ELEMENTARY FUNCTIONAL ANALYSIS NOTES 1

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Abstract. We collect some basic results of functional analysis with applications to elliptic partial differential equations in mind.

Contents

1. The Banach fixed point theorem

A *distance function*, or a *metric*, on a set M is a function $\rho : M\times M \to \mathbb{R}$ that is symmetric: $\rho(u,v) = \rho(v,u)$, nonnegative: $\rho(u,v) \geq 0$, nondegenerate: $\rho(u,v) = 0 \Leftrightarrow u = v$, and satisfies the triangle inequality: $\rho(u,v) \leq \rho(u,w)+\rho(w,v)$. Then a *metric space* is a set with a metric. If a sequence $\{u_n\}$ in M satisfies $\rho(u_n,u) \to 0$ as $n \to \infty$ for some $u \in M$, we say that the sequence *converges* to u, and write $u_n \to u$ in M. It is obvious that convergent sequences are *Cauchy*, meaning that $\rho(u_n,u_m) \to 0$ as $n,m \to \infty$. In general, however, Cauchy sequences do not have to converge in the space, as can be seen from, e.g, the example $M = \mathbb{Q}$ and $\rho(x,y) = |x-y|$. If the metric space M is such that every Cauchy sequence converges to an element of M, we call it a *complete metric space*. A mapping $\phi : M \to W$ between two metric spaces is called *continuous* if $u_n \to u$ in M implies $\phi(u_n) \to \phi(u)$ in W. With ϱ denoting the metric of W, if

$$\varrho(\phi(u),\phi(v)) \leq k\rho(u,v), \qquad u,v \in M, \tag{1}$$

with some constant $k \in \mathbb{R}$, then we say that ϕ is *Lipschitz continuous*. In this setting, ϕ is called a *nonexpansive* mapping if $k \leq 1$, and a *contraction* if $k < 1$.

Theorem 1. *Let M be a non-empty, complete metric space, and let $\phi : M \to M$ be a contraction. Then ϕ has a unique fixed point, i.e., there is a unique $u \in M$ such that $\phi(u) = u$.*

Proof. Uniqueness follows easily from nondegeneracy of the metric. For existence, starting with some $u_0 \in M$, define the sequence $\{u_n\}$ by $u_n = \phi(u_{n-1})$ for $n = 1,2,\ldots$. Then this sequence is Cauchy, because

$$\rho(u_n,u_{n+1}) = \rho(\phi(u_{n-1}),\phi(u_n)) \leq k\rho(u_{n-1},u_n) \leq \ldots \leq k^n\rho(u_0,u_1), \tag{2}$$

and so

$$\rho(u_n,u_m) \leq \rho(u_n,u_{n+1}) + \ldots + \rho(u_{m-1},u_m) \leq (k^n + \ldots + k^{m-1})\rho(u_0,u_1) \leq \frac{k^n\rho(u_0,u_1)}{1-k}, \tag{3}$$

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for $n < m$. Since M is complete, there is $u \in M$ such that $u_n \to u$, which is a good candidate for the fixed point we are looking for. Indeed, we have

$$\rho(u, \phi(u)) \leq \rho(u, u_n) + \rho(\phi(u_{n-1}), \phi(u)) \leq \rho(u, u_n) + k\rho(u_{n-1}, u) \to 0, \tag{4}$$

as $n \to \infty$, showing that $u = \phi(u)$. □

2. A convex minimization problem

Let X be a vector space over $\mathbb{R}$. Then a *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ satisfying the positive homogeneity: $\|\alpha u\| = |\alpha|\|u\|$, the triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$, and a nondegeneracy condition: $\|u\| = 0 \Rightarrow u = 0$. Equipping X with a norm makes X a *normed space*. Note that a norm naturally defines the corresponding metric $\rho(u, v) = \|u - v\|$. Then a complete normed space is called a *Banach space*.

The subset $C \subseteq M$ of the metric space M is called *closed* if $u_n \to u$ in M with $\{u_n\} \subset C$ implies $u \in C$. The subset $C \subseteq X$ of the linear space X is called *convex* if $u, v \in C$ implies $\lambda u + (1 - \lambda)v \in C$ for $\lambda \in (0, 1)$.

