McGill University

Department of Mathematics and Statistics

Part A Examination

Applied Mathematics β

Date: Friday, September 15, 1995

Time: 13:00 - 17:00 hrs

Answer any six of the nine questions.

Partial Differential Equations

1. An elastic membrane subject to uniform gas pressure satisfies the equation

$$\psi_{tt} + P = a^2\nabla^2\psi,$$

where a^2 and P are constants.

Find the displacement of a circular membrane of radius b, if it is clamped on the circumference, i.e. $\psi(b,t) = 0$.

The initial displacement is $f(r)$ and the initial velocity is $g(r)$.

2. Obtain the Green's function and then solve the general Neumann problem for a semi-infinite bar

$$\alpha^2\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial t} = h(x,t); \quad 0 < x < \infty, \quad t > 0$$

with (i) $\psi(x,0) = f(x)$ and (ii) $\psi_x(0,t) = g(t)$. You may assume that α^2 is constant.

3. Solve

$$\psi_{xx} + \psi_{yy} = 0; \quad x > 0, \quad y > 0$$

$$\psi(0,y) = T_0$$

$$\psi_y(x,0) = h[\psi(x,0) - T_1]$$

$$\lim_{y\to\infty} \psi(x,y) = T_0,$$

and interpret physically.

McGill University
Department of Mathematics and Statistics
Part A Examination
Applied Mathematics β

Date: Friday, May 10, 1996
Time: 13:00 - 17:00 hrs

The best SIX answers will account for your final grade

PARTIAL DIFFERENTIAL EQUATIONS

1. Use the method of characteristics to solve the PDE

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xyu, \; x > 0, \quad \text{with} \quad u = x^2 \text{ on } y = 0, \; 1 < x < 4.$$

Indicate with a sketch the region of the xy-plane over which the solution of this problem is determined uniquely.

2. The nonlinear equations of shallow water theory are

$$h_t + uh_x + hu_x = 0,$$
$$u_t + uu_x + gh_x = 0,$$

where h is the height of the water above a horizontal bottom, u is its velocity and g is the gravitational constant. Given that this system is totally hyperbolic, determine the equation of the characteristics and the ODEs that apply along them. Finally, show that the latter can be integrated to obtain Riemann invariants, i.e., quantities that are constant along each characteristic.
One of several ways to proceed is the following. Form a linear combination of the above ODEs, multiplying the first by λ_1 and the second by λ_2, say. Then, choose λ_1 and λ_2 such that in the combined equation both u and h are differentiated in the same direction. Show that two such directions exist, namely, $dx/dt = u \pm (gh)^{1/2}$ and, hence, that the wave propagation speeds satisfy $c^2 = gh$.
Use any solution procedure with which you are comfortable, i.e., you do not have to use the method outlined above.

3. Use a Fourier transform to obtain the Green's function for Laplace's equation in the upper half plane $y \geq 0, \; -\infty < x < \infty$.

be a (global) optimal solution of the program. Is it true that there exists u^* in R^m with only non-negative components such that (x^*, u^*) is a saddle point of the Lagrangian

$$L(x, u) = f(x) + \sum_{i=1}^{m} u_i f^i(x) \ ?$$

Give a proof or a counter-example.

(iii) Does your answer to (ii) change if f is convex?

Partial Diffferential Equations

4. Use the method of characteristics to solve the PDE

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = -xu, \quad \text{with} \quad u(0, y) = y, \quad 0 < y < 5.$$

Indicate with a sketch the region of the xy-plane over which the solution of this problem is determined uniquely.

5. For a linear PDE of the form

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y)$$

the characteristics (if they exist) are solutions of the ODE

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}\ .$$

Classify the following PDEs, transform them to canonical form and then obtain their general solution making use of the above information:

(i) $u_{xy} + yu_{yy} + u_y = 0$; and

(ii) $u_{xx} - 2u_{xy} + u_{yy} = 1$.

6. Find the Green's function and then solve the Neumann problem

$$\alpha^2\psi_{xx} - \psi_t = h(x, t); \quad 0 < x < \infty, \quad t > 0\ .$$

(i) $\psi(x, 0) = f(x)$ and (ii) $[\psi_x(x, t)]_{x=0} = q(t)$. ($\alpha^2$ is a positive constant).

Useful information: $\displaystyle\int_0^\infty e^{-a^2x^2} \cos bx dx = \frac{\sqrt{\pi}e^{-b^2/4a^2}}{2a}.$

7. Given the I.V.P. problem

$$\begin{cases} y' &= f(x,y) \\ y(0) &= y_0 \end{cases}$$

consider the linear multi-step method:

$$y_{j+4} - y_j + \alpha(y_{j+3} - y_{j+1}) = h[\beta(f_{j+3} - f_{j+1}) + \gamma f_{j+2}].$$

(a) Determine (α, β, γ) such that the method has order 3.

(b) Determine whether the resultant method is stable, convergent or not.

8. Given the I.V.P. problem

$$\begin{cases} y' &= f(x,y) \\ y(0) &= y_0 \end{cases}$$

Determine the absolute stable region $(\wedge)$ for the method

$$y_{j+1} - y_j = \frac{h}{2}(3f_j - f_{j-1}).$$

9. Given the elliptic equation

$$U_{xx} + U_{yy} + U = 0$$

in the region as shown in Figure 1.

