The Moser-Harnacke inequality and its applications to regularity results of de Giorgi, Nash, and Moser.

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1 Introduction

1.1 Outline of paper

In this paper, I will give an overview of basic regularity theory for (elliptic) partial differential equations, mainly concerning Hölder spaces C^α. I will then devote a significant portion of the text to proving Moser's version of a Harnack inequality. Finally, I will discuss the extraordinary regularity results of De Giorgi, Nash, and Moser that were discovered in the mid 1950s and early 1960s, and then some of the applications of these results to existence and uniqueness theorems for various partial differential equations.

1.2 Notation

We will use a notation borrowed from [2]. We call $\Omega \subset \mathbb{R}^n$ a *domain* if it is an open and connected set. If Ω is a domain and X is an open subset of Ω, then we write $\Omega \subset\subset$ to mean that $\overline{X} \subset \Omega$ and $\overline{X}$ is compact. If u is a sufficiently differentiable function, we write

$$D_i u = D_{x_i} u = \frac{\partial u}{\partial x_i}, \qquad D_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

for the partial differential operator. If α is a multi-index, $\alpha = (\alpha_1, \ldots, \alpha_n)$ then we write D^α if we mean

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

By $C^0(\Omega)$ we mean the set of continuous functions $u : \Omega \to \mathbb{R}$, and if $k \geq 1$ we denote $C^k(\Omega)$ by the set of continuous functions from Ω to $\mathbb{R}$ such that the derivatives $D^\alpha u$ for $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq k$ exist and are continuous. For a C^1 domain, we write $\nu = (\nu_1, \ldots, \nu_n)$ to mean the outward unit normal along $\partial\Omega$.

2 Overview of Hölder spaces

Hölder spaces are a fundamental tool in studying partial differential equations, and they will come into use particularly in the later section when we develop the regularity results of De Giorgi, Nash, and Moser. Following Jost [3], we will first prove the Moser-Harnack inequality, after which we will discuss some applications of this inequality to elliptic partial differential equations, and then finally discuss regularity results for variational problems. We first begin with a definition for the Hölder seminorms and Hölder spaces.

Definition 1 (Hölder semi-norms and Hölder space $C^{m,\mu}$)**.** Let $\mu \in (0,1]$ and $K \subset \mathbb{R}^n$ be compact. Given $u \in C^0(K)$, we define the Hölder semi-norm $[u]_{\mu,\Omega}$ by

$$[u]_{\mu,\Omega} = \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu}$$

For $m \geq 0$ and $\mu \in (0,1]$, we say $u \in C^{m,\mu}(\Omega)$ if $u \in C^m(\Omega)$ and $[D^m]_{\mu,K} < \infty$ for all $K \subset \Omega$.

We define the Hölder norms as follows.

Definition 2 (C^k and Hölder norms)**.** Let Ω be a bounded domain. We define the C^0 norm $|\cdot|_{0,\Omega}$ and C^m norm $|\cdot|_{m,\Omega}$ by

$$|u|_{0,\Omega} = \sup_{\Omega} |u|, \qquad \|u\|_{C^m(\overline{\Omega})} = |u|_{m;\Omega} := \sum_{k \leq m} |D^k u|_{0;\Omega}$$

The Hölder norm $|\cdot|_{m,\mu,\Omega}$ is defined by

$$\|u\|_{C^{m,\mu}(\overline{\Omega})} = |u|_{m,\mu;\Omega} := \sum_{k \leq m} |D^k u|_{0;\Omega} + [D^m u]_{\mu;\Omega}$$

Definition 3 (Sobolev space $W^{k,p}$)**.** If $\Omega \subset \mathbb{R}^n$ is a domain, we say that an integrable function $v : \Omega \to \mathbb{R}$ is the α-th weak derivative of u, written $v = D^\alpha u$, if

$$\int_\Omega v\phi dx = (-1)^{|\alpha|} \int_X u D^\alpha \phi dx \qquad\qquad \text{for all } \phi \in \mathscr{D}(\Omega)$$

For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define the Sobolev space $W^{k,p}$ as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p \text{ for all } |\alpha| \leq k\}$$

The corresponding norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is given by

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}$$

Lemma 1. *Let $f_1, f_2 \in C^\alpha(\Omega) := C^{0,\alpha}$ where $\Omega \subset \mathbb{R}^n$. Then we have that $f_1 \cdot f_2 \in C^\alpha(\Omega)$, and moreover we have the estimate*

$$|f_1 f_2|_{C^\alpha(\Omega)} \leq |f_2|_{C^\alpha(\Omega)} \left[\sup_\Omega |f_1| \right] + |f_1|_{C^\alpha(\Omega)} \left[\sup_\Omega |f_2| \right]$$

Proof. The proof of this is quite simple. We need only use the triangle in the following:

$$\frac{|f_1(x)f_2(x) - f_1(y)f_2(y)|}{|x-y|^\alpha} \leq \frac{|f_1(x) - f_1(y)|}{|x-y|^\alpha} |f_2(x)| + \frac{|f_2(x) - f_2(y)|}{|x-y|^\alpha} |f_1(x)|$$

The result follows immediately since $f_1, f_2 \in C^\alpha$. □

3 Moser-Harnack inequality

We follow Jost [3] and Moser [4], [5] here. We want to understand the weak solutions to the homogeneous equation $Lu = 0$ where L is defined by

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a^{ij}(x) \frac{\partial}{\partial x_i} u(x) \right) = 0$$

where the coefficients a^{ij} are measurable and bounded, i.e., there is finite $\Lambda > 0$ such that

$$\sup_{i,j,x} |a^{ij}(x)| \leq \Lambda < \infty$$

and that the coefficients also satisfy the ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \tag{1}$$

where $0 < \lambda \leq \Lambda$, and $\sup(\cdot)$ is understood as ess $\sup(\cdot)$. The notion of subsolutions and supersolutions will be used in establishing the Harnack inequalities.

Definition 4 (Sub- and supersolutions)**.** Let $u \in W^{1,2}(\Omega)$. We call u a *weak subsolution* (resp. *supersolution*) of L, denoted $Lu \geq 0$ (resp. $Lu \leq 0$) if for all positive functions $\phi \in H_0^{1,2}(\Omega)$, we have that

$$\int_X \sum_{i,j} a^{ij}(x) D_i u D_j \phi dx \leq 0 \tag{2}$$

(resp. $\int \sum \geq 0$ for supersolution). All the inequalities are assumed to hold except possibly on sets of measure zero.

From this definition, it is clear from the following relation that if $f \in C^2(\mathbb{R})$ is convex, and u is a subsolution ($Lu \geq 0$), then $f \circ u$ is also a subsolution:

$$L(f \circ u) = \sum_{i,j} \frac{\partial}{\partial x_j}\left(a^{ij} f'(u) \frac{\partial u}{\partial x_j}\right) = f'(u)Lu + f''(u)\sum_{i,j} a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \tag{3}$$

In fact, if the first and second derivatives of f are integrable, we can perform integration by parts to deduce the following lemma:

Lemma 2. *Suppose $f \in C^2(\mathbb{R})$ is convex and u is a weak subsolution for L. Then $f \circ u$ is in fact a weak subsolution for L provided we can integrate f so that the chain rule for weak derivatives holds.*

Proof. Supposing that f is sufficiently integrable, we have that$D_i(f \circ u) = f'(u)D_i(u)$ and $D_i(f' \circ u) = f''(u)D_i u$ for each $i = 1, \ldots, n$. We then have that

$$\begin{aligned}\int_\Omega \sum_{i,j} a^{ij} D_i(f \circ u) D_j \phi &= \int_\Omega \sum_{i,j} f'(u) D_i u D_j \phi \\ &= \int_\Omega \sum_{i,j} a^{ij} D_i u D_j(f'(u)\phi) - \int_\Omega \sum_{i,j} a^{ij} D_i u f''(u) D_j u \phi\end{aligned}$$

Since f is convex and because of the ellipticity condition (1), if u is a weak subsolution and $f'(u)$ is positive, then we have that

$$\int_\Omega \sum_{i,j} a^{ij} D_i(f \circ u) D_j \phi \leq 0$$

Therefore $f \circ u$ is a weak subsolution. □

We need one more lemma before beginning the Moser iteration.

Lemma 3. *If $u \in W^{1,2}(\Omega)$ is a weak subsolution of L and k is some real number, then the function v defined by*

$$v(x) = \max(u(x), k)$$

is also a weak subsolution to L.