Theorem 2. *Let X be a Banach space, and let $C \subseteq X$ be a closed convex set. Let $E : C \to \mathbb{R}$ be a continuous mapping satisfying*

$$\mu := \inf_{u \in C} E(u) > -\infty, \qquad u, v \in C, \tag{5}$$

and

$$E(\tfrac{1}{2}u + \tfrac{1}{2}v) + f(\|u - v\|) \leq \tfrac{1}{2}E(u) + \tfrac{1}{2}E(v), \qquad u, v \in C, \tag{6}$$

for some strictly increasing continuous function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$. Then there is a unique $u \in C$ such that $E(u) = \mu$.

Proof. For $u, v \in C$ we have

$$\mu \leq E(\tfrac{1}{2}u + \tfrac{1}{2}v) \leq \tfrac{1}{2}E(u) + \tfrac{1}{2}E(v) - f(\|u - v\|), \tag{7}$$

which implies

$$f(\|u - v\|) \leq \tfrac{1}{2}E(u) + \tfrac{1}{2}E(v) - \mu. \tag{8}$$

It is immediate that E has at most one minimizer in C.

Let $\{u_n\} \subset C$ be a minimizing sequence of E in C, meaning that

$$E(u_n) \to \mu. \tag{9}$$

From (8) it follows that $\{u_n\}$ is Cauchy, hence there is $u \in X$ such that $u_n \to u$. Since C is closed we have $u \in C$, and $E(u) = \mu$ by continuity of E. □

3. Hilbert spaces

An *inner product* on a vector space H over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ that is symmetric: $\langle u, v \rangle = \langle v, u \rangle$, positive definite: $\langle u, u \rangle \geq 0$, nondegenerate: $\langle u, u \rangle = 0 \Rightarrow u = 0$, and linear in both arguments. Such an H with an inner product is called an *inner product space*, or sometimes a *pre-Hilbert space*. The inner product generates the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, thus making H a normed (and therefore metric) space. A complete inner product space is called a *Hilbert space*.

An important property of inner products is the *Cauchy-Bunyakovsky-Schwarz inequality*

$$\langle u, v \rangle \leq \|u\|\|v\|, \qquad u, v \in H, \tag{10}$$

where H is a pre-Hilbert space. This can be proven by observing that

$$\|u\|^2 + 2\langle u, v \rangle t + \|v\|^2 t^2 = \langle u + tv, u + tv \rangle \geq 0, \tag{11}$$

so the discriminant $D = 4\langle u, v \rangle^2 - 4\|u\|^2\|v\|^2$ must be nonpositive.

Exercise 1. Prove the parallelogram law

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2, \tag{12}$$

as well as the polarization identity

$$4\langle u, v\rangle = \|u+v\|^2 - \|u-v\|^2, \tag{13}$$

where u and v are elements of a Hilbert space.

One can show that a linear operator $A : X \to Y$ between two normed spaces is continuous iff it is *bounded*, i.e., iff

$$\|A\| = \|A\|_{X\to Y} := \sup_{u\in X\setminus\{0\}} \frac{\|Au\|_Y}{\|u\|_X} < \infty, \tag{14}$$

where $\|A\|$ is called the *operator norm* of A. We denote by $B(X,Y)$ the space of all bounded linear operators between X and Y. A *linear functional* on X is simply a linear operator mapping X to $\mathbb{R}$. Then the *dual* of X, denoted by $X' \equiv B(X,\mathbb{R})$, is the space of all bounded linear functionals on X.

Exercise 2. Let X and Y be normed spaces. Show that

(a) a linear operator between X and Y is continuous iff it is bounded.
(b) the operator norm indeed defines a norm.
(c) $B(X,Y)$ is a normed linear space.
(d) if Y is Banach, so is $B(X,Y)$. In particular, X' is always Banach.

Theorem 3. *Let H be a Hilbert space, and let $C \subseteq H$ be a closed convex set. Let $\ell \in H'$, and let $A : H \to H'$ be a bounded linear operator satisfying*

$$(Au - Av)(u-v) \geq \alpha\|u-v\|^2, \qquad u, v \in C, \tag{15}$$

for some constant $\alpha > 0$. Then the function

$$E(u) = (Au + \ell)(u), \qquad u \in C, \tag{16}$$

has a unique minimum in C.