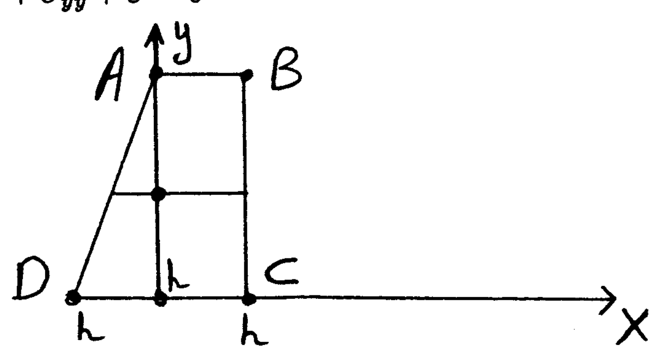
the boundary conditions are

$U(x,y) = y \quad (x,y) \in (AD)$

$U(x,y) = y \quad (x,y) \in (BC)$

$U(x,y) = 1 \quad (x,y) \in (AB)$

$\dfrac{\partial U}{\partial y} = 0 \quad (x,y) \in (DC)$



(a) Set $h = \frac{1}{2}$. Write down the finite difference scheme for the above defined B.V.P. in a matrix form.

(b) Using the Jacobi iteration method twice, find the approximation numerical solution.

(c) Discuss the convergence of the iteration.

2. Consider the program

(P)

$$\begin{aligned} &\text{Min} x_1 \\ &\text{s.t.} \\ &\quad x_1 \quad x_2 = 0 \\ &\quad x_1 - x_2 = 0. \end{aligned}$$

(a) Construct the corresponding Lagrangian function (with the leading co-efficient $\lambda_0 = 1$). Show that the Method of Lagrange does not identify the origin $x^* = (0,0)$ as a candidate for a local optimum of (P). Why not?

(b) State a general second-order optimality condition that can be used in this situation to verify that a feasible point is an isolated local optimum. Using this condition verify that the origin is an isolated local optimum. Using this condition verify that the origin is an isolated local optimum for the program (P).

3. Solve the problem

$$\min_{\text{s.t.}} \int_0^1 (\dot{x}^2 + 2x)dr$$
$$x(0) = x(1) = 0$$

using the Euler-Lagrange equation.

PARTIAL DIFFERENTIAL EQUATIONS

4. Solve Laplace's equation for the steady-state temperature distribtion in a circular cylinder of radius a and height H subject to the following boundary conditions:

$$u_z(r,0) = u_z(r,H) = 0 \quad \text{and} \quad u(a,z) = z.$$

Laplace's equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

5. Employ a Green's function to convert the following boundary value problem into an integral equation:

$$x^2 y'' + xy' + (\lambda x^2 - 1)y = 0,$$
$$y(0) = y(1) = 0.$$

(Hint: Dispose, temporarily, of the most troublesome term by treating it as a nonhomogeneous term.)

6. Consider the propagation of a nonlinear wave as described by

$$u_t + uu_x + \gamma u = 0, \quad u(x,0) = f(x),$$

where γ is a positive constant. Make a rough sketch of the characteristics on an xt diagram and show that wave breaking will occur eventually only if $f'(x) < -\gamma$ somewhere.

McGILL UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
APPLIED β

Date: Friday, May 15, 1998

Time: 9:00 A.M. - 1:00 P.M.

Room: BURN 1120

PARTIAL DIFFERENTIAL EQUATIONS

1. (a) Let D be a finite domain with piecewise smooth bounding surface S. Explain how you would solve the problem

$$\nabla^2 u = f(\mathbf{x}); \quad \mathbf{x} \in D.$$

subject to $u = h(\mathbf{x})$, $\mathbf{x} \in S$. Further, if you know the eigenfunctionsassociated with the domain, the Laplacian and the homogeneous Dirichlet boundary condition, what more can you do?

   (b) Let D be the square $0 < x < 1,\ 0 < y < 1$. Obtain the solution of

$$u_t = \nabla^2 u + 1, \quad \mathbf{x} \in D$$

subject to $u(\mathbf{x}, 0) = 0; \quad u(\mathbf{x}, t) = 0;\ \mathbf{x} \in S$. Give the dominant term in the solution for $t \gg 1$.

2. Let D be the semi-infinite strip $0 < x < \infty,\ 0 < y < 1$. Solve the problem

$$\nabla^2 u = f(x) \qquad \mathbf{x} \in D$$

subject to

$$\begin{cases} u = & 0 & \text{at } y = 1, \\ u = & 1 & \text{at } x = 0, \\ u = & f(x) & \text{at } y = 0. \end{cases}$$

where

$$f(x) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

3. (a) Let $f : R^n \to R$ be a twice continuously differentiable function. If $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite, then x^* is an isolated local minimum of f. Prove this statement. Is the reverse claim true? Give a proof or a counter-example.

   (b) Using an optimality condition check whether $x^* = (1, 0, 0.5)^T$ is an optimal solution of the program

$$\begin{aligned} \text{Max} \quad & x_1/x_3 \\ \text{s.t.} \quad & \\ & x_1 + x_2 \leq 1 \\ x_1 + x_2 x_3 \geq 1, \; & x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0.5. \end{aligned}$$

PARTIAL DIFFERENTIAL EQUATIONS

1. Show that the system

$$\begin{aligned} u_t + u u_x &= 0 \\ v_t + u v_x + v u_x &= 0 \end{aligned}$$

is not totally hyperbolic. Be an optimist and show that, nonetheless, the system can still be solved using the method of characteristics. The reason is that the first equation is independent of the second. Take as initial conditions $u(x, 0) = f(x)$ and $v(x, 0) = g(x)$.

2. Suppose that we wish to solve the Poisson equation

$$\nabla^2 u = h(x, y).$$

   (a) Green's theorem in two dimensions takes the form

$$\int\int_R (v\nabla^2 u - u\nabla^2 v) d\xi d\eta = \oint_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds.$$

   Letting $v = G$, the Green's function, write the solution $u(x, y)$ in terms of integrals for the case $\lim_{x^2+y^2 \to \infty} u = 0$.

   (b) Derive the free-space Green's function $G = \frac{1}{2\pi} \log R$ explaining each step.