Proof. We can write v as a composition of functions $v = f \circ u$ where $f(x) := max(x, k)$, where $k \in \mathbb{R}$ is as in the statement of the lemma. For such an f there is a sequence (f_n) of convex, twice differentiable functions such that $f_n \to f$ and f_n is equal to f for all x outside of the interval $(k - \frac{1}{n}, k + \frac{1}{n})$, and such that $|f_n'(x)| \leq 1$ for all x. We then have that $f_n \circ u \to f \circ u$ in $W^{1,2}$ norm, so that we have for any $0 \leq \phi \in H_0^{1,2}$,

$$\int_\Omega \sum_{i,j} a^{ij} D_i v D_j \phi = \lim_{n \to \infty} \int_\Omega \sum_{i,j} D_i(f_n \circ u) D_j \phi$$

Finally, by (2), this means that the above quantity is a a weak subsolution, since each f_n are convex. □

3.1 Moser iteration and estimates

The goal of this section is to establish Moser's result [5] on general Harnack inequalities. We will first prove two more general theorems (Theorems 1 and 2) from which we will deduce the more well-known forms of the inequalities appearing in these theorems.

Let us now denote the average mean integral $\fint$ by

$$\fint_\Omega \phi dx = \frac{1}{\text{vol}(\Omega)} \int_\Omega \phi dx$$

We will be using the average mean integral in order to have our constants be independent of the size of the ball that we are integrating over.

The two main results of this section are due to Moser [5]. We follow Jost's presentation here [3]. The theorems are as follows.

Theorem 1. *If u is a subsolution to L in the ball $D(x_0, 4R) \subset \mathbb{R}^n$ for some $R > 0$, then for any $p > 1$ we have*

$$\sup_{D(x_0,R)} u \leq c_1 \left(\frac{p}{p-1}\right)^{2/p} \left(\fint_{D(x_0,2R)} (\max(0, u(x)))^p \, dx\right)^{1/p} \tag{4}$$

If u is a positive function, then (4) takes the form

$$\sup_{D(x_0,R)} u \leq c_1 \left(\frac{p}{p-1}\right)^{2/p} \left(\fint_{D(x_0,2R)} u^p dx\right)^{1/p}$$

The constant c_1 depends only on the dimension n and on the ratio of the ellipticity bounds, $\frac{\Lambda}{\lambda}$.

Theorem 2. *If u is a positive supersolution to L in the ball $D(x_0, 4R) \subset \mathbb{R}^n$, then for any dimension $n \geq 3$ and for any $p \in \left(0, \frac{d}{d-2}\right)$, we have that*

$$\left(\fint_{D(x_0,2R)} u^p dx \right)^{1/p} \leq \frac{c_2}{\left(\frac{n}{n-2} - p\right)^2} \inf_{D(x_0,R)} u \tag{5}$$

where $c_2 = c_2\left(n, \frac{\Lambda}{\lambda}\right)$.

These two theorems immediately imply the more familiar looking version of the Harnack inequalities: if u is a positive weak solution to $Lu = 0$ in the ball $D(x_0, 4R)$ in $\mathbb{R}^n$, then there is some constant C depending only on n and $\frac{\Lambda}{\lambda}$ such that

$$\sup_{D(x_0,R)} u \leq C \inf_{D(x_0,R)} u \tag{6}$$

This local result extends to general domains in $\mathbb{R}^n$ in the following manner. If u is a positive weak solution to $Lu = 0$ in some $\Omega \subset \mathbb{R}^n$, then for any $X \subset\subset \Omega$ (i.e., $\bar{X} \subset \Omega$ and $\bar{X}$ is compact), we have that

$$\sup_X u \leq C' \inf_X u \tag{7}$$

where C' depends on n, Ω, X, and $\frac{\Lambda}{\lambda}$. To see this, let $\{B_i\}_{i=1}^N$ be a finite subcover of $\bar{X}$, with the balls $B_i \subset \Omega$, each of radius R, such that $B_i \cap B_{i+1}$ is non-empty for all i. Then if $y_1, y_2 \in X$, we can take $y_1 \in B_k$ and $y_2 \in B_{k+n}$ for some positive integer m. Then by applying (6) to each of the balls B_i, we get that

$$\begin{aligned}
u(y_1) &\leq \sup_{B_k} u(x) \leq C \inf_{B_k} u(x) \\
&\leq C \sup_{B_{k+1}} u \qquad\qquad \text{because } B_i \cap B_{i+1} \text{ is non-empty} \\
&\leq C^2 \inf_{B_{k+1}} u \\
&\leq C^2 \sup_{B_{k+2}} u \leq \dots \\
&\leq C^{m+1} \inf_{B_{k+m}} u \leq C^{m+1} u(y_2)
\end{aligned}$$

Therefore (7) holds for general domains $\Omega \subset \mathbb{R}^n$.

We can now begin to start proving Theorems (1) and (2). If u is positive and $x_0 \in \mathbb{R}^n$, then we define $\phi(p, R)$ as

$$\phi(p, R) := \left(\fint_{B(x_0,R)} u^p dx \right)^{1/p}$$

The following two lemmas will be necessary for the whole proof of the two theorems.

Lemma 4. *We have the following behaviour of $\phi(p,R)$ as $p \to \pm\infty$.*

$$\lim_{p\to\infty} \phi(p,R) = \sup_{B(x_0,R)} u \tag{8}$$

$$\lim_{p\to-\infty} \phi(p,R) = \inf_{B(x_0,R)} u \tag{9}$$

Proof. Let $p' > p$ be arbitrary. If $u \in L^{p'}(\Omega)$, then we have that $\phi(p,R)$ is an increasing function of p for fixed R by Hölder's inequality:

$$\begin{aligned}\left(\fint_\Omega u^p dx\right)^{1/p} &\le \frac{1}{(\mathrm{vol}(\Omega))^{1/p}} \left(\int_\Omega dx\right)^{\frac{p'-p}{pp'}} \left(\int_\Omega (u^p)^{p'/p}\right)^{1/p'} \\ &= \left(\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega u^{p'}\right)^{1/p'}\end{aligned}$$

We also have that $\phi(p,R)$ is bounded above by $\phi(\infty,R) := \lim_{p\to\infty}\phi(p,R)$ since

$$\phi(p,R) \le \left(\frac{1}{R^n}\int_{B(x_0,R)} (\text{ess sup}\, u)^p\right)^{1/p} = \phi(\infty,R)$$

However, by definition of ess sup, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if we denote the set A as

$$A = \{x \in B(x_0,R) : u(x) \ge \sup_{B(x_0,R)} u - \varepsilon\}$$

then the measure of A is strictly greater than δ. We therefore have that

$$\begin{aligned}\phi(p,R) &\ge \left[\frac{1}{R^n}\int_A u^p dx\right]^{1/p} \\ &\ge \left(\frac{\delta}{R^n}\right)^{1/p} (\sup u - \varepsilon)\end{aligned}$$

Therefore we have that for any $\varepsilon > 0$, $\lim_{p\to\infty}\phi(p,R) \ge \sup u - \varepsilon$. Therefore we also have that $\lim_{p\to\infty}\phi(p,R) \ge \sup u$, which implies (8). By applying the above limit to the function $\frac{1}{u}$ instead of u, we get (9). □

We now prove the final lemma necessary for the proof of the desired theorems.

Lemma 5. *a) If u is a positive subsolution to L in Ω, then for $q > \frac{1}{2}$ assume that $v = u^q \in L^2(\Omega)$. Then for any $\eta \in H_0^{1,2}$ we have that*

$$\int_\Omega \eta^2 |Dv|^2 \le \frac{\Lambda^2}{\lambda^2}\left(\frac{2q}{2q-1}\right)^2 \int_\Omega |D\eta|^2 v^2 \tag{10}$$

b) If u is a supersolution, then this inequality is true when $q < \frac{1}{2}$.