Moreover, $u \in C$ is the minimizer of E if and only if

$$(Au)(h) + (Ah)(u) + \ell(h) \geq 0, \qquad \textit{for all} \quad h \in H \quad \textit{such that} \quad u + h \in C. \tag{17}$$

Proof. Let us show that E satisfies the conditions of Theorem 2. For $u, v \in C$ we have

$$E(u) - E(v) = (Au - Av)(u-v) + (Au - Av)v + (Av)(u-v) + \ell(u-v), \tag{18}$$

so one can bound $E(u)$ from below as

$$E(u) \geq E(v) + \alpha\|u-v\|^2 - (2\|A\|\|v\| + \|\ell\|)\,\|u-v\| \geq E(v) - (2\|A\|\|v\| + \|\ell\|)^2/(2\alpha). \tag{19}$$

We also have

$$E(\tfrac{1}{2}u + \tfrac{1}{2}v) = (\tfrac{1}{2}Au + \tfrac{1}{2}Av)(\tfrac{1}{2}u + \tfrac{1}{2}v) = \tfrac{1}{2}E(u) + \tfrac{1}{2}E(v) - \tfrac{1}{4}(Au - Av)(u-v), \tag{20}$$

implying that

$$E(\tfrac{1}{2}u + \tfrac{1}{2}v) + \tfrac{\alpha}{4}\|u-v\|^2 \leq \tfrac{1}{2}E(u) + \tfrac{1}{2}E(v). \tag{21}$$

The first part of the theorem is established.

For the second part, we write (18) as

$$E(u+h) = E(u) + (Ah)(u) + (Au)(h) + \ell(h) + (Ah)(h), \tag{22}$$

for $u \in C$ and $u + h \in C$. Since the last term is of second order in h, the minimality of $E(u)$ implies the property (17). The converse direction follows upon noting the positivity of the last term. $\square$

Given a subset $C \subseteq X$ of the linear space X, we call $TC = \{\alpha(u-v) : u, v \in C, \alpha \in \mathbb{R}\}$ the *tangent space* of C. It is obvious that TC is a linear subspace of X. The subset C is called an *affine subspace* of X if $u, v \in C$ implies $\lambda u + (1-\lambda)v \in C$ for $\lambda \in \mathbb{R}$.

Corollary 4 (Symmetric elliptic equation)**.** *In addition to the conditions of the preceding theorem, assume that C is an affine subspace of H, and that A satisfies*

$$(Au)(h) = (Ah)(u), \qquad \forall u \in C, \quad \forall h \in TC. \tag{23}$$

Then there exists a unique $u \in C$ satisfying

$$(2Au)(h) + \ell(h) = 0, \qquad \text{for all} \quad h \in TC. \tag{24}$$

Corollary 5 (Ritz-Galerkin approximation)**.** *In the setting of Theorem 3, suppose that the minimizer $u \in C$ of E is in the interior of C. Let $C' \subset C$ be a closed convex set, and let $u + h \in C'$ be the minimizer of E in C'. Then h minimizes the quantity $(Ah)(h)$ over the set $\{h : u + h \in C'\}$.*

Every $u \in H$ defines a bounded linear functional via $v \mapsto \langle u, v\rangle$. Let us denote this correspondence by $J : H \to H'$, i.e., $J(u)(v) = \langle u, v\rangle$. The following corollary states that this mapping J is invertible.

Corollary 6 (Riesz representation theorem)**.** *Let H be a Hilbert space and let $f \in H'$. Then there exists a unique $u \in H$ such that*

$$\langle u, v\rangle = f(v), \qquad v \in H. \tag{25}$$

Moreover, we have $\|u\| = \|f\|$.

In view of this result, we write the action $f(u)$ of $f \in H'$ on $u \in H$ also as $\langle f, u\rangle$ or $\langle u, f\rangle$.

Corollary 7 (Orthogonal projection)**.** *Let $C \subset H$ be an affine subspace of the Hilbert space H, and let $x \in H$ be given. Then there exists a unique $y \in C$ satisfying*

$$\|x - y\| = \inf_{z \in C} \|x - z\|, \tag{26}$$

and moreover this y is characterized by the condition

$$\langle x - y, z\rangle = 0, \qquad \text{for all} \quad z \in TC. \tag{27}$$

Theorem 8. *Let H be a Hilbert space, and let $f : H \to H$ be a Lipschitz continuous map satisfying*

$$\langle f(u) - f(v), u - v\rangle \geq \alpha\|u - v\|^2, \qquad u, v \in H, \tag{28}$$

for some constant $\alpha > 0$. Then there is a unique $u \in H$ such that $f(u) = 0$.