   (c) Suppose that the region of interest is the half-plane $x > 0$ and $\partial u/\partial x = f(y)$ on the boundary $x = 0$. Write down the solution for $u(x, y)$ in terms of integrals involving $G(x, y; \xi, \eta)$, $f(y)$ and the non-homogeneous term in Poisson's equation.

3. Transform the following PDE to canonical form and then solve it:

$$x u_{xx} + u_{xy} + u_x = 0.$$

A. Introduce a small artifical diffusivity term in the original equation, namely

$$U_x + bU_y = \frac{b^2}{2}hU_{yy}.$$

B. Apply the forward and central difference for discretizing the resulting equation.

(b) Derive the truncation error for the Lax-Wendroff (L-W) scheme. Is this scheme consistent with the original PDE?

(c) Derive the amplification factor with Von Neumann stability analysis.

(d) Discuss the dispersion and dispassion of (L-W) method with different parameter ρ.

DIFFERENTIAL EQUATIONS

1. (a) Write down the definition of the Fourier transform and its inverse. Applying this to an odd function derive the formulae for the Fourier sine transform and its inverse.

   (b) Solve the following initial boundary value problem (in which k is a positive constant):

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,t) = k\dfrac{\partial^2 u}{\partial x^2}(x,t), \text{ for } x \geq 0, \ t \geq 0; \\ \\ u(0,t) = h(t), \text{ for } t \geq 0; \\ u(x,t) \text{ and } u_x(x,t) \to 0 \text{ as } x \to \infty; \\ u(x,0) = 0, \text{ for } x \geq 0. \end{cases}$$

(Note: $\displaystyle\int_0^\infty e^{-\alpha y^2} \cos \omega y \, dy = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$).

2. Let $T(x,t)$ denote the ground temperature at depth x below the surface, at time t. Suppose that the surface temperature varies with time in the form:

$$T(0,t) = T_0 + A_0 \cdot \cos \omega t \qquad (t > 0) \tag{1}$$

where T_0 is the average temperature, A_0 is the amplitude of temperature fluctuation, and ω is the frequency (T_0, A_0, ω are known). Suppose that the temperature distribution underground is subject to the heat conduction equation:

$$\frac{\partial T}{\partial t} = \kappa_T \frac{\partial^2 T}{\partial X^2} \tag{2}$$

and that, as $x \to \infty$

$$T \to T_\infty \tag{3}$$

where T_∞ is a constant to be determined.

Find

(a) the underground temperature distribution $T(x,t)$,

(b) T_∞, the temperature in the deep underground,

(c) the depth x_*, where the temperature fluctuation is reduced to 1% .

3. Consider an infinite non-homogeneous string, which is composed of two different materials joined at $x = 0$. The displacements u_1 and u_2 of the string segments are subject to the wave equations:

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2 \frac{\partial^2}{\partial x^2}\right) u_1(x,t) = 0, \qquad x < 0$$
$$\left(\frac{\partial^2}{\partial t^2} - c_2^2 \frac{\partial^2}{\partial x^2}\right) u_2(x,t) = 0, \qquad x > 0$$

Suppose that starting from $t = 0$, there is an incoming right running wave

$$u_1(x,t) = \begin{cases} \Phi\left(t - \dfrac{x}{c_1}\right) & x < c_1 t \\ 0 & x > c_1 t \end{cases}$$

This incoming wave will be reflected at $x = 0$ and also transmitted into the region $(x > 0)$. Determine

(a) the transmission wave and the reflection wave,

(b) the transmission rate T and reflection rate R of the wave energy, supposing that the wave energy is measured by its magnitude.

PARTIAL DIFFERENTIAL EQUATIONS

4. Consider the initial value problem for the heat equation

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } I\!R^n \times (0,\infty) \\ u &= g \quad \text{on } I\!R^n \times \{t=0\}. \end{cases}$$

(a) Setting $f = 0$, use the Fourier transform to find the fundamental solution of the heat equation in $I\!R^n$.

(b) Extend your previous argument to derive a formula for the solution of the given inhomogeneous problem.

(c) Does the solution make sense for $f \equiv 1$? What is $lim_{t\to\infty} u(x,t)$?

(A useful identity: $\dfrac{1}{(2\pi)^{n/2}} \displaystyle\int_{I\!R^n} e^{ix\cdot y - t|y|^2}\, dy = \dfrac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}$, $t > 0$.)

5. Consider the boundary-value problem

$$\begin{cases} -\frac{1}{x}u_x + \frac{1}{y}u_y - u^2 &= 0, \quad \text{in } \Omega := \{(x,y)\,|\, x>0, y>0\}, \\ u &= g, \quad \text{on } \Gamma := \{(x,y)\,|\, x>0, y=0\}. \end{cases}$$

(a) Use the method of characteristics to find $u(x,y)$.

(b) Give a condition on $g(x)$ which guarantees the existence of a unique continuously differentiable solution on all of Ω.

(c) Sketch the base characteristics in the (x,y) plane.

6. Consider the wave equation

$$\begin{cases} u_{tt} &= c^2 u_{xx} \quad \text{in } I\!R \times (0,\infty) \\ u &= u_0 \quad \text{on } I\!R \times \{t=0\} \\ u_t &= u_1 \quad \text{on } I\!R \times \{t=0\}. \end{cases}$$

(a) Derive the d'Alembert formula for the solution explicitly in terms of the initial data u_0 and u_1.

(b) In case u_0 and u_1 both vanish outside the interval $(-L, L)$ show that the solution always vanishes when $|x| > L + ct$, taking $c > 0$.