Proof. We know that for (a), $f(u)$ is a subsolution by Lemma 2; for (b), $f(u)$ is a supersolution. So define $\phi = f'(u) \cdot \eta^2$; then $\phi \in H_0^{1,2}(\Omega)$ and so we have that

$$\int_\Omega \sum_{i,j} a^{ij}(x) D_i u D_j \phi = \int_\Omega \sum_{i,j} a^{ij} D_i u D_j u f''(u) \eta^2 + \int_\Omega \sum_{i,j} a^{ij} D_i u f'(u) 2\eta D_j \eta \tag{11}$$

$$= 2|q|(2q-1) \int_\Omega \sum_{i,j} D_i u D_j u u^{2q-2} \eta^2 + 4|q| \int_\Omega \sum_{i,j} a^{ij} D_i u u^{2q-1} \eta D_j \eta$$

$$\begin{cases} \leq 0 & \text{case (a)} \\ \geq 0 & \text{case (b)} \end{cases}$$

Now, recall Young's inequality: if a, b are positive real numbers and p, q are conjugate exponents, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. For case (a) we can apply Young's inequality to the last term: for any $\varepsilon > 0$ we have that, using the ellipticity estimate,

$$2|q|(2q-1)\lambda \int_\Omega |Du|^2 u^{2q-2} \eta^2 \leq 2|q|\Lambda\varepsilon \int_\Omega |Du|^2 u^{2q-2} \eta^2 + \frac{2|q|\Lambda}{\varepsilon} \int_\Omega u^{2q} |D\eta|^2$$

Since this inequality holds for all $\varepsilon > 0$, we can take $\varepsilon = \frac{\lambda}{\Lambda} \frac{2q-1}{2}$ for case (a) (and ε^{-1} for case (b)) to get that

$$\int_\Omega |Du|^2 u^{2q-2} \eta^2 \leq \frac{\Lambda^2}{\lambda^2} \frac{4}{(2q-1)^2} \int_\Omega u^{2q} |D\eta|^2 \quad \Leftrightarrow \quad \int_\Omega |Dv|^2 \eta^2 \leq \frac{\Lambda^2}{\lambda^2} \int_\Omega v^2 |D\eta|^2$$

□

We can now begin with the proofs of Theorems 1 and 2. Note that in each of the theorems, the inequalities are scaling and translation invariant, so we can assume that the problem is concerned with a ball of radius 1 centred at the origin; namely, $x_0 = 0$ and $R = 1$. Also note that by Lemma 3, we can consider the case when u is a positive function, for otherwise we could consider functions of the form

$$v_k(x) = \max\{u(x), k\}$$

where $k > 0$ is an arbitrary positive constant, and applying the following proof for positive functions and letting $k \downarrow 0$.

For brevity we will denote $B_r := B(0, r) \subset \mathbb{R}^n$, and let r' be a number such that $0 < r' < r \leq 2r'$. We define cutoff function $\eta \in H_0^{1,2}(B-r)$ such that

$$\begin{aligned} \eta &\equiv 1 && \text{on } B_{r'} \\ \eta &\equiv 0 && \text{on } B_r^c \\ |D\eta| &\leq \frac{2}{r-r'} \end{aligned} \tag{12}$$

So let us define $v = u^q$ again, and assume $v \in L^2(\Omega)$. Recall the following form of the Sobolev embedding theorem:

Theorem 3 (Sobolev embedding theorem)**.** *For $1 \le p < n$ and $u \in H^{1,p}(B(x_0,R))$ (where $B(x_0,R) \subset \mathbb{R}^n$), we have that*

$$\left(\fint_{B(x_0,R)} |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le c\left[R^p \fint_{B(x_0,R)} |Du|^p + \fint_{B(x_0,R)} |u|^p\right]^{1/p}$$

and c depends only on p and n

Thus for $n \ge 3$ we have that

$$\left(\fint_{B_{r'}} v^{\frac{2d}{d-2}}\right)^{\frac{d-2}{d}} \le c\left(r'^2 \fint_{B_{r'}} |Dv|^2 + \fint_{B_{r'}} v^2\right) \tag{13}$$

Putting (10), (12), (13) together then gives us that

$$\left(\fint_{B_{r'}} v^{\frac{2d}{d-2}}\right)^{\frac{d-2}{d}} \le C \fint_{B_r} v^2 \tag{14}$$

Here, C is a constant that is bounded above by

$$C \le c_1\left[\left(\frac{r'}{r-r'}\right)^2 \left(\frac{2q}{2q-1}\right)^2 + 1\right] \tag{15}$$

Therefore we have that $v \in L^{\frac{2n}{n-2}}(\Omega)$. We iterate this step to get that larger and larger power of u are integrable. ("Moser iteration"!) So let $s = 2q$ and assume that $|s| \ge \mu > 0$ for some lower bound μ, whose exact value is to be determined. What matters is that it is strictly bounded away from zero. Then, since $r \le 2r'$ by construction, we have that

$$C \le c_2 \left(\frac{r'}{r-r'}\right)^2 \left(\frac{s}{s-1}\right)^2 \tag{16}$$

where c_2 depends on μ as well. Since $v = u^{s/2}$, for $s \ge \mu$ the relations (14) and (16) imply that

$$\phi\left(\frac{ns}{n-2}, r'\right) = \left(\fint_{B_{r'}} v^{\frac{2n}{n-2}}\right)^{\frac{n-2}{ns}} \le c_3 \left(\frac{r'}{r-r'}\right)^{2/s} \left(\frac{s}{s-1}\right)^{2/s} \phi(s,r) \tag{17}$$

where $c_3 := c_2^{1/s}$. When $s \le -\mu$ we similarly have that

$$\phi\left(\frac{ns}{n-2}, r'\right) \ge \left(\frac{s}{s-1}\right)^{-2/s} \frac{1}{c_3} \left(\frac{r'}{r-r'}\right)^{-2/|s|} \phi(s,r) \ge \left(\frac{r'}{r-r'}\right)^{-2/|s|} \phi(s,r) \tag{18}$$

since $s \leq -\mu$. We can now perform the iteration we spoke of earlier. The idea is that we can appropriately bound the integrals of larger power of u by smaller powers of u. So let us define numbers s_m, r_m, r'_m for $m \in \mathbb{N}$ as

$$s_m = p\left(\frac{n}{n-2}\right)^m, \qquad r_n = 1 + 2^{-n}, \qquad r'_n := r_{n+1} > \frac{r_n}{2}$$

Then we can use (16) to get that for any $m \in \mathbb{N}$,

$$\begin{aligned}\phi(s_{m+1}, r_{m+1}) &\leq c_3 \left[\frac{1+2^{-m-1}}{2^{-m-1}} \cdot \frac{p\left(\frac{n}{n-2}\right)^m}{p\left(\frac{n}{n-2}\right)^m - 1}\right]^{\frac{2}{p\left(\frac{n}{n-2}\right)^m}} \phi(s_m, r_m) \\ &:= c_4^{m\left(\frac{n}{n-2}\right)^{-m}} \phi(s_m, r_m)\end{aligned}$$

Thus we define iteratively $\phi(s_{m+1}, r_{m+1})$ by

$$\phi(s_{m+1}, r_{m+1}) \leq c_4^{\sum_{k=1}^m k\left(\frac{n}{n-2}\right)^{-k}} \phi(s_1, r_1) \leq c'_4 \left(\frac{p}{p-1}\right)^{2/p} \phi(p, 2) \tag{19}$$

But Lemma 4 allows us to take the limit $m \to \infty$ to deduce that

$$\text{ess sup}\, u := \phi(\infty, 1) \leq C'' \left(\frac{p}{p-1}\right)^{2/p} \left[\fint_{B(0,2)} u^p dx\right]^{1/p}$$

which is precisely Theorem 1.

We now begin with proving Theorem 2. We will assume that u is strictly positive for this proof; say, there is some $\varepsilon > 0$ such that $u > \varepsilon > 0$ on the domain. This will allow $\phi(s, r)$ to be finite when s is negative. After proving the theorem for $u > \varepsilon > 0$, if we want to prove the theorem for $v \geq 0$, we can just apply the theorem to $v + \varepsilon > 0$ and then take the limit as $\varepsilon \downarrow 0$. So we continue with the previous Moser iteration for when $s \leq -\mu$. Letting $r_m = 2 + 2^{-m}$ again, (18) implies that

$$\phi(-\mu, 3) \leq k_0 \phi(-\infty, 2) \leq k_0 \phi(-\infty, 1) \tag{20}$$

for some constant k_0. We iterate this procedure until we get the relation

$$\phi(p, 2) \leq k_1 \phi(\mu, 3) \tag{21}$$

for some constant k_1. So we need only prove that

$$\phi(\mu, 3) \leq k_2 \phi(\mu, 3) \tag{22}$$

To prove this, we will need the John-Nirenberg theorem, which we present without proof (see Theorem 9.1.2 of Jost [3] for a full proof):

Theorem 4. *Let $B_0 := B(x_0, R_0) \subset \mathbb{R}^n$ be a ball and let $u \in W^{1,1}(B(x_0, R_0))$. Suppose that for all $B = B(x, R) \subset \mathbb{R}^n$ we have that*

$$\int_{B_0 \cap B} |Du| \le R^{d-1}$$

Then there is some $\alpha > 0$ and $\beta_0 < \infty$ such that

$$\int_{B_0} e^{\alpha|u-u_0|} \le \beta_0 R_0^d$$

where u_0 is the mean of u on B_0, i.e.,

$$u_0 = \frac{1}{vol(B(0,1))} \int_{B_0} u$$

Thus the inequality

$$\int_{B_0} e^{\alpha u} \int_{B_0} e^{-\alpha u} = \int_{B_0} e^{\alpha(u-u_0)} \int_{B_0} e^{-\alpha(u-u_0)} \le \beta_0^2 R_0^{2n}$$

holds.