Proof. Define the mapping $\phi : H \to H'$ by $\phi(u) = u - \omega f(u)$, where $\omega > 0$ is a parameter to be chosen later. Then $\phi(u) = u$ if and only if $f(u) = 0$, so the proof is established upon showing that ϕ is a contraction for some choice of ω. By using the Lipschitz continuity and the strong montonicty (28), we infer

$$\begin{aligned} \|\phi(u) - \phi(v)\|^2 &= \|u - v\|^2 + \omega^2\|f(u) - f(v)\|^2 - 2\omega\langle f(u) - f(v), u - v\rangle \\ &\leq (1 + \beta^2\omega^2 - 2\alpha\omega)\|u - v\|^2, \end{aligned} \tag{29}$$

where $\beta > 0$ is the Lipschitz constant of f. Now we see that, e.g., the choice $\omega = \alpha/\beta^2$ ensures that ϕ is a contraction. □

Corollary 9 (Lax-Milgram lemma)**.** *Let H be a Hilbert space, and let $\ell \in H'$. Let $A : H \to H'$ be a bounded linear operator satisfying*

$$\langle Au, u\rangle \geq \alpha\|u\|^2, \qquad u \in H, \tag{30}$$

for some constant $\alpha > 0$. Then there is a unique $u \in H$ such that $Au = \ell$.

4. Baire's theorem and its consequences

In a metric space X with metric ρ, we define the *ball* centered at $x \in X$, of radius r, to be the set $B_r(x) \equiv B(x,r) = \{y \in X : \rho(x,y) < r\}$. A subset $S \subseteq X$ is called *open* if for any $x \in S$, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. Then S is closed iff its complement $X \setminus S$ is open. Indeed, S is *not closed* means that there is a sequence $\{x_n\} \subset S$ such that $x_n \to x \in X \setminus S$. On the other hand, $X \setminus S$ is *not open* means that there is some $x \in X \setminus S$ and a sequence $\{x_n\} \subset S$ such that $x_n \to x$. The following lemma clarifies to what extent continuity of a function is determined by the metrics we put on the domain and target spaces.

Lemma 10. *Let $\phi : X \to Y$ be a map between two metric spaces. Then the followings are equivalent.*

a) ϕ is continuous.
b) Whenever $U \subset Y$ is open, its preimage $\phi^{-1}(U) = \{x \in X : \phi(x) \in U\}$ is open.
c) The preimage of any closed $U \subset Y$ is closed.

Proof. The parts b) and c) are easily seen to be equivalent, since $\phi^{-1}(U) \cup \phi^{-1}(Y \setminus U) = X$ is a disjoint union. Now let ϕ be continuous, and let $U \subset X$ be closed. Suppose that $\{x_n\} \subset \phi^{-1}(U)$ is a sequence with $x_n \to x \in X$. Then from continuity we have $\phi(x_n) \to \phi(x)$, and from closedness of U we infer $\phi(x) \in U$. This establishes that a) implies c).

Suppose that b) holds. Then for any $\varepsilon > 0$ and $y = \phi(x)$ with $x \in X$, the preimage of $B_\varepsilon(y)$ contains a ball $B_\delta(x)$ with $\delta = \delta(\varepsilon, x) > 0$. In other words, the δ-closeness in X implies the ε-closeness in Y, which is continuity. □

It is immediate from the definition that the intersection of any collection of closed sets is again closed. The *closure* $\overline{S}$ of a subset $S \subset X$ is the intersection of all closed sets $C \subseteq X$ such that $C \supseteq S$.

Exercise 3. Show that $\overline{S} = \{x \in X : \{x_n\} \subset S \text{ and } x_n \to x\}$.