(c) If in addition to the hypothesis in (b) one assumes that $\int_{-L}^{L} u_1(x)\, dx = 0$ show that the solution also vanishes identically in the cone $|x| < ct - L$

(Hint: Draw pictures)

PARTIAL DIFFERENTIAL EQUATIONS

4. Consider the PDE below with given initial curve ℓ parametrized by t:

$$\begin{cases} uu_x + u_y = 1 \\ \ell: \ x_0(t) = t, \quad y_0(t) = t, \quad u_0(t) = t/2, \quad 0 \le t \le 1 \end{cases}$$

   (a) Write down the characteristic equations for this problem, and solve for $u(x, y)$ by the method of characteristics. Draw the base characteristics. For what regions of points $(x, y) \in \mathbb{R}^2$ can you find a unique, continuously differentiable solution by this method?

   (b) Now suppose $\ell : x_0(t) = t^2$, $y_0(t) = 2t$, $u_0(t) = t$, $t \in [0, 1]$. Can you find a solution to this new initial value problem by the method of characteristics? If so, what is it?

5. Consider the boundary value problem posed on $\Omega \subset \mathbb{R}^n$, a bounded, simply connected region with smooth boundary Γ. Find $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{cases} \Delta u - k^2 u = f, & x \in \Omega \\ \lambda u + \frac{\partial u}{\partial n} = g, & x \in \Gamma. \end{cases}$$

   Here $\vec{n}$ is the outward normal on Γ, λ, k are real constants, and f, g are continuous functions.

   (a) Under what conditions on λ, if any, are you guaranteed unique solutions to this BVP?

   (b) State and prove a uniqueness theorem concerning solutions to this problem.

6. Use the Fourier Transform Method to solve the following boundary value problem of the modified Helmholtz equation:

$$u_{xx} + u_{yy} - k^2 u = f(x, y), \quad (x, y) \in \mathbb{R}^2,$$

   with the boundary condition:

$$u(x, y) \to 0 \text{ as } (|x|, |y|) \to \infty.$$

   Here we assume that $k > 0$ and that the given function $f(x, y)$ has a Fourier Transform. *Hint*: Use the following properties of the zeroth order Bessel function and modified Bessel function:

$$J_0(z) = J_0(-z) = \frac{1}{2\pi} \int_0^{\infty} \exp[iz \cos \phi] \, d\phi,$$

   and

$$K_0(kr) = -\int_0^{\infty} \frac{\rho J_0(\rho r)}{\rho^2 + k^2} \, d\rho.$$

4. (a) Solve the initial value problem

$$v_t + cv_x = f(x,t), \qquad v(x,0) = F(x), \qquad x \in \mathbb{R}, t \geq 0,$$

where f and F are specified continuous functions, and c is a positive constant.

(b) Draw the base characteristics associated with the problem above.

(c) Find the solution explicitly when $f(x,t) = xt, \qquad F(x) = \sin(x)$.

5. Consider the following initial-boundary value problem: *Find* $u \in C^2((0,l) \times (0,\infty)) \cup C^1([0,1] \times [0,\infty))$ *such that*

$$(IBVP) \begin{cases} u_t - c^2 u_{xx} &= 0, & 0 < x < l, t > 0, \\ u(x,0) &= f(x) & 0 \leq x \leq l, \\ u_x(0,t) &= u_x(l,t) = 0, & t > 0. \end{cases}$$

Here $c > 0$ and $l > 1$ are real constants and f is a given continuous function.

(a) State and prove a theorem concerning uniqueness of solutions to (IBVP).

(b) Solve $(IBVP)$

(c) Find $\lim_{t\to\infty} u(x,t)$.

6. (a) Use Fourier transforms to find a representation formula for the solution $u(x,t)$ of the initial value problem

$$\begin{cases} bu_t - u_{xx} &= \quad 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) &= g(x) & \leq x \in \mathbb{R}. \end{cases}$$

Here b is a positive constant. Clearly identify the fundamental solution (free-space Green's function) . (HINT: $\int_\infty^\infty e^{-ixy} e^{-py^2}\, dy \;=\; \sqrt{\frac{\pi}{p}} e^{-\frac{y^2}{4p}}$)

(b) Solve the heat equation

$$\begin{cases} u_t - u_{xx} &= \quad 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) &= 1 & \leq x \in \mathbb{R}. \end{cases}$$

(c) What happens in the case $b < 0$, $g(x) \equiv 1$?

(d) Write down the fundamental solution for Schrödinger's equation, $iu_t - u_{xx} = 0$. For which range of t is there a well defined solution in case the initial data satisfies $\int_\infty^\infty |g(y)|^2\, dy < \infty$?

Applied Partial Differential Equations

Q1) Use the method of characteristics to solve the PDE

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = xe^u, \; y > 0, \qquad \text{with } u = y \text{ on } x = 0, \; 0 < y < 5.$$

Indicate with a sketch the region of the xy-plane over which the solution is determined uniquely.

Q2) (a) Find a series solution of the following initial boundary value problem:

$$\begin{cases} u_t(x,y,t) - \Delta u(x,y,t) &= 0, \quad 0 < x < L, 0 < y < M, t > 0, \\ u_x(0,y,t) &= 0, \quad 0 < y < M, t > 0, \\ u_x(L,y,t) &= 0, \quad 0 < y < M, t > 0, \\ u_y(x,0,t) &= 0, \quad 0 < x < L, t > 0, \\ u_y(x,M,t) &= 0, \quad 0 < x < L, t > 0, \\ u(x,y,0) &= xy \quad 0 < x < L, 0 < y < M. \end{cases}$$

(b) What can you say about the convergence of the series?

(c) What happens to the solution as $t \longrightarrow \infty$?

