MATH 581 ASSIGNMENT 5

DUE TUESDAY APRIL 2

1. Consider the Cauchy problem

$$\partial_t u = P(D_x)u, \qquad u|_{t=0} = f,$$

where u is a vector function with m components, P is a (possibly complex) $m \times m$ matrix of n-variable polynomials, and f is a given (vector) function. The operator $P(D_x)$ is called *Petrowsky well-posed* (PWP) if the eigenvalues $\{\lambda_j(\xi)\}$ of $P(\xi)$ satisfy

$$\sup_{\xi\in\mathbb{R}^n} \max_j \operatorname{Re} \lambda_j(\xi) < \infty.$$

Even though Petrowsky well-posedness is an algebraic condition as defined, it is easy to convert it into an analytic well-posedness condition (but there are many such equivalent conditions and the algebraic definition above seems to be the cleanest choice for PWP itself). In particular, we have seen in class that $P(D_x)$ being PWP is essentially equivalent to the existence of an L^2-solution for the Cauchy problem with sufficiently smooth initial data. So one can have a loss of smoothness in the sense that the solution is less regular than the initial data. If we do not allow this, we are led to the following notion.

The Cauchy problem is called *strongly well-posed* if for any $f \in L^2$, there exists a solution $u \in \mathscr{C}(\overline{\mathbb{R}}_+, L^2)$, which satisfies the estimate

$$\|u(t)\|_{L^2} \leq Ce^{\alpha t}\|f\|_{L^2}, \qquad t \geq 0,$$

with some constants α and C. We want to investigate how these two types of well-posedness conditions behave under zero-order perturbations.

a) Show that $P(D_x)$ is strongly well-posed if and only if

$$\sup_{\xi\in\mathbb{R}^n} |e^{tP(\xi)}| \leq Ce^{\alpha t} \qquad t \geq 0,$$

with some constants α and C, where $|\cdot|$ is understood as a matrix norm.

b) Show that if $P(D_x)$ is PWP but not strongly well-posed, then there exists a matrix B such that $P(D_x) + B$ is not PWP.

c) Show that if $P(D_x)$ is strongly well-posed, then so is $P(D_x) + B$ for any a matrix B.

2. Consider the Cauchy problem

$$\partial_t u = \sum_{k=1}^{n} A_k \partial_k u + Bu + f, \qquad u|_{t=0} = g,$$

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where u is a vector function with m components, all A_k and B are $m \times m$ matrices, and $f \in \mathscr{C}(\mathbb{R}, H^s(\mathbb{R}^n))$ and $g \in H^s(\mathbb{R}^n)$ are given (vector) functions, with some $s \in \mathbb{R}$. In each of the following cases, prove that there exists a unique solution $u \in \mathscr{C}(\mathbb{R}, H^s(\mathbb{R}^n))$, which satisfies the estimate

$$\|u(t)\|_{H^s} \leq Ce^{\alpha|t|}\|g\|_{H^s} + C\int_0^t e^{\alpha|t-\tau|}\|f(\tau)\|_{H^s}\mathrm{d}\tau, \qquad t \in \mathbb{R},$$

with some constants α and C.

a) *Symmetric hyperbolic* case: All A_k are Hermitian.
b) *Strictly hyperbolic* case: For all nonzero $\xi \in \mathbb{R}^n$, the eigenvalues of $P(\xi) = \sum_{k=1}^n A_k\xi_k$ are real and distinct.

3. Maxwell's equations for 3 dimensional electromagnetism in vacuum are

$$\partial_t E = \nabla \times B, \qquad \partial_t B = -\nabla \times E, \tag{1}$$

and

$$\nabla \cdot E = 0, \qquad \nabla \cdot B = 0, \tag{2}$$

where $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ are the electric and magnetic field, respectively. Show that the system (1) is symmetric hyperbolic. Then show that the constraints (2) are preserved by the evolution, i.e., that if one starts with initial data satisfying the constraints (2), and if E and B evolve according to (1), then (2) will be satisfied for all time.

4. For isotropic and homogeneous materials, the elastodynamics equation is given by

$$\partial_t^2 u = \mu\Delta u + \lambda\nabla(\nabla \cdot u),$$

where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the displacement field, and μ and λ are real parameters. In components, it reads

$$\partial_t^2 u_k = \mu\Delta u_k + \lambda\partial_k(\partial_1 u_1 + \ldots \partial_n u_n), \qquad k = 1, \ldots, n.$$

We consider the corresponding Cauchy problem with the initial data $u|_{t=0} = f$ and $\partial_t u|_{t=0} = g$. Determine the values of the parameters μ and λ for which the Cauchy problem is well-posed in the following sense: For any initial data $(f, g) \in H^s \times H^{s-1}$ with some $s \in \mathbb{R}$, there exists a unique solution $u \in \mathscr{C}(\mathbb{R}, H^s) \cap \mathscr{C}^1(\mathbb{R}, H^{s-1})$, satisfying

$$\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \leq C(\|f\|_{H^s} + \|g\|_{H^{s-1}}), \qquad t \in \mathbb{R},$$

with some constant $C > 0$.

5. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary, and recall that $H_0^s(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ with respect to the H^s norm. Recall also that $H^s(\Omega) = \{w|_\Omega : w \in H^s(\mathbb{R}^n)\}$, with the norm

$$\|u\|_{H^s(\Omega)} = \inf_{\{w \in H^s(\mathbb{R}^n) : w|_\Omega = u\}} \|w\|_{H^s}.$$

Prove the followings.

a) Let $X = \{v \in H^s(\mathbb{R}^n) : \operatorname{supp} v \subset \mathbb{R}^n \setminus \Omega\} \subset H^s(\mathbb{R}^n)$ and let $P : H^s(\mathbb{R}^n) \to X$ be the H^s-orthogonal projection onto X. Then

$$(w|_\Omega, v|_\Omega) = \langle w - Pw, v - Pv\rangle_{H^s},$$

is an inner product on $H^s(\Omega)$, making it a Hilbert space.

b) The (topological) dual of $H_0^s(\Omega)$ is isometric to $H^{-s}(\Omega)$, and vice versa.

c) For $0 < s < 1$, the norm $\|u\|_{H^s(\Omega)}$ on $H^s(\Omega)$ is equivalent to the *Sobolev-Slobodeckij norm*

$$[u]_{s,\Omega} = \left(\|u\|^2_{L^2(\Omega)} + \iint\limits_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{2}}.$$