Theorem 11 (Baire)**.** *Let X be a complete metric space, and let $\{C_n\}$ be a countable collection of closed subsets of X such that $\bigcup_n C_n = X$. Then at least one of C_n contains an open ball, i.e., there exist n, $x \in X$, and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset C_n$.*

Proof. Suppose that C_n does not contain any open ball, for any n. This means that any open ball B in X contains a point from $X \setminus C_n$, and so $B \cap (X \setminus C_n)$ contains a nontrivial closed ball, because $X \setminus C_n$ is open. Applying this with B equal to a ball of radius 1, we obtain $x_1 \in X$ and $r_1 \in (0,1)$ such that $\overline{B(x_1,r_1)} \subset X \setminus C_1$. Similarly, there are $x_2 \in X$ and $r_2 \in (0, \frac{1}{2})$ such that $\overline{B(x_2,r_2)} \subset B(x_1,r_1) \cap (X \setminus C_2)$, and so on, we get a sequence of balls $B(x_n,r_n)$ such that $r_n \in (0, \frac{1}{n})$ and $\overline{B(x_n,r_n)} \subset B(x_{n-1},r_{n-1}) \cap (X \setminus C_n)$. In particular, we have $x_n \in \overline{B(x_k,r_k)}$ for $n > k$, hence $\{x_n\}$ is Cauchy, and by completeness, there is $x \in X$ such that $x_n \to x$ in X. By closedness, we have $x \in \overline{B(x_n,r_n)}$, and since $\overline{B(x_n,r_n)} \subset X \setminus C_n$, we have shown that there is $x \in X$ such that $x \notin C_n$ for all n. □

This proof can be slightly modified to get the following forms of the Baire theorem.

- A complete metric space cannot be written as a countable union of nowhere dense sets.
- The intersection of a countable collection of open dense subsets of a complete metric space is again dense.

Exercise 4. Prove the above statements.

Theorem 12 (Uniform boundedness)**.** *Let X be a complete metric space, and let F be a collection of continuous functions $f : X \to [0,\infty)$ such that*

$$\sup_{f \in F} f(x) < \infty, \tag{31}$$

for each $x \in X$. Then there is a nonempty open set $B \subset X$ such that

$$\sup_{x\in B}\sup_{f\in F} f(x) < \infty. \tag{32}$$

In other words, pointwise boundedness of continuous functions on a complete metric space implies uniform boundedness on a nonempty open set.

Proof. The sets

$$C_n = \bigcap_{f\in F}\{x \in X : f(x) \leq n\}, \tag{33}$$

are closed, and $\bigcup_n C_n = X$, so by Baire's theorem at least one of C_n contains an open ball. □

Theorem 13 (Banach-Steinhaus)**.** *Let X and Y be a Banach and normed spaces, respectively, and let $\mathfrak{A}$ be a collection of bounded linear operators $A : X \to Y$ such that*

$$\sup_{A\in\mathfrak{A}} \|Ax\| < \infty, \tag{34}$$

for each $x \in X$. Then we have

$$\sup_{A\in\mathfrak{A}} \|A\| < \infty. \tag{35}$$

In other words, pointwise boundedness of linear operators on a Banach space implies uniform boundedness.

Proof. For each $A \in \mathfrak{A}$, define the function $f_A : X \to [0, \infty)$ by $f_A(x) = \|Ax\|$. We apply Theorem 12 to the collection $F = \{f_A : A \in \mathfrak{A}\}$ to conclude that there is a ball $B_\varepsilon(z)$ with $\varepsilon > 0$ (and $z \in X$) such that

$$\alpha := \sup_{x\in B_\varepsilon(z)}\left(\sup_{A\in\mathfrak{A}}\|Ax\|\right) < \infty. \tag{36}$$

Now if $x \in B_\varepsilon(0)$, then we can bound $\|Ax\|$ by using the triangle inequality as

$$\|Ax\| = \|A(z + x) - Az\| \leq \|A(z + x)\| + \|Az\| \leq 2\alpha. \tag{37}$$

Finally, for arbitrary $x \in X$, a simple scaling argument gives

$$\|Ax\| = \frac{2\|x\|}{\varepsilon}\left\|A\left(\frac{\varepsilon x}{2\|x\|}\right)\right\| \leq \frac{4\alpha}{\varepsilon}\|x\|, \tag{38}$$

meaning that $\|A\| \leq 4\alpha/\varepsilon$ independent of $A \in \mathfrak{A}$. □

A mapping is called *open* if it sends open sets to open sets. Note that in view of Lemma 10, openness is "continuity in the wrong direction", in the sense that if exists, the inverse of a continuous mapping is open. To get some rough feeling of what open mappings do, if a set is "expanding in all possible directions", then the image of this process under an open mapping will look similar, "expanding in all possible directions". As an example of this behavior, if a linear operator between normed spaces is open, it must be surjective. Indeed, under an open linear mapping $T : X \to Y$, an open neighborhood of $0 \in X$ goes to an open neighborhood U of $0 \in Y$, and for any $y \in Y$ there is $\alpha \neq 0$ such that $\alpha y \in U$. The following theorem says that the converse is also true when the two spaces are complete.

