MATH 581 ASSIGNMENT 1

DUE WEDNESDAY JANUARY 30

1. Consider the Laplace equation $\Delta u = 0$ on the unit disk, given in polar coordinates by $\mathbb{D} = \{(r,\theta) : r < 1\}$. Specify the Cauchy data
$$u(1,\theta) = f(\theta), \qquad \partial_r u(1,\theta) = g(\theta),$$
where f and g are 2π-periodic real analytic functions. Then show that a real analytic solution exists for all $\theta \in \mathbb{R}$ and $|r-1|$ sufficiently small. Investigate what happens to the solution as $r \to 0$ and $r \to \infty$, if f and g are of the form
$$a_0 + \sum_{n=1}^{m} a_n \cos n\theta + b_n \sin n\theta,$$
i.e., trigonometric polynomials.
2. On $B_r \subset \mathbb{R}^2$ with some $r > 0$, consider the system
$$\begin{cases} u_x \cdot u_{yy} = 0, \\ u_y \cdot u_{yy} = 0, \\ u_{xx} \cdot u_{yy} = |u_{xy}|^2 - f, \end{cases}$$
with the Cauchy data
$$u(x,0) = v(x), \qquad u_y(x,0) = w(x), \qquad -r < x < r,$$
where $u : B_r \to \mathbb{R}^3$ is the unknown, $f \in C^\omega(B_r)$, $v, w \in C^\omega(B_r, \mathbb{R}^3)$, and the dot denotes the dot product between two vectors in $\mathbb{R}^3$. Assuming that the vectors $v_x(0)$, $v_{xx}(0)$, $w(0)$ are linearly independent, prove that there exists a solution u analytic in a neighbourhood of 0. This is the analytic core of the Janet-Cartan isometric embedding theorem, specialized to 2 dimensions. Please do not generalize the problem, i.e., solve it in 2 dimensions as stated.
3. Consider the linear *system*
$$Au \equiv \sum_{|\alpha| \leq q} a_\alpha \partial^\alpha u = f,$$
where a_α are $m \times m$ real matrices whose entries are real analytic functions in an open neighbourhood of 0 in $\mathbb{R}^n$, and similarly f is an $\mathbb{R}^m$-valued analytic function near 0.

Date: Winter 2013.

a) Assume that there are constants $M < \infty$ and $r > 0$ such that

$$\max\{|\partial^\beta a_\alpha(0)|, |\partial^\beta f(0)|\} \leq M\beta! r^{-|\beta|},$$

for all α with $|\alpha| \leq q$, and for all β, where $|\cdot|$ denotes the maximum norm in $\mathbb{R}^{m\times m}$ or in $\mathbb{R}^m$. Assuming that $\{x_n = 0\}$ is noncharacteristic near 0, prove that there exists $\rho > 0$ depending only on M and r, such that there exists $u \in C^\omega(B_\rho, \mathbb{R}^m)$ satisfying $Au = f$ in B_ρ and $\partial^\alpha u|_{B_\rho\cap\{x_n=0\}} = 0$ for all α with $|\alpha| \leq q - 1$.

b) State and prove a local Holmgren theorem for these linear systems.

4. Let p be a nontrivial polynomial of n variables.
   a) Prove that the set $\{\xi \in \mathbb{R}^n : p(\xi) = 0\}$ is closed and of measure zero.
   b) Let $f \in C^\omega(\Omega)$ with $\Omega \subset \mathbb{R}^n$ an open neighbourhood of 0. Show that there is an open neighbourhood of 0, on which the equation $p(\partial)u = f$ has a solution.
5. Let p be a nontrivial polynomial of n variables, and let $H \subset \mathbb{R}^n$ be a (closed) half-space.
   a) Show that if $u \in C^\infty(\mathbb{R}^n)$ satisfies $p(\partial)u = 0$ in $\mathbb{R}^n$ and $\operatorname{supp} u \subset H$, and if the boundary of H is noncharacteristic for the constant coefficient operator $p(\partial)$, then $u \equiv 0$. Provide a counterexample when ∂H is characteristic and p is a nonconstant homogeneous polynomial.
   b) Show that if we require that u is compactly supported, then the noncharacteristic condition on ∂H can be removed, i.e., prove that if $u \in C_c^\infty(\mathbb{R}^n)$ satisfies $p(\partial)u = 0$ in $\mathbb{R}^n$ and $\operatorname{supp} u \subset H$ then $u \equiv 0$. Imply that if $u \in C_c^\infty(\mathbb{R}^n)$ then $\operatorname{supp} u$ is contained in the convex hull of $\operatorname{supp} p(\partial)u$.
6. Let u be a C^2 solution of the n-dimensional wave equation $\partial_t^2 - \Delta u = 0$, and assume that u and all its first derivatives vanish on the line segment $\{(0,t) \in \mathbb{R}^{n+1} : 0 < t < T\}$. By using Holmgren's theorem, determine the region of $\mathbb{R}^{n+1}$ where u must vanish.