The system of differential equation

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = \dfrac{v^q(x)}{|x|^{\sigma_1}} & \text{in } \mathbb{R}^n \setminus \{0\} \\ (-\Delta)^{\alpha/2} v(x) = \dfrac{u^p(x)}{|x|^{\sigma_2}} & \text{in } \mathbb{R}^n \setminus \{0\} \end{cases}$$

is closely related to the integral system

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{v^q(y)}{|x-y|^{n-\alpha} |y|^{\sigma_1}} \,\mathrm{d}y \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha} |y|^{\sigma_2}} \,\mathrm{d}y. \end{cases} \tag{14}$$

Defining

$$r_0 = \frac{n(pq-1)}{\alpha(1+p) + (\sigma_2 + \sigma_1 p)} \quad \text{and} \quad s_0 = \frac{n(pq-1)}{\alpha(1+q) + (\sigma_2 + \sigma_1 q)},$$

we have the following result:

Theorem 11 (Theorem 7 in [Vil15]). *Suppose* $(u, v) \in L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$ *are positive solutions to system* (14). *Then* u, v *are radially symmetric and monotone decreasing about the origin.*

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