Theorem 14 (Open mapping)**.** *Let $A : X \to Y$ be a bounded linear operator between two Banach spaces. Then A is surjective iff it is open.*

Proof. Suppose that A is surjectve. This implies $Y = \bigcup_{n\in\mathbb{N}} \overline{A(B_n(0))}$, and hence by Baire, there is a nonempty ball $B_\delta(y) \subset Y$ and $n \geq 1$ such that $B_\delta(y) \subset \overline{A(B_n(0))}$. By choosing $x \in X$ such that $y = Ax$, and $\varepsilon > 0$ so large that $B_r(x) \supset B_\varepsilon(0)$, we can guarantee $B_\delta(y) \subset \overline{A(B_\varepsilon(x))}$. By linearity, with $\alpha = \delta/\varepsilon$ we have $B_{\alpha r}(Ax) \subset \overline{A(B_r(x))}$ for all $x \in X$ and all

$r > 0$. If the inclusion did not have the closure in the right hand side, this statement is exactly what we wanted.

Now we shall remove the closure. Let $z \in B_{\alpha r}(Ax)$, and fix some $\varepsilon \in (0, 1)$. Then there is $x_0 \in B_r(x)$ such that $\|z - Ax_0\| < \alpha\varepsilon$, which implies that $z \in B_{\alpha\varepsilon}(Ax_0) \subset \overline{A(B_\varepsilon(x_0))}$. This means that there is $x_1 \in B_\varepsilon(x_0)$ such that $\|z - Ax_1\| < \alpha\varepsilon^2$. By iterating, we get a sequence $\{x_n\}$ in X satisfying $\|x_n - x_{n-1}\| < \varepsilon^n$ and $\|z - Ax_n\| < \alpha\varepsilon^n$ for $n \in \mathbb{N}$. From the latter property we have $z = Ax_*$ with $x_* = \lim x_n$, and from the former we infer $\|x_*\| < r_* := r + \varepsilon/(1-\varepsilon)$, meaning that $B_{\alpha r}(Ax) \subset A(B_{r_*}(x))$. If we squeze this argument we can get the result with $r_* = r$, but what we have is already sufficient for establishing the theorem. □

In view of Lemma 10, the open mapping theorem implies that if the inverse A^{-1} exists, then it must be continuous. Since continuity is equivalent to boundedness for linear operators on normed spaces, we obtain the following.

Corollary 15 (Bounded inverse). *Let $A : X \to Y$ be an invertible bounded linear operator between two Banach spaces. Then the inverse $A^{-1} : Y \to X$ is bounded.*

A map $T : X \to Y$ can be identified with its *graph*

$$\operatorname{graph}(T) = \{(x, Tx) : x \in X\} \subset X \times Y. \tag{39}$$

Suppose that (X, ρ) and (Y, σ) are complete metric spaces, and equip $X \times Y$ with the metric $\rho + \sigma$. If T is continuous, obviously $\operatorname{graph}(T)$ is closed, since $(x_n, Tx_n) \to (x, y)$ in $X \times Y$ implies $y = Tx$. In the linear world, the converse is also true.

Theorem 16 (Closed graph). *Let $A : X \to Y$ be a linear operator between two Banach spaces. Then A is bounded iff its graph is closed.*

Proof. Suppose that $\operatorname{graph}(A)$ is closed in $X \times Y$, i.e., that it is a Banach space. Define the two operators $\pi_1 : \operatorname{graph}(A) \to X$ and $\pi_2 : \operatorname{graph}(A) \to Y$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, respectively. It is clear that the both operators are bounded linear, and that π_1 is invertible. Since we have $A = \pi_2\pi_1^{-1}$, the claim follows from an application of Corollary 15 to π_1. □