So let us define functions $v := \log u$ and $\phi := \frac{1}{u}\eta^2$ where η is a cut-off function, $\eta \in H_0^{1,2}(B(0,4))$. Since u is a supersolution, then have that

$$0 \le \int_{B(0,4)} \sum_{i,j} a^{ij} D_i\phi D_j u = -\int_{B(0,4)} \eta^2 \sum_{i,j} a^{ij} D_i v D_j v + \int_{B(0,4)} 2\eta \sum_{i,j} a^{ij} D_i\eta D_j v$$

Thus using the Cauchy-Schwarz inequality and the ellipticity estimate we get that

$$\lambda \int_{B(0,4)} \eta^2 |Dv|^2 \le \int_{B(0,4)} \eta^2 \sum_{i,j} a^{ij} D_i v D_j v \le 2 \int_{B(0,4)} \eta \sum_{i,j} a^{ij} D_i\eta D_j v$$
$$\le 2\Lambda \left[\int_{B(0,4)} \eta^2 |Dv|^2\right]^{1/2} \left[\int_{B(0,4)} |D\eta|^2\right]^{1/2}$$

We thus have that

$$\int_{B(0,4)} \eta^2 |Dv|^2 \le 4\left(\frac{\Lambda}{\lambda}\right)^2 \int_{B(0,4)} |D\eta|^2 \tag{23}$$

In order to apply Theorem 4, we need to bound the integral of $|Dv|$ by a constant times R^{n-1}. So let $B(x, R) \subset B(0, 7/2)$ be any ball, and choose cut-off function η such that

$$\eta \equiv 1 \text{ on } B(x, R)$$
$$\eta \equiv 0 \text{ outside } B(0,4) \cap B(x, 2R)$$
$$|D\eta| \le \frac{6}{R}$$

Thus (23) implies that there is some constant c such that

$$\fint_{B(x,R)} |Dv|^2 \leq c\frac{1}{R^2}$$

and from here we can apply Hölder's inequality to get that

$$\int_{B(x,R)} |Dv| \leq \mathrm{vol}(B(x,R))\sqrt{c}R^{n-1}$$

Now we can apply Theorem 4: let α be as in the theorem, and define $\mu := \frac{\alpha}{\sqrt{c}\mathrm{vol}(B(0,1))}$, and then apply the theorem to the function w defined by

$$w = \frac{1}{\sqrt{c}\mathrm{vol}(B(0,1))}v = \frac{1}{\sqrt{c}\mathrm{vol}(B(0,1))}\log u$$

to get that

$$\int_{B(0,3)} u^{\mu} \int_{B(0,3)} u^{-\mu} \leq \beta^2$$

which finally gives the desired inequality: $\phi(\mu, 3) \leq \beta^{2/\mu}\phi(-\mu, 3)$ Having proved (22), we deduce the theorem from (20) and (21).

4 Applications of Moser-Harnack inequality

With the inequalities proved in the previous section, we are in place to demonstrate the Hölder continuity of weak solutions to the elliptic equation $Lu = 0$. Both de Giorgi and Nash proved the following result, but we will follow Moser's proof which is based on the Moser-Harnack inequality.

Theorem 5. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to $Lu = 0$, i.e.,*

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left[a^{ij}(x)\frac{\partial}{\partial x_j}u(x)\right] = 0 \tag{24}$$

where the coefficients $a^{ij}(x)$ are measurable and satisfy the ellipticity conditions: for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, we have that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j, \qquad |a^{ij}(x)| \leq \Lambda \tag{25}$$

where $0 < \lambda < \Lambda < \infty$ as before. Then there is some $\alpha \in (0, 1)$ such that $u \in C^{\alpha}(\Omega)$, i.e., u is Hölder continuous in Ω. Thus for any $X \subset\subset \Omega$, there is $\alpha \in (0, 1)$ and constant $c > 0$ such that for any $x, y \in X$,

$$|u(x) - u(y)| \leq c|x - y|^{\alpha} \tag{26}$$

where c depends only on the difference $\sup_X u - \inf_X u$, and α depends only on the dimension n, the ellipticity ratio $\frac{\Lambda}{\lambda}$, and X

Proof. Let $x \in \Omega$. For positive radius $R > 0$ and ball $B(x, R) \subset \Omega$, we define $M(R)$ and $m(R)$ as

$$M(R) = \sup_{B(x,R)} u, \qquad m(R) = \inf_{B(x,R)} u$$

We claim that both $m(R)$ and $M(R)$ are finite. To see this, define for any $k > 0$

$$v(x) = \max\{u(x), k\}$$

Then by Lemma 3, we know that v is a positive subsolution to L. Then v is locally bounded by Theorem 1, and by the proof used in the Harnack inequality, we get that u is bounded as well.

So now let us define the oscillation of u in the ball:

$$\omega(R) := M(R) - m(R)$$

Notice that $\omega(r)$ is an increasing function. If we can prove the following inequality

$$\omega(r) \leq c \left(\frac{r}{R}\right)^{\alpha} \omega(R) \qquad\qquad 0 < r \leq \frac{R}{4} \tag{27}$$

for some $\alpha \in (0, 1)$, then this will prove that u is Hölder continuous, as for any $y \in B(x, r)$ we would have that

$$u(x) - u(y) \leq \sup_{B(x,r)} u - \inf_{B(x,r)} u = \omega(r) \leq c\frac{\omega(R)}{R^{\alpha}}|x - y|^{\alpha} \tag{28}$$

Now, for any $\varepsilon > 0$, we have that the functions

$$M(R) - u + \varepsilon > 0$$

$$u - m(R) + \varepsilon >$$

are positive solutions to $Lu = 0$ in $B(x, R)$. We thus have the inequalities

$$\begin{aligned} M(R) - m\left(\frac{R}{4}\right) &= \sup_{B\left(x,\frac{R}{4}\right)} (M(R) - u) \leq c' \inf_{B\left(x,\frac{R}{4}\right)} (M(R) - u) \\ &= c'\left[M(R) - M\left(\frac{R}{4}\right)\right] \end{aligned}$$

$$\begin{aligned} M\left(\frac{R}{4}\right) - m(R) &= \sup_{B\left(x,\frac{R}{4}\right)} (u - m(R)) \leq c' \inf_{B\left(x,\frac{R}{4}\right)} (-M(R) + u) \\ &= -c'\left[m(R) - m\left(\frac{R}{4}\right)\right] \end{aligned}$$

Adding these together we get that

$$M\left(\frac{R}{4}\right) - m\left(\frac{R}{4}\right) \leq \frac{c' - 1}{c' + 1}\left[M(R) - m(r)\right] \tag{29}$$

□

So define $\kappa = \frac{c'-1}{c'+1}$. Then $\kappa < 1$, so that we have $\omega\left(\frac{R}{4}\right) \leq \kappa\omega(R)$. Therefore $\omega\left(\frac{R}{4^2}\right) \leq \kappa^2\omega(R)$, and inductively for any $m \in \mathbb{N}$, we have that

$$\omega\left(\frac{R}{4^m}\right) \leq \kappa^m \omega(R) \tag{30}$$

So let $r > 0$ be such that $\frac{R}{4^{n+1}} \leq r \leq \frac{R}{4^n}$, and choose $\alpha > 0$ such that

$$\kappa \leq \frac{1}{4^\alpha}$$

Then this proves (27), since we have

$$\begin{aligned}
\omega(r) &\leq \omega\left(\frac{R}{4^m}\right) && (\omega(r) \text{ is increasing}) \\
&\leq \kappa^m \omega(R) && (\text{inequality (30)}) \\
&\leq \left(\frac{1}{4^m}\right)^\alpha \omega(R) \leq \left(\frac{4}{4R}\right)^\alpha \omega(R) && (\text{since } r \in \left(\frac{R}{4^{m+1}}, \frac{R}{4^m}\right)) \\
&= 4^{-\alpha}\left(\frac{r}{R}\right)^\alpha \omega(R)
\end{aligned}$$

So we have shown that any weak solution $u \in W^{1,2}$ to the elliptic $Lu = 0$ is Hölder continuous for some $\alpha \in (0,1)$. This regularity result has significant applications, the first of which is a stronger version of the maximum principle, and the second of which is an analogue of Liouville's theorem.

