MATH 581 ASSIGNMENT 4

DUE MONDAY MARCH 24

1. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\{u_k\} \subset \mathscr{D}'(\Omega)$ be a sequence such that for each $\varphi \in \mathscr{D}(\Omega)$, $\langle u_k, \varphi\rangle$ is convergent as $k \to \infty$. Show that the map $\varphi \mapsto \lim\langle u_k, \varphi\rangle$ is a distribution on Ω.
2. Find all fundamental solutions of the ordinary differential operators $\partial_x + \lambda$ and $\partial_x^2 + \lambda$ in $\mathbb{R}$, where $\lambda \in \mathbb{C}$ is a constant. How many of those are supported in $[0, \infty)$?
3. Find a fundamental solution of the wave operator $\partial_t^2 - \Delta$ with support in the half space $\{(x, t) : x \in \mathbb{R}^n,\ t \geq 0\}$, where $n = 1, 2, 3$. What is the support of each?
4. For $u \in \mathscr{E}'$ and $v \in L^1_{\text{loc}}(\mathbb{R}^n)$, we defined $u * v \in \mathscr{D}'$ by
$$\langle u * v, \varphi\rangle = \langle u, \tilde{v} * \varphi\rangle, \qquad \varphi \in \mathscr{D}.$$
Recall also the notations $\tilde{v}(z) = v(-z)$ and $(\tau_x\phi)(y) = \phi(y - x)$.
a) Show that if $u \in \mathscr{E}'$ and $v \in \mathscr{E}$ then $u * v \in \mathscr{E}$ and $(u * v)(x) = u(\tau_x\tilde{v})$ for $x \in \mathbb{R}^n$.
b) Show that for fixed $u \in \mathscr{E}'$, the mapping $v \mapsto u * v : \mathscr{E} \to \mathscr{E}$ is continuous.
5. *a)* Let $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$. Show that
$$\tau_a(u * v) = (\tau_a u) * v = u * (\tau_a v), \qquad a \in \mathbb{R}^n,$$
where for distributions, the translation is defined by
$$\langle \tau_a u, \varphi\rangle = \langle u, \tau_{-a}\varphi\rangle.$$
b) For any distribution u, show that
$$\tau_a u = \delta_a * u,$$
where $\delta_a \equiv \tau_a\delta$ is the Dirac mass concentrated at $a \in \mathbb{R}^n$.
6. Let $u \in \mathscr{E}'$, $\phi \in \mathscr{E}$, and $\psi \in \mathscr{D}$. Prove that
$$u * (\phi * \psi) = (u * \phi) * \psi = (u * \psi) * \phi.$$
Show that
$$1 * (\delta' * \vartheta) \neq (1 * \delta') * \vartheta,$$
where 1 is the function identically 1 in $\mathbb{R}$, and ϑ is the Heaviside step function.
7. Prove that $\mathscr{D}(\Omega)$ is dense in $\mathscr{D}'(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open.
8. Let u and v be the surface measures of the spheres $\{x \in \mathbb{R}^3 : |x| = a\}$ and $\{|x| = b\}$, respectively. Compute $u * v$, and determine its singular support.
9. *a)* Are the operators from Problems 2 and 3 hypoelliptic?
b) Describe all fundamental solutions of the Cauchy-Riemann operator $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, as well as those of $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

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c) Schwartz's theorem implies that the Cauchy-Riemann operator is analytic-hypoelliptic, i.e., that if $\Omega \subset \mathbb{C}$ is open and if $f \in \mathscr{D}'(\Omega, \mathbb{C})$ satisfies $\partial_{\bar{z}} f = 0$ on Ω then f is *real* analytic in Ω (with Ω considered as a subset of $\mathbb{R}^2$). Show that f is in fact complex analytic (i.e., holomorphic) in Ω.

d) Show that there exists a real analytic, but no complex analytic function f in $\mathbb{C} \setminus \{0\}$ such that $\partial_z f = \frac{1}{z}$ in $\mathbb{C} \setminus \{0\}$.