Q3) Consider the following regular Sturm-Liouville eigenvalue problem in a bounded region $G \in \mathbb{R}^3$: *find* M, λ *such that*

$$-\nabla \cdot (p\nabla M) + qM = \lambda \rho M \qquad \text{in G} \tag{1}$$

$$M = 0, \qquad \text{on } \partial G \tag{2}$$

Here, $p(x) > 0, \rho(x) > 0, q(x) \geq 0$ in G. Consider the Rayleigh quotient $\frac{E(u)}{\|u\|_0^2}$, where

$$E(u) = \int_G \left(p|\nabla u|^2 + qu^2\right)\, dx, \qquad \|u\|_0^2 = \int_G \rho u^2 \, dx.$$

Prove the following:

- Each eigenvalue λ_n and the corresponding eigenfunction M_n satisfy the relationship
$$\lambda_n = \frac{E(M_n)}{\|M_n\|_0^2}.$$
- The minimum of the Rayleigh quotient over all admissable functions $w(x)$ is the smallest eigenvalue λ_1:
$$\lambda_1 = \min\left\{\frac{E(w)}{\|w\|_0^2}\right\},$$
and the minimizer $w_*(x)$ is the corresponding eigenfunction $M_1(x)$.

Applied Partial Differential Equations

Q1) Consider the PDE

$$\frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e^y \frac{\partial u}{\partial y}\right) = 0, \qquad x^2 + y^2 < 2,$$

with boundary condition $u(x,y) = x^2$ on $x^2+y^2 = 2$. Write the weak form of the PDE. Show that the PDE has a unique weak solution (i.e., prove existence and uniqueness of solutions).

Q2) Let $u(x,y)$ be a harmonic function on the bounded domain Ω which lies in the first quadrant of $\mathbb{R}^2$ and is bounded by the two axes and the parabola $y = 4 - x^2$. Suppose also that along the boundary of Ω, $u(0,y) = 0$, $u(x, 4-x^2) = 0$ and $u(x,0) = x(4-x^2)$. Show that on Ω, $0 < u(x,y) < x(4 - x^2 - y^2)$. State precisely any theorems you use.

Q3) (a) Use Fourier transforms to solve the following initial boundary value problem

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t) + \alpha u_x(x,t), & \text{for } x \in \mathbb{R}, \quad t \geq 0, \\ u(x,0) &= u^0(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

where $u^0(x) \in L^2(\mathcal{R})$.

(b) Verify that $\|u(\cdot,t)\| \leq \|u^0\|$ for all $t \geq 0$ and that $\lim_{t\to\infty} \|u(\cdot,t)\| = 0$, where $\|u\|$ denotes the norm in $L^2(\mathbb{R})$.

Hint: $\displaystyle\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-p\xi^2}\, d\xi = \sqrt{\frac{\pi}{p}} e^{-x^2/4p}.$

Q4) The initial value problem

$$\begin{cases} 2uu_x + u_t &= 0 \\ u(x,0) &= \begin{cases} 0 & \text{if } \tau > 0 \\ \sqrt{|\tau|} & \text{if } \tau \leq 0. \end{cases} \end{cases}$$

has a shock along a curve of the form $x = Ct^2$. Find the solution and the precise location of the shock.

Applied Partial Differential Equations

Q1) Show that the following initial value problem has at most one solution $u(x,t)$:

$$\begin{cases} u_{tt} = [p(x)u_x]_x + f(x,t); \quad x \in (0,L), t > 0 \\ u(0,t) = g(t), \quad u_x(L,t) = h(t) \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x). \end{cases}$$

(You may assume any regularity you need for your argument).

Q2) Consider the PDE

$$uu_x + u_y = 1.$$

Suppose the integral surface of the differential equation passes through the curve C whose parametric equations are

$$x(\tau) = \frac{\tau^2}{2}, \qquad y(\tau) = \tau, \qquad u(\tau) = \tau, \qquad 0 \le \tau \le 1.$$

(a) Check the non-characteristic condition. What can you infer about the existence and uniqueness of solutions?

(b) Locate all solutions.

(c) Sketch the base characteristics for two solutions in the $x - y$ plane, and indicate carefully the region of validity of the solutions.

(d) Does the solution describe a shock or a rarefaction wave?

Q3) Let $\Omega := \{(r,\theta) | r < 1, \ 0 \le \theta \le 2\pi\}$ denote the open unit disk in $\mathbb{R}^2$, with boundary C.

(a) Consider the boundary value problem

$$-\triangle u = 0, \text{ in } \Omega, \qquad \frac{\partial u}{\partial r} = \sin^2\theta \text{ on} C.$$

Show that this problem does not possess a weak solution.

(b) Now consider the generalized Neumann problem for Poisson's equation:

$$-\triangle u = f, \text{ in } \Omega, \qquad \frac{\partial u}{\partial r} = g \text{ on } C.$$

i. Write down the weak form of this problem, carefully identifying the solution space H in which solutions will be sought. Ensure that if a weak solution $u \in H$ exists, it will be unique.

ii. What conditions on f and g are *necessary* for a weak solution to this problem to exist?

Q4) Find the Fourier transform of $e^{-|x|}$. Use this to solve the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-s|} u(s)ds = g(x)$$

for u.

Applied Partial Differential Equations Module

Problem 1) Consider Burger's equation,

$$u_t + uu_x = 0,$$

with the following initial conditions:

$$u(x,0) = \begin{cases} 1 & \text{for } |x| > 1 \\ |x| & \text{for } |x| \le 1. \end{cases}$$

Sketch the (weak) solution $u(x,t)$ for all $t > 0$. Identify the time of shock formation and give an explicit parametrization $s(t)$ of the shock (hint: try a parametrization of the form $s(t) = t + 1 + C\sqrt{t+1}$). Write down $u(x,t)$ explicitly for all $(x,t) \in \mathbb{R} \times (0,\infty)$.

Problem 2) Show that $F(x,y) = -\dfrac{1}{2\pi a_1 a_2} \ln |(x/a_1, y/a_2)—$ satisfies $-LF(x,y) = \delta_0$ where the differential operator L is defined by

$$Lu = a_1^2 \frac{\partial^2 u}{\partial x^2} + a_2^2 \frac{\partial^2 u}{\partial y^2},$$

with a_1 and a_2 both positive. (You may assume that $E(x,y) = -1/2\pi \ln |(xy)|$ satisfies $- \triangle E(x,y) = \delta_0$.)