Theorem 6. *Let $u \in W^{1,2}(\Omega)$ be a weak subsolution to L, i.e., $Lu \geq 0$ weakly. Let the coefficients $\{a^{ij}\}$ of L satisfy the ellipticity estimates*

$$\lambda|\xi|^2 \leq \sum_{i,j} a^{ij}(x)\xi_i\xi_j, \qquad |a^{ij}(x)| \leq \Lambda$$

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$. Suppose that for some open ball $B(y_0, R) \subset\subset \Omega$ we have

$$\sup_{B(y_0,R)} u = \sup_\Omega u \tag{31}$$

Then we have that u is constant on the whole domain Ω.

Proof. We first note that if (31) is true, then there is another ball $B(x_0, r_0)$ with $B(x_0, 4r_0) \subset \Omega$ such that

$$\sup_{B(x_0,r_0)} = \sup_\Omega u \tag{32}$$

Moreover, by Theorem 5, we can take $\sup_\Omega u$ to be finite since $\sup_{B(y_0,R)} u < \infty$. If M is a number such that $M > \sup_\Omega u$, then $M - u$ is a positive subsolution to L, and hence we can apply Theorem 2; taking the limit we get that

$$M = \sup_\Omega u \tag{33}$$

Again by Theorem 2, relations (32) and (33) imply that

$$\fint_{B(x_0,2r_0)} (M-u) \le c \inf_{B(x_0,r_0)} (M-u) = 0$$

Since M is equal to the supremum of u over the domain, we also have that $u \le M$, and hence in the ball $B(x_0, 2r_0)$, we have that

$$u = M \ (\text{on } B(x_0, 2r_0)) \tag{34}$$

Now we have found that u is constant in a ball of radius $2r_0$; we would like to extend this result to the whole domain. So let $y \in \Omega$ be arbitrary. Then there is a sequence of balls $B_i := B(x_i, r_i)$ for $i = 0, \dots, N$ such that $B(x_i, 4r_i) \subset \Omega$, and $B_{i-1} \cap B_i \neq \emptyset$ for $i = 1, \dots, N$, and that $y \in B_N$. Since $B_0 \cap B_1 \neq \emptyset$, and since we have already shown that $u = M$ on $B(x_0, 2r_0)$, we thus have that

$$\sup_{B_1} u = M$$

and hence by the same argument as before, we get that $u = M$ on the ball $B(x_1, 2r_1)$. Evidently we can iterate this process for each ball to obtain that

$$u = M \text{ on } B(x_N, 2r_N)$$

Since $y \in B(x_N, r_N)$, we get that $u(y) = M$, and since $y \in \Omega$ was arbitrary, we get that $u \equiv M$ on all of Ω. $\square$

One last result of the Harnack inequality is the following.

Theorem 7. *Let u be a* bounded *weak solution to $Lu = 0$ defined on all of $\mathbb{R}^n$, where again L has measurable coefficients a^{ij} that satisfy the ellipticity condition*

$$\lambda|\xi| \sum_{i,j} a^{ij}(x)\xi_i\xi_j, \qquad |a^{ij}(x)| \le \Lambda$$

for some constants $0 < \lambda \le \Lambda < \infty$, and all $x, \xi \in \mathbb{R}^n$. Then u is a constant function.

Proof. Since u is bounded, we know that its supremum and infimum over $\mathbb{R}^n$ are finite, and so if α is an constant such that

$$\alpha < \inf_{\mathbb{R}^n} u$$

then we know that $u - \alpha$ is a positive subsolution to $Lu = 0$ on all of $\mathbb{R}^n$. Thus we know that for any $R > 0$ and such an α,

$$0 \le \sup_{B(0,R)} -\alpha \le c \left[\inf_{B(0,R)} u - \alpha \right]$$

Therefore, taking the limit as $R \to \infty$, we get that $\alpha = \inf_{\mathbb{R}^n} u$. We thus have that

$$0 \le \sup_{\mathbb{R}^n} u - \alpha \le c \left[\inf_{\mathbb{R}^n} -\alpha \right] = 0$$

which implies that u is constant on all of $\mathbb{R}^n$. $\square$

5 Regularity for variational problems

We will prove a special case of de Giorgi's work [1], again following Jost [3]. The focus will be on elliptic Euler-Lagrange equations, and the main result is the following; the proof of which will take considerable work.

Theorem 8. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function satisfying the following for some constants $K, \Lambda < \infty$ and $\lambda > 0$ for all $y \in \mathbb{R}^n$:*

(i) $\left|\frac{\partial F}{\partial y_i}(y)\right| \leq K|y|$ *for each* $i = 1, \ldots, n$

(ii) $\lambda|\xi|^2 \leq \sum_{i,j} \frac{\partial^2 F(y)}{\partial y_i \partial y_j}\xi_i\xi_j \leq \Lambda|\xi|^2$ *for all* $\xi \in \mathbb{R}^n$

and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in W^{1,2}(\Omega)$ be a minimizier of the variation problem

$$I(v) := \int_\Omega F(Dv(x))dx$$

In particular, for all $\phi \in H_0^{1,2}(\Omega)$, we have that

$$I(u) \leq I(u + \phi) \tag{35}$$

Then u itself is a $C^\infty(\Omega)$ function.

The variational problem makes sense because (i) implies that F is bounded by some constant multiple of $(1+|y|^2)$, and since Ω is bounded this implies that for any $v \in W^{1,2}(\Omega)$, we have that

$$I(v) = \int_\Omega F(Dv) < \infty$$

So we first get the Euler-Lagrange equations for the minimizer of the functional I. In the rest of the problem we will use the notation $F_{y_i} := \frac{\partial F}{\partial y_i}$.

Lemma 6. *In the settings of Theorem 8, we have for all $\phi \in H_0^{1,2}(\Omega)$ that*

$$\int_\Omega \sum_{i=1}^n F_{y_i}(Du)D_i\phi = 0 \tag{36}$$

where Du is the vector $Du = (D_1u_1, \ldots, D_nu_n)$.

Proof. By property (i) in the referred theorem, we have that

$$\int_\Omega \sum_{i=1}^n F_{y_i}(Dv)D_i\phi \leq nK \int_\Omega |Dv||D\phi| \leq dK \|Dv\|_{L^2} \|D\phi\|_{L^2} < \infty$$

Thus by the Lebesgue differentiation theorem we can differentiate through the integral sign to compute

$$\frac{d}{dt}I(u + t\phi) = \int_\Omega \sum_{i=1}^n F_{y_i}(Du + tD\phi)D_i\phi \tag{37}$$

Since u is the minimizer, this implies that

$$\frac{d}{dt}\, I(u+t\phi)|_{t=0} = 0 \tag{38}$$

Substituting $t = 0$ into (37) with (38) gives the desired inequality. $\square$

With the help of this lemma, we now need only prove the following theorem:

Theorem 9. *Let $A^j : \mathbb{R}^n \to \mathbb{R}$ be smooth functions for $i = 1, \dots, n$, satisfying three conditions for some constants $K, \Lambda < \infty$ and positive constant $\lambda > 0$ for all $y \in \mathbb{R}^n$:*

(i) $|A^i(y)| \le K|y|$ *for all* $i = 1, \dots, n$.

(ii) $\lambda|\xi|^2 \le \sum_{i,j=1}^n \frac{\partial A^i(y)}{\partial y_j}\xi_i\xi_j$ *for all* $\xi \in \mathbb{R}^n$.

(iii) $\left|\frac{\partial A^i(y)}{\partial y_j}\right| \le \Lambda$

Then if $u \in W^{1,2}(\Omega)$ is a weak solution to $\sum \partial x_j A^j(Du) = 0$ in $\Omega \subset \mathbb{R}^n$, i.e., for all $\phi \in H_0^{1,2}(\Omega)$ we have

$$\int_\Omega \sum_{i=1}^n A^i(Du) D_i\phi = 0 \tag{39}$$

Then $u \in C^\infty(\Omega)$.

We will prove this theorem after a few lemmas. We will see that the most essential part of the proof relies on Theorem 5, which we recall were initially proved by de Giorgi and Nash.

Lemma 7. *Assuming the setup of Theorem 9, for any $X \subset\subset \Omega$ we have that $u \in W^{2,2}(X)$, and the inequality*

$$\|u\|_{W^{2,2}(X)} \le c\,\|u\|_{W^{1,2}(\Omega)}$$

where c depends on λ, Λ, and $\operatorname{dist}(X, \partial\Omega)$.

Proof. Let e_j denote the j-th unit vector in $\mathbb{R}^n$, and let h be such that

$$|h| < \operatorname{dist}(\operatorname{supp}\phi, \partial\Omega)$$

so that $\phi_{k,-h}(x) := \phi(x - he_k) \in H_0^{1,2}(\Omega)$. We then have that

$$\begin{aligned}
0 &= \int_\Omega \sum_{i=1}^n A^i(Du(x)) D_i\phi_{k,-h}(x)dx = \int_\Omega \sum_{i=1}^n A^i(Du(x)) D_i\phi(x - he_k)dx \\
&= \int_\Omega \sum_{i=1}^n A^i(Du(y + he_k)) D_i\phi(y)dy \\
&= \int_\Omega \sum_{i=1}^n A^i((Du)_{k,h}) D_i\phi
\end{aligned}$$

We therefore have, by subtracting (39) from the above, that

$$\int \sum_{i=1}^{n} \left[A^i(Du(x+he_k)) - A^i(Du(x))\right] D_i\phi(x) = 0 \tag{40}$$

We therefore have that for a.e. $x \in \Omega$,

$$A^i(Du(x+he_k)) - A^i(Du(x)) = \int_0^1 \frac{d}{dt} A^i(tDu(x+he_k) + (1-t)Du(x))dt \tag{41}$$

$$= \int_0^1 \left[\sum_{j=1}^{d} A^i_{y_j}(tDu(x+he_k) + (1-t)Du(x))D_j(u(x+he_k) - u(x))\right] dt$$

We can now define coefficients a_h^{ij} as the following:

$$a_h^{ij}(x) := \int_0^1 A^i_{y_j}\left[tDu(x+he_k) + (1-t)Du(x))\right] dt$$

We can then use (**??**) to rewrite (48) as

$$\int_\Omega \sum_{i,j} a_h^{ij}(x) D_j \left(\Delta_k^h u(x)\right) D_i\phi(x)dx = 0 \tag{42}$$

where $\Delta_k^h u(x)$ is notation for the forward difference

$$\Delta_k^h u(x) := \frac{u(x+he_k) - u(x)}{h}$$

Notice that the coefficients a_h^{ij} also satisfy the ellipticity conditions. Then let $\eta \in C_0^1(X')$ where X' is such that

$$X \subset\subset X' \subset\subset \Omega$$

with both of $\operatorname{dist}(X', \partial\Omega)$ and $\operatorname{dist}(X, \partial X')$ greater than $\frac{1}{4}\operatorname{dist}(X, \partial\Omega)$, such that η is bounded as follows:

$$0 \le \eta \le 1$$
$$\eta(x) := 1 \text{ for } x \in X$$
$$|D\eta| \le \frac{8}{\operatorname{dist}(X, \partial\Omega)}$$

and $|2h| < \operatorname{dist}(X', \partial\Omega)$. We then continue with (**??**) to deduce that

$$\lambda \int_\Omega |D\Delta_k^h u|^2 \eta^2 \le \int_\Omega \sum_{i,j=1}^{n} a_h^{ij} \left(D_j \Delta_k^h u\right) (D_i \Delta_k^h u)\eta^2$$

$$= -\int_\Omega \sum_{i,j=1}^{n} a_h^{ij} D_j \Delta_k^h u \cdot 2\eta \left(D_i\eta\right) \Delta_k^h u$$

But from here we can apply Young's inequality to get that the above is bounded by, for any $\varepsilon > 0$,

$$\cdots \leq \varepsilon\Lambda \int_\Omega |D\Delta_k^u|^2 + \frac{\Lambda}{\varepsilon}\int_\Omega |\Delta_k^h u|^2 |D\eta|^2$$

In particular, we can take $\varepsilon = \lambda/2\Lambda$ to get that

$$\int_\Omega |D\Delta_k^h|^2\eta^2 \leq c\int_{X'} |\Delta_k^h u|^2 \leq c\int_\Omega |Du|^2$$

Therefore we have shown that $\left\|D\Delta_k^h u\right\|_{L^2(X)} \leq c\left\|Du\right\|_{L^2(\Omega)}$. We thus deduce that $D^2u \in L^2(X)$, and hence

$$\left\|D^2 u\right\|_{L^2(X)} \leq c\left\|Du\right\|_{L^2(\Omega)} \tag{43}$$

It follows that $u \in W^{2,2}(X)$, as was to be shown. □

Note that Δ_k^h is an approximation to the derivative, so that in the limit as $h \to 0$, if we let a^{ij} and v be defined as

$$a^{ij}(x) = A^i_{y_j}(Du(x))$$
$$v = D_k u$$

Then we get that

$$\int_\Omega \sum_{i,j=1}^n a^{ij}(x) D_j v D_i \phi = 0 \text{ for all } \phi \in H_0^{1,2}(\Omega)$$

Therefore, applying Theorem 5 to v, we get the following lemma.

Lemma 8. *Assuming the conditions of Theorem 5, we have that $Du \in C^\alpha(\Omega)$ for some Hölder exponent $\alpha \in (0, 1)$. In particular, this means that*

$$u \in C^{1,\alpha}(\Omega)$$

for some $0 < \alpha < 1$.

Therefore for each $k = 1, \ldots, n$, we have that $v = D_k u$ is a solution to the divergence-type equation

$$\sum_{i,j=1}^n D_i\left(a^{ij}(x) D_j v\right) = 0 \tag{44}$$

where te coefficients $a^{ij}(x)$ again satisfy the ellipticity requirements

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}\xi_i\xi_j, \qquad |a^{ij}(x)| \leq \Lambda$$

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$. But by the above lemma, we know that since the A^i are smooth and Du is Hölder continuous, we also know that the coefficients a^{ij} are Hölder continuous as well. So in order to prove Theorem 9, we need to develop some results for these particular types of equations. We will bring in a few lemmas in order to do so. The first of which is known as the Caccioppoli inequality.

Lemma 9. *Let (A^{ij}), $i, j = 1, \ldots, n$ be a matrix such that $|A^{ij}| \leq \Lambda$ for each i, j as well as*

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} A^{ij}\xi_i\xi_j$$

for all $\xi \in \mathbb{R}^n$. Say $u \in W^{1,2}(\Omega)$ is a weak solution to the differential equation

$$\sum_{i,j=1}^{n} D_j(A^{ij}D_iu) = 0 \tag{45}$$

inside Ω. Then for any $x_0 \in \Omega$ and radius r with $r < R < \operatorname{dist}(x_0, \partial\Omega)$, we have that

$$\int_{B(x_0,r)} |Du|^2 \leq \frac{c}{(R-r)^2} \int_{B(x_0,R)\setminus B(x_0,r)} |u-k|^2 \tag{46}$$

for any $k \in \mathbb{R}$.

Proof. We define a cut-off function $\eta \in H_0^{1,2}(B(x_0,R))$ by confining $0 \leq \eta \leq 1$ with the following conditions

$$\eta = 1 \text{ on } B(x_0,R) \Rightarrow D\eta \equiv 0 \text{ on } B(x_0,R)$$

$$|D\eta| \leq \frac{2}{R-r}$$

Then let ϕ be a test function defined by

$$\phi = (u-\mu)\eta^2$$

so that we can get

$$\begin{aligned} 0 &= \int_\Omega \sum_{i,j=1}^{n} A^{ij}D_iuD_j\left((u-\mu)\eta^2\right) \\ &= \int_\Omega \sum_{i,j=1}^{n} A^{ij}D_iuD_ju\eta^2 + \int_\Omega 2\sum_{i,j=1}^{n} A^{ij}D_iu(u-\mu)\eta D_j\eta \end{aligned}$$

From this we can use the fact that we are dealing with elliptic coefficients and the fact that $D\eta = 0$ on the ball $B(x_0,r)$ to deduce from Young's inequality that

$$\begin{aligned} \lambda\int_{B(x_0,R)} |Du|^2\eta^2 &\leq \int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij}D_iuD_ju\eta^2 \\ &\leq \varepsilon\Lambda n\int_{B(x_0,R)} |Du|^2\eta^2 + \frac{\Lambda}{\varepsilon}n\int_{B(x_0,R)\setminus B(x_0,r)} |D\eta|^2|u-\mu|^2 \end{aligned}$$

for any $\varepsilon > 0$. In particular, we can take $\varepsilon = \frac{1}{2}\frac{\lambda}{\Lambda n}$ to get that

$$\int_{B(x_0,R)} |Du|^2\eta^2 \leq \frac{c}{(R-r)^2}\int_{B(x_0,R)\setminus B(x_0,r)} |u-\mu|^2$$

Using the fact that $\int_{B(x_0,r)} |Du|^2 \leq \int_{B(x_0,R)} |Du|^2\eta^2$, the lemma is now proved. □

We now will show the Campanato inequalities.