Problem 3) (a) Find the Fourier transform of $exp(-2|x|)$ for $x \in \mathbb{R}$.

(b) Use Fourier transforms to solve

$$u_{xx} + 4u_{yy} = 0, \qquad x \in \mathbb{R}, 0 < y < 1,$$

with $u(x,0) = exp(-2|x|), u(x,1) = 0, u(x,y) =\longrightarrow 0$ as $|x| \to \infty$, uniformly in y.

Here, we define the Fourier transform of an $L^1(\mathbb{R}^n)$ function, f, as $\hat{f}(s) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(-ix \cdot s) f(x)\, dx$. The inverse transform is defined via $\check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(ix \cdot s) \hat{f}(s)\, ds$.

Problem 4) Let D be the unit disk in $\mathbb{R}^2$, ∂D it's boundary.

(a) Show that the Neumann problem

$$-\triangle u = 0, \ \vec{x} \in D, \qquad \frac{\partial u}{\partial n} = 1, \ \vec{x} \in \partial D$$

has no solution $u \in C^2(D) \cap C^1(\bar{D})$.

(b) Now consider

$$-\triangle u_\epsilon = 0, \ \vec{x} \in D, \qquad \frac{\partial u_\epsilon}{\partial n} + \epsilon u_\epsilon = 1, \ \vec{x} \in \partial D, \qquad 0 < \epsilon << 1.$$

Find u_ϵ. Can one solve this problem using a regular perturbation series in ϵ? Justify your answer carefully.

Numerical Analysis Module

Problem 1) For this question do not worry about issues of floating point error.

(a) Let A be a 2×2 real matrix. Give the entries of a real orthogonal 2×2 matrix Q such that QA is upper triangular. The entries of Q will depend on the entries of A.

(b) Let A be an $n \times n$ matrix, and suppose $1 \leq i < j \leq n$ Using your answer to part (a), give the entries a real orthogonal $n \times n$ matrix Q such that QA has its jith component equal to 0, and such that QA only differs from A in rows i and j. You do not need to justify your answer.

(c) Using only matrices of the form described in (b), write down an algorithm that computes the upper triangular matrix R of a QR factorization of an $n \times n$ matrix A You do not need to justify your answer.

Problem 2) (a) Consider approximating the solution to the linear system $Ax = b$, where A is non-singular, with the simple iterative method

$$x^{(k+1)} = Bx^{(k)} + f.$$

We say that the method is stable if $\|B\| < 1$ for some induced matrix norm $\|\cdot\|$, that it is consistent if $f = (I - B)A^{-1}b$, and that it is convergent if $x^{(k)} \to A^{-1}b$ for all $x^{(0)}$. Show that consistency and stability imply convergence.

(b) Let $A = L + \epsilon E$ where L is non-singular upper triangular and E is some matrix. Describe a consistent simple iterative method for solving $Ax = b$ that costs only $\mathcal{O}(n^2)$ flops per iteration. Show that for fixed A and b your method is convergent when ϵ is small enough.

Applied PDE Module

Problem 1) Consider the problem

$$\begin{aligned} -\nabla\cdot[a(x)\nabla u(\mathbf{x})] &= f(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Omega, \\ \nu\cdot\nabla u(\mathbf{x}) &= g(\mathbf{x}) \quad \text{for } \mathbf{x}\in\partial\Omega. \end{aligned}$$

where Ω is a bounded domain with smooth boundary, ν is the normal to the boundary and $a(x)$ is a positive differentiable function on the closure of Ω.

(a) State a necessary condition for this problem to have a solution.

(b) Assuming (a), prove that the above problem has at most one solution in $C^2(\overline{\Omega})$.

(c) Use the divergence theorem to motivate and then carefully specify the weak formulation of this problem.

(d) Assuming the condition in (a), *briefly outline* the proof of the existence of weak solutions.

Problem 2) (a) Derive formulae for the solutions $u(x,t)$ and $v(x,t)$ of the system

$$\begin{aligned} \partial_t u + 3\partial_x v &= 0, \\ \partial_t v + 3\partial_x u &= 0, \end{aligned} \quad \text{subject to} \quad \begin{aligned} u(x,0) &= u^0(x), \\ v(x,0) &= v^0(x). \end{aligned}$$

(b) Evaluate the solutions at $(x,t) = (1,1)$ and $(x,t) = (2,0.5)$ in the case that

$$u^0(x) = \begin{cases} 0 & \text{for } x<0 \\ 2 & \text{for } x\geq 0 \end{cases}, \text{ and } v^0(x) = \begin{cases} 2 & \text{for } x<1 \\ 4 & \text{for } x\geq 1 \end{cases}.$$

(c) Note that the solutions are identically constant over certain regions in the (x,t)−plane. Sketch these regions. Verify that the Rankine Hugoniot condition holds across the boundary between the regions containing $(x,t) = (1,1)$ and $(x,t) = (2,0.5)$.

Problem 3) The forced vibration of an infinitely long uniform beam is described by

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{EI}P(x,t),$$

where $u(x,t)$ is the lateral deflection of the beam, and $c, E, \& I$ are constants. The forcing $P(x,t)$ is localized and impulsive, ie,

$$P(x,t) = \delta(x)\delta(t)$$

where δ denotes the Dirac measure.

Use Fourier transforms to find $u(x,t)$, given that $u(x,0) = 0$. Evaluate all the integrals, and interpret the results physically.

You may use the identity

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-\xi^2 a - i\xi x)d\xi = \frac{1}{\sqrt{2a}}\exp(-x^2/4a),$$

where a can be complex.