Lemma 10. *Under the setting of Lemma 9, we also have the following two inequalities.*

$$\int_{B(x_0,r)} |u|^2 \leq c_3\left(\frac{r}{R}\right)^n \int_{B(x_0,R)} |u|^2 \tag{47}$$

$$\int_{B(x_0,r)} \left|u - u_{avg_{B(x_0,R)}}\right|^2 \leq c_4\left(\frac{r}{R}\right)^{n+2} \int_{B(x_0,R)} \left|u - u_{avg_{B(x_0,R)}}\right|^2 \tag{48}$$

Proof. It is of no loss of generality to assume that $r < \frac{R}{2}$. So choose $k > n$; by the Sobolev embedding theorem, we then have that

$$W^{k,2}(B(x_0,R)) \subset C^0(B(x_0,R))$$

So $u \in W^{k,2}(B(x_0,\frac{R}{2}))$, and hence we have

$$\begin{aligned}\int_{B(x_0,r)} |u|^2 &\leq c_5 r^n \sup_{B(x_0,r)} |u|^2 \leq c_6\frac{r^n}{R^{n-2k}}\|u\|_{W^{k,2}(B(x_0,\frac{R}{2}))} \\ &\leq c_3\frac{r^n}{R^n}\int_{B(x_0,R)} |u|^2\end{aligned}$$

We therefore have (47). Since the equation in question has constant coefficients, we also know that Du is a solution, so that when $r < \frac{R}{2}$ we get that

$$\int_{B(x_0,r)} |Du|^2 \leq c_7\frac{r^d}{R^d}\int_{B(x_0,\frac{R}{2})} |Du|^2 \tag{49}$$

and so by the Poincarè inequality, we get that

$$\int_{B(x_0,r)} |u - u_{avg}|^2 \leq c_8 r^2 \int_{B(x_0,r)} |Du|^2 \tag{50}$$

Lemma 9 then implies

$$\int_{B(x_0,\frac{R}{2})} |Du|^2 \leq \frac{c_9}{R^2}\int_{B(x_0,R)} |u-u_{avg}|^2 \tag{51}$$

The inequalities (49), (50), (51) together prove the lemma. □

Using Campanato's inequalities, we can derive the desired regularity result.

Theorem 10. *Let $a^{ij}(x)$ be C^α functions on $\Omega \subset \mathbb{R}^n$ for $i, j = 1, \ldots, n$, satisfying the elliptic bounds*

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j, \qquad |a^{ij}(x)| \leq \Lambda$$

for each $\xi \in \mathbb{R}^n$ and $x \in \Omega$, and $i, j = 1, \ldots, n$, for some $0 < \lambda \leq \Lambda < \infty$. We then have that any weak solution v to

$$\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i v\right) = 0 \tag{52}$$

is a $C^{1,\alpha'}(\Omega)$ function for any $\alpha' \in (0, \alpha)$.

Proof. For arbitrary $x_0 \in \Omega$ we rewrite a^{ij} as

$$a^{ij} = a^{ij}(x_0) + \left(a^{ij}(x) - a^{ij}(x_0)\right)$$

Then if we define $A^{ij} := a^{ij}(x_0)$, equation (**??**) turns into

$$\sum_{i,j=1}^{n} D_j(A^{ij} D_i v) = \sum_{i,j=1}^{n} D_j \left((a^{ij}(x_0) - a^{ij}(x)) D_i v\right) = \sum_{j=1}^{n} D_j(f^j(x))$$

where we define f^j as the sum

$$f^j(x) := \sum_{i=1}^{d} \left((a^{ij}(x_0) - a^{ij}(x)) D_i v\right) \tag{53}$$

We therefore have the following equality for each $\phi \in H_0^{1,2}(\Omega)$:

$$\int_\Omega \sum_{i,j=1}^{n} A^{ij} D_i v D_j \phi = \int_\Omega \sum_{j=1}^{n} f^j D_j \phi \tag{54}$$

From here, we proceed by taking some ball in $B(x_0, R) \subset \Omega$, and letting $w \in H^{1,2}$ be the weak solution inside the ball to

$$\sum_{i,j=1}^{n} D_j(A^{ij} D_i w) = 0 \text{ inside } B(x_0, R); \qquad w \equiv v \text{ on } \partial B(x_0, R) \tag{55}$$

Such a function exists by the Lax-Milgram lemma. Then we know that w is the solution to the differential equation for all $\phi \in H_0^{1,2}$ inside the ball:

$$\int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij} D_i w D_j \phi = 0 \tag{56}$$

Recall that we are trying to find some $z = w - v$ such that

$$
\begin{aligned}
B(\phi, z) :&= \int \sum A^{ij} D_i z D_j \phi = -\int \sum A^{ij} D_i v D_j \phi \\
&=: F(\phi)
\end{aligned}
$$

for all $\phi \in H_0^{1,2}(B(x_0, R))$.

Now, noting that (55) is a linear differential equation with constant coefficients, we know that w is a solution implies that $D_k w$ is as well for each k. Thus we get that

$$
\int_{B(x_0,r)} |Dw|^2 \leq c_1 0 \left(\frac{r}{R}\right)^n \int_{B(x_0,R)} |Dw|^2 \tag{57}
$$

and since w and v are equal on the boundary of the ball $B(x_0, R)$, we can set $\phi = v - w$ to be a test function in (56) to get that

$$
\int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij} D_i w D_j w = \int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij} D_i w D_j v \tag{58}
$$

We then use the Cauchy-Schwarz inequality together with (**??**) and (**??**) to get that

$$
\int_{B(x_0,R)} |Dw|^2 \leq \left(\frac{n\Lambda}{\lambda}\right)^2 \int_{B(x_0,R)} |Dv|^2 \tag{59}
$$

So then (54) and (56) give us that for any $\phi \in H_0^{1,2}(B(x_0, R))$, we have

$$
\int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij} D_i(v - w) D_j \phi = \int_{B(x_0,R)} \sum_{i,j=1}^{n} f^j D_j \phi
$$

Since this holds for any ϕ, we can take $\phi := v - w$ to get that

$$
\begin{aligned}
\int_{B(x_0,R)} |D(v - w)|^2 &\leq \frac{1}{\lambda} \int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij} D_i(v - w) D_j(v - w) \\
&= \frac{1}{\lambda} \int_{B(x_0,R)} \sum_{j=1}^{n} f^j D_j(v - w) \\
&\leq \frac{1}{\lambda} \left[\int_{B(x_0,R)} |D(v - w)|^2\right]^{1/2} \left[\int_{B(x_0,R)} \sum_{j=1}^{n} |f^j|^2\right]^{1/2} \quad \text{(by Cauchy-Schwarz)}
\end{aligned}
$$

We thus have that

$$
\int_{B(x_0,R)} |D(v - w)|^2 \leq \frac{1}{\lambda^2} \int_{B(x_0,R)} \sum_{j=1}^{n} |f^j|^2 \tag{60}
$$

Putting all of the previous inequalities together, we have by (57) and (59) that for any $0 < r \le R$,

$$\int_{B(x_0,r)} |Dv|^2 \le 2\int_{B(x_0,R)} |Dw|^2 + 2\int_{B(x_0,R)} |D(v-w)|^2$$
$$\le c_{11}\left(\frac{r}{R}\right)^d \int_{B(x_0,R)} |Dv|^2 + 2\int_{B(x_0,R)} |D(v-w)|^2$$

Therefore we have that

$$\begin{aligned}\int_{B(x_0,r)} |D(v-w)|^2 &\le \int_{B(x_0,R)} |D(v-w)|^2 && \text{(since } r \le R\text{)}\\ &\le \frac{1}{\lambda^2}\int_{B(x_0,R)} \sum_{j=1}^{n} |f^j|^2 && \text{(by (60))}\\ &\le \frac{1}{\lambda^2}\sup_{i,j}\left|a^{ij}(x_0) - a^{ij}(x)\right|^2 \int_{B(x_0,R)} |Dv|^2 && \text{(by (53))}\\ &\le CR^{2\alpha}\int_{B(x_0,R)} |Dv|^2 && \text{(since } a^{ij} \in C^\alpha\text{)}\end{aligned} \tag{61}$$

Finally, this means that we have the good estimate

$$\int_{B(x_0,R)} |Dv|^2 \le \gamma\left[\left(\frac{r}{R}\right)^n + R^{2\alpha}\right]\int_{B(x_0,R)} |Dv|^2 \tag{62}$$

We need only worry about the $R^{2\alpha}$ term in the above. We can make this term bounded by the following lemma.

Lemma 11. *Let $\sigma(r)$ be a positive increasing function such that for any $0 < r \le R \le R_0$ with $\mu > \nu$ and $\delta \le \delta_0(\gamma, \mu, \nu)$,*

$$\sigma(r) \le \gamma\left(\left(\frac{r}{R}\right)^\mu + \delta\right)\sigma(R) + \kappa R^\nu$$

If δ_0 is small enough, then again for $0 < r \le R \le R_0$ we have that

$$\sigma(r) \le \gamma_1\left(\frac{r}{R}\right)^\nu \sigma(R) + \kappa_1 r^\nu$$

where $\gamma_1 = \gamma_1(\gamma, \mu, \nu)$ and $\kappa_1 = \kappa_1(\gamma, \mu, \nu, \kappa)$.

Proof of Lemma. Let $t \in (0,1)$ and $R < R_0$. By assumption, we thus have that

$$\sigma(tR) \le \gamma t^\mu\left(1 + \delta t^{-\mu}\right)\sigma(R) + \kappa R^\nu$$

So let t be such that $t^\lambda = 2\gamma t^\mu$, with $\nu < \lambda < \mu$, and assume that $\delta_0 t^{-\mu} \le 1$. We then have that

$$\sigma(tR) \le t^\lambda \sigma(R) + \kappa R^\nu$$

We can continue this inequality iteratively to get for any $m \in \mathbb{N}$,

$$\begin{aligned}\sigma(t^{m+1}(R) &\leq t^\lambda \sigma(t^m R) + \kappa t^{m\nu} R^\nu \\ &\leq t^{(m+1)\lambda}\sigma(R) + \kappa t^{m\nu} R^\nu \sum_{j=0}^{m} t^{j(\lambda-\nu)} \\ &\leq \gamma_0 t^{(m+1)\nu}\left[\sigma(R) + \kappa R^\nu\right]\end{aligned}$$

So let $m \in \mathbb{N}$ be large enough such that $t^{m+2}R < r \leq t^{m+1}R$, and then we get the desired inequality:

$$\sigma(r) \leq \sigma\left(t^{m+1}(R)\right) \leq \gamma_1 \left(\frac{r}{R}\right)^\nu \sigma(R) + \kappa_1 r^\nu$$

□

This lemma will allow us to deal with the $R^{2\alpha}$ term in (62), but we will prove one last lemma before doing so.

Lemma 12. *Let $f \in L^2$. Then if we denote f_{av} as the average of f over the ball $B(x_0, R)$, then we have that*

$$\int_{B(x_0,R)} |f - f_{av}|^2 = \inf_{\beta \in \mathbb{R}} \int_{B(x_0,R)} |f - \beta|^2$$

Proof. The function $F(\beta) := \int_\Omega |g-\beta|^2$ is convex and differentiable since $f \in L^2$. Its derivative is given by

$$F'(\beta) = 2\int_\Omega (\beta - f)$$

and so $F'(0) = 0$ when $\beta = \frac{1}{\text{vol}(\Omega)} \int_\Omega f$. Since F is convex, this critical point is a minimizer of the functional. □

Finally we return to the proof of Theorem 10. Let us use Lemma 11 in equation (62) for $0 < r \leq R \leq R_0$ and $R_0^{2\alpha} \leq \delta_0$ to get that for any $\varepsilon > 0$,

$$\int_{B(x_0,R)} |Dv|^2 \leq c_3 \left(\frac{r}{R}\right)^{n-\varepsilon} \int_{B(x_0,R)} |Dv|^2 \tag{63}$$

Repeating this procedure, we get that

$$\int_{B(x_0,R)} |Dw - (Dw)_{avg}|^2 \leq c_4 \left(\frac{r}{R}\right)^{n+2} \int_{B(x_0,R)} |Dw - (Dw)_{avg}|^2 \tag{64}$$

where the average is taken over the ball $B(x_0, R)$. From Lemma 12, we also have that

$$\int_{B(x_0,R)} |Dw - (Dw)_{avg}|^2 \leq \int_{B(x_0,R)} |Dw - (Dv)_{avg}|^2$$

By (58), this means that

$$\int_{B(x_0,R)} |Dw - (Dv)_{av}|^2 \leq \frac{1}{\lambda}\int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij}\left(D_i w - (D_i v)_{av}\right)\left(D_j w - (D_j v)_{av}\right)$$
$$= \frac{1}{\lambda}\int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij}(D_i w - (D_i v)_{av})(D_j v - (D_j v)_{av})$$
$$+ \frac{1}{\lambda}\int_{B(x_0,R)} \sum_{i,j=1}^{n} A^{ij}(D_i v)_{av}(D_j v - D_j w)$$

Since $u - v \in H_0^{1,2}(B(x_0,R))$ and $A^{ij}(D_i v)_{av}$ is constant the last term is zero, and so by Cauchy-Schwarz we get that

$$\int_{B(x_0,R)} |Dw - (Dw)_{av}|^2 \leq \frac{\Lambda^2}{\lambda^2} n^2 \int_{B(x_0,R)} |Dv - (Dv)_{av}|^2 \tag{65}$$

So by Hölder inequality and (61), we get that

$$\int_{B(x_0,r)} |Dv - (Dv)_{av}|^2 \leq 3\int_{B(x_0,r)} |Dw - (Dw)_{av}|^2$$
$$+ 3\int_{B(x_0,r)} |Dv - Dw|^2 + 3\int_{B(x_0,r)} \left[(Dv)_{av} - (Dw)_{av}\right]^2$$
$$\leq 3\int_{B(x_0,r)} |Dw - (Dw)_{av}|^2 + 6\int_{B(x_0,r)} |Dv - Dw|^2$$
$$\leq 3\int_{B(x_0,r)} |Dw - (Dw)_{av}|^2 + c_5 R^{2\alpha}\int_{B(x_0,r)} |Dv|^2 \tag{66}$$

where all the averages here are taken over the ball $B(x_0, r)$. Putting this all together, (63), (64), (65), and (66) give us that

$$\int_{B(x_0,r)} |Dv - (Dv)_{av}|^2 \leq c_6\left(\frac{r}{R}\right)^{n+2}\int_{B(x_0,r)} |Dv - (Dv)_{av}|^2 + c_7 R^{2\alpha}\int_{B(x_0,R)} |Dv|^2$$
$$\leq c_6\left(\frac{r}{R}\right)^{n+2}\int_{B(x_0,R)} |Dv - (Dv)_{av}|^2 + c_8 R^{n-\varepsilon+2\alpha}$$

We then use Lemma 11 to finally get that

$$\int_{B(x_0,r)} |Dv - (Dv)_{av}|^2 \leq c_9\left(\frac{r}{R}\right)^{n-\varepsilon+2\alpha}\int_{B(x_0,R)} |Dv - (Dv)_{av}|^2 + c' r^{n-\varepsilon+2\alpha} \tag{67}$$

Campanato's theorem thus proves the theorem. □

We can now finally complete the proof of Theorem 9.

Proof. Let $v = Du$ and use Theorem 10 to deduce that $v \in C^{1,\alpha'}$ for any $\alpha' < \alpha$ non-zero. Therefore we have that $u \in C^{2,\alpha'}$ for any $0 < \alpha' < \alpha$. We can then differentiate with respect to x^k and use that each of the derivatives

$$D_i D_k u, \qquad j, k = 1, \dots, m$$

satisfy the same equation, so that we can apply the theorem again to deduce that $D^2 u \in C^{1,\alpha''}$, and so that $u \in C^{3,\alpha''}$. Evidently we can iterate this process to deduce that $u \in C^{k,\alpha_k}$ for each natural number k, with $\alpha_k \in (0, 1)$ for all k. This means $u \in C^\infty$. □

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