Problem 4) Consider a solid sphere $\Omega \subset \mathbb{R}^3$ of radius a, centered at the origin. Let the sphere have constant mass density μ. By Newton's law, the gravitational force $\vec{F}$ exerted by this sphere on a unit mass at point ξ is given by

$$\vec{F}(\xi) = c\int_{\Omega} \frac{\mu(x-\xi)}{|x-\xi|^3}\,dx.$$

Associated with this force is a gravitational potential u, which is harmonic outside Ω, vanishes at infinity, is $C^1(\mathbb{R}^3)$, and satisfies

$$\Delta u = \nabla \cdot \vec{F} \text{ in } \Omega.$$

(a) Evaluate the gravitational potential, which is defined by

$$u(\xi) := c\int_{\mathbb{R}^3} \frac{\mu}{|x-\xi|}\,dx.$$

(b) Evaluate the force $\vec{F}$, by first showing that $\vec{F} = \nabla u$.

Partial Differential Equations

1. Find the solution $u = u(x,t)$ of

$$2xtu_x + u_t = u, \;\; u(x,0) = f(x)$$

where $f \in C^1(-\infty, \infty)$ and prove that when $f(0) = 0$ one has $lim_{t\to\infty} u(x,t) = 0$.

2. Assume the fact that on $\mathbb{R}^3$ $\Delta \frac{1}{r} = -4\pi\delta$ in the distribution sense where $r = (x_1^2 + x_2^2 + x_3^2)^{-1/2}$. Now consider the operator $Lu = a_1^2\partial_1^2 u + a_2^2\partial_2^2 u + a_3^2\partial_3^2 u$ and the function $E(x_1, x_2, x_3) = (x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2)^{-1/2}$ where $a_1,\ a_2$ and a_3 are positive constants and ∂_i denotes partial differentiation wrt x_i. Prove that

$$LE = -4\pi a_1 a_2 a_3 \delta$$

in the distribution sense.

3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Prove that there is a constant $C > 0$ depending only on Ω such that

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega).$$

4. Let $\{u_k(x,t)\}$ be a monotone sequence of solutions of heat equation $u_t - \Delta_x u = 0$ in $\mathbb{R}^n \times (0,\infty)$. Suppose there is a point $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$ such that $\{u_k(x_0,t_0)\}_{k=1}^\infty$ is bounded. Prove that for any compact subset $K \subset \mathbb{R}^n$, any $T \geq 0$, the sequence $\{u_k(x,t)\}_{k=1}^\infty$ is uniformly convergent in $K \times [t_0, t_0 + T]$, and the limit function satisfies the heat equation.

Partial Differential Equations

1. Show that the distribution T_N defined by

$$< T_N, \phi >= \int_{\mathbf{R}^n} N\phi \, dx, \quad \phi \in C_c^\infty(\mathbf{R}^n),$$

where

$$N(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

is for $n \geq 3$ a fundamental solution of the Laplacian Δ in $\mathbf{R}^n$, that is

$$\Delta T_N = \delta,$$

where δ denotes the Dirac measure in $\mathbf{R}^n$. Hint: You may use without proof the following lemma: If $f \in L^1(\mathbf{R}^n)$ and $g \in L^p(\mathbf{R}^n)$, then $f * g_\epsilon \to af$ in the L^p norm, where

$$a = \int_{\mathbf{R}^n} g \, dx, \quad g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon).$$

2. Suppose that $u(x,t)$ is C^2 for $0 \leq t \leq T$, $x \in \mathbf{R}^n$, and that u solves the wave equation

$$\partial_t^2 u - \Delta u = 0.$$

Suppose furthermore that $u = 0 = \partial_t u$ on the ball

$$B = \{(x,0) \,|\, |x - x_0| \leq t_0\},$$

contained in the hyperplane $t = 0$, where $x_0 \in \mathbf{R}^n$ and $0 \leq t_0 \leq T$. Show that u vanishes identically in the backward solid cone

$$\Omega = \{(x,t) \,|\, 0 \leq t \leq t_0, \, |x - x_0| \leq t_0 - t\}.$$

Hint: Calculate the time derivative $\frac{dE}{dt}$ of the "energy"

$$E(t) = \frac{1}{2} \int_{B_t} [(\partial_t u)^2 + |\nabla_x u|^2] dx$$

where

$$B_t = \{x \,|\, |x - x_0| \leq t_0 - t\},$$

and show that

$$\frac{dE}{dt} \leq 0,$$

using the divergence theorem. Impose then the Cauchy data.

3. The governing equations for the one-dimensional, isentropic flow of a compressible gas are

$$c_t + u\,c_x + \frac{\gamma-1}{2}\,c\,u_x = 0$$
$$u_t + u\,u_x + \frac{2}{\gamma-1}\,c\,c_x = 0\,,$$

where $c, u,$ and γ are, respectively, the sound speed, the velocity in the x direction and the ratio of specific heats (a constant).

Try to find a solution of this system using the method of characteristics. You may use any procedure with which you are comfortable, but one approach is the following. Form a linear combination of the two PDEs by multiplying the second one by λ and adding it to the first. Then, choose λ such that in the combined equation both c and u are differentiated in the *same* direction (i.e., dx/dt is the same for c as it is for u).

For each real value of λ, determine the ODE that applies along the characteristics. Finally, obtain integrals ("Riemann invariants") that are constant along each characteristic. Use two or three adjectives to classify the above system.

4. Find a similarity solution of the following PDE using the procedure outlined in (a) and (b) below:

$$A_t = A_{xx} + A, \quad -\infty < x < \infty,\ t > 0\,.$$

(a) Noting the resemblance of the above equation to the heat equation, make the change of independent variables $(t, x) \to (t, \eta)$, where $\eta = x^2/4t$.

(b) Taking advantage of its linearity, solve the resulting PDE by employing the separation of variables

$$A(t, \eta) = f(t)e^{-\eta}.$$

(c) Employ a Fourier transform to solve the same problem and take as initial condition $A(x, 0) = f(x)$. For what choice of $f(x)$ will you recover the answer obtained in part (b)? (See next page for Fourier transform table.)

Partial Differential Equations Module

[PDE. 1]

(a) Using the method of characteristics, solve

$$u_t + (u_x)^2 = 0, \qquad u(x,0) = x^2.$$

(b) Find the entropy solution to Burgers' equation $u_t + uu_x = 0$ with initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1 \\ 1/2 & \text{if } -1 < x < 1 \\ 3/2 & \text{if } 1 < x < 2 \\ 1 & \text{if } x > 2. \end{cases}$$

[PDE. 2]

(a) *Write down* the Fundamental Solution $\Phi(\cdot)$ to $-\triangle$ on $\mathbf{R}^3$ *and write down* the PDE it solves in the sense of distributions.

(b) If $\Omega \subset \mathbf{R}^3$ is a bounded open set with smooth boundary, define the *usual* Green's function (for homogeneous Dirichlet boundary conditions) for $-\triangle$ on Ω in terms of Φ and a *corrector* function. For $f \in C_c^2(\Omega)$, *write* the solution to

$$-\triangle u = f \text{ in } \Omega \qquad u = 0 \text{ on } \partial\Omega$$

in terms of the Green's function.

(c) For the rest of this question, we let the dimension $n = 1$. *Find* the Fundamental Solution $\Phi(\cdot)$ of $-$ 1D Laplacian on $\mathbf{R}$ (i.e. $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$). *Prove* that if $f \in C_c^2(\mathbf{R})$, then the solution to

$$-\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x), \qquad x \in (-\infty, \infty),$$

is given by

$$u(x) = \int_{-\infty}^{\infty} \Phi(x-y)\, f(y) dy.$$

(d) *Find* the Green's function for $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $(-1,1)$ with Dirichlet boundary conditions (i.e. associated with boundary conditions $u(-1) = u(1) = 0$).

(e) Does there exist a Green's function (often referred to as a Neumann function) for $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $(-1,1)$ with Neumann boundary conditions?

[PDE. 3]

Let Ω be a bounded. connected open set in $\mathbf{R}^3$ with smooth boundary $\partial\Omega$ which is the union of two nonempty disjoint surfaces, say $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Suppose $f \in L^2(\Omega)$.

(a) Consider the BVP

$$\begin{cases} -\triangle u = f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \Gamma_1, \\ u = 0 & \text{on} \quad \Gamma_2. \end{cases}$$

Write down a weak formulation of this BVP and explain why it is the weak formulation.

Hint: The Hilbert space to work in is

$$H = \left\{ u \in H^1(\Omega) \;\middle|\; u = 0 \text{ in the sense of trace on } \Gamma_0 \right\}.$$

Note that the Neumann condition is absent from the space but it will (naturally!) come in by noting that $H_0^1(\Omega) \subset H$, so if a statement holds for all $v \in H$, it certainly holds for all $v \in H_0^1(\Omega)$.

(b) *Prove* that for all $f \in L^2(\Omega)$, there is a weak solution $u \in H$. You may assume that the usual Poincaré inequality holds on H (it does).

[PDE. 4]

Let Ω be a bounded, connected open set in $\mathbf{R}^n$ with smooth boundary. Let X denote the closed subspace of $H^1(\Omega)$ that does NOT contain nonzero constant functions. Using the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$, *prove* that there exists a constant $C < \infty$ (independent of u) such that for $u \in X$,

$$\int_\Omega |u|^2 \mathrm{d}x \leq C \int_\Omega |Du|^2 \mathrm{d}x.$$

Partial Differential Equations Module

[PDE. 1]

(a) Using the method of characteristics, solve

$$xu_x + 2yu_y + u_z = 3u, \qquad u(x,y,0) = g(x,y).$$

(b) Consider the Burgers equation $u_t + uu_x = 0$ with initial data

$$u(x,0) = 0 \text{ if } |x| \geq 1 \qquad \textit{and} \qquad u(x,0) = 1 - |x| \text{ if } |x| \leq 1.$$

By sketching the characteristics, describe the entropy solution, i.e. the solution with the property that at its discontinuities (shocks), the entropy inequality is satisfied. Clearly indicate on your sketch of the characteristics where the shock is. Is the shock a line? What is the equation of the shock? What happens to $u(\cdot,t)$ as $t \to \infty$?

[PDE. 2]

Define the Green's function $G(x,y)$ for the Dirichlet problem involving $-\triangle$ in three space dimensions and a region Ω. What can you say about the sign of G i.e. is it always positive, always negative, the answer depends on the source point, just can't say? Prove your answer.

[PDE. 3]

Let Ω be a connected subset of R^n with smooth boundary.

(a) Give a definition for a weak solution $u \in H^1(\Omega)$ of the Neumann problem

$$-\triangle u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Clearly motivate why this is a natural definition.

(b) Prove that there is a weak solution to this problem iff

$$\int_\Omega f \mathrm{d}x = 0.$$

[PDE. 4]

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary.

(a) For $f \in L^2(\Omega)$, show that there exists a $u_f \in H_0^1(\Omega)$ such that

$$(f, u_f)_{L^2(\Omega)} = \|u_f\|^2_{H_0^1(\Omega)}.$$

Hint: For $v \in H_0^1(\Omega)$ define the functional

$$T_f(v) := (f,v)_{L^2(\Omega)} = \int_\Omega f(x)v(x)dx.$$

Show that T_f is a bounded linear functional and apply the Riesz representation theorem.

(b) The H^{-1} norm of f is given by

$$\|f\|^2_{H^{-1}(\Omega)} := \|u_f\|^2_{H_0^1(\Omega)}.$$

Of course this norm applies to a wider class of distributions which include L^2, but using just the restriction to L^2, prove that $L^2(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$. You may use the fact that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.