MATH 581 ASSIGNMENT 7

DUE MONDAY APRIL 16

1. For $s \in \mathbb{R}$, the (Bessel potential) *Sobolev space* $H^s(\mathbb{R}^n)$ is the set of those $u \in \mathscr{S}'(\mathbb{R}^n)$ with $\|u\|_{H^s} := \|\langle D\rangle^s u\|_{L^2} < 0$, where the *Bessel potential* $\langle D\rangle^s u$ of u is defined by
$$\widehat{\langle D\rangle^s u}(\xi) = \langle\xi\rangle^s \hat{u}(\xi) \equiv (1+|\xi|^2)^{s/2}\hat{u}(\xi).$$
Prove the followings.
a) $\langle D\rangle^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometry.
b) For $k \geq 0$ integer, $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.
c) $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
d) The (topological) dual of $H^s(\mathbb{R}^n)$ is isometric to $H^{-s}(\mathbb{R}^n)$.
2. Prove the followings.
a) If $s = \frac{n}{2} + k + \alpha$ with $0 < \alpha < 1$ and $k \geq 0$ an integer, then $H^s(\mathbb{R}^n) \hookrightarrow C^{k,\alpha}(\mathbb{R}^n)$.
b) The *trace operator* $\gamma : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^{n-1})$ defined by
$$(\gamma u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0),$$
has a unique extension to a bounded linear operator $\gamma : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.
c) If $u \in H^s(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$, then $\varphi u \in H^s(\mathbb{R}^n)$ with
$$\|\varphi u\|_{H^s} \leq C\|u\|_{H^s},$$
where
$$C = 2^{|s|/2}\int_{\mathbb{R}^n} \langle\xi\rangle^{|s|}|\hat{\varphi}(\xi)|\mathrm{d}\xi.$$
Hint: Verify *Peetre's inequality*
$$\langle\xi\rangle^{2s} \leq 2^{|s|}\langle\xi-\eta\rangle^{2|s|}\langle\eta\rangle^{2s},$$
for $\xi, \eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$.
d) Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with $\mathrm{d}\phi \in W^{\ell,\infty}(\mathbb{R}^n)$ and $\mathrm{d}(\phi^{-1}) \in W^{\ell,\infty}(\mathbb{R}^n)$ for all ℓ. Then the pullback $\phi^* : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is a linear homeomorphism.
3. For a domain $\Omega \subset \mathbb{R}^n$, we define
$$H^s(\Omega) = \{u \in \mathscr{D}'(\Omega) : u = w|_\Omega \text{ for some } w \in H^s(\mathbb{R}^n)\},$$
with the norm
$$\|u\|_{H^s(\Omega)} = \inf_{\{w \in H^s(\mathbb{R}^n): w|_\Omega = u\}} \|w\|_{H^s}.$$
Similarly, define
$$\mathscr{D}(\overline{\Omega}) = \{u : u = w|_\Omega \text{ for some } w \in \mathscr{D}(\mathbb{R}^n)\}.$$

Date: Winter 2012.

a) Show that the restriction operator $w \mapsto w|_\Omega : H^s(\mathbb{R}^n) \to H^s(\Omega)$ is continuous, and that $\mathscr{D}(\overline{\Omega})$ is dense in $H^s(\Omega)$.

b) Show that there exists a sequence $\{\lambda_k\}$ satisfying

$$\sum_{k=0}^{\infty} 2^{jk}\lambda_k = (-1)^j, \qquad j \in \mathbb{N}_0.$$

c) Define the *Seeley extension operator* $E : \mathscr{D}(\overline{\mathbb{R}^n_+}) \to \mathscr{D}(\mathbb{R}^n)$ by

$$(Eu)(x) = \begin{cases} u(x) & \text{if } x_n \geq 0, \\ \sum_{k=0}^{\infty} \lambda_k u(x_1, \ldots, x_{n-1}, -2^k x_n) & \text{if } x_n < 0. \end{cases}$$

Prove that indeed E maps $\mathscr{D}(\overline{\mathbb{R}^n_+})$ into $\mathscr{D}(\mathbb{R}^n)$, and that $E : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$ is bounded for $s \geq 0$.

d) Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. By using coordinate transformations and partitions of unity, construct a bounded extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ for $s \geq 0$.

4. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. Prove the followings.

a) If $s = \frac{n}{2} + k + \alpha$ with $0 < \alpha < 1$ and $k \geq 0$ an integer, then $H^s(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$.

b) If Ω is bounded and $s > t \geq 0$, then the embedding $H^s(\Omega) \hookrightarrow H^t(\Omega)$ is compact.

c) Let $\{U_k\}$ be a finite open cover of a neighbourhood of Ω, and let $\{\varphi_k\}$ be a smooth partition of unity subordinate to $\{U_k\}$. Then

$$\|u\|^2_{H^s(\Omega)} \eqsim \sum_k \|\varphi_k u\|^2_{H^s(U_k \cap \Omega)}, \qquad \text{for} \quad u \in H^s(\Omega).$$

In particular, the membership $u \in H^s(\Omega)$ is equivalent to $\varphi_k u \in H^s(U_k \cap \Omega)$ $\forall k$.

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the second order elliptic operator given in divergence form

$$Lu = -\sum_{j,k=1}^{n} \partial_j(A_{jk}\partial_k u) + \sum_{j=1}^{n} B_j \partial_j u + Cu,$$

with smooth coefficients: $A_{jk}, B_j, C \in C^\infty(\overline{\Omega}, \mathbb{R}^{m\times m})$. Here u is understood to be a vector function on Ω with m (real) components. Assume that L is *strongly elliptic*, i.e.,

$$\sum_{j,k=1}^{n} \xi_j \xi_k [\eta^T A_{jk}(x)\eta] \geq c|\xi|^2|\eta|^2, \qquad \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^m,\ x \in \overline{\Omega},$$

for some constant $c > 0$. Formally integrating $\langle Lu, v\rangle$ by parts with $v \in \mathscr{D}(\Omega)$, we are led to the bilinear form

$$a(u,v) = \int_\Omega (\partial_j v)^T A_{jk}\partial_k u + v^T B_j \partial_j u + v^T C u,$$

where the summation convention is assumed. Prove the followings.

a) The bilinear form a is coercive in $H_0^1(\Omega)^m$, i.e., the Gårding inequality

$$a(u,u) \geq c\|u\|_{H^1}^2 - c_1\|u\|_{L^2}^2, \qquad u \in H_0^1(\Omega)^m,$$

is valid for some constants $c > 0$ and $c_1 \geq 0$.

b) For $\lambda \in \mathbb{R}$ sufficiently large, and for $f \in L^2(\Omega)^m$, there exists a unique $u \in H_0^1(\Omega)^m$ such that $Lu + \lambda u = f$. *Hint*: Lax-Milgram lemma.

c) Interior regularity theorem: If $Lu = f$ with $u \in H_0^1(\Omega)^m$ and $f \in H^s(\Omega)^m$ then $u \in H^{s+2}(U)^m$ for any open U with $\overline{U} \subset \Omega$.

6. With reference to the preceding problem, in linear elasticity, one has $m = n$ and

$$Lu = -\mu\Delta - (\mu + \lambda)\nabla(\nabla \cdot u),$$

where the real constants μ and λ are called *Lamé coefficients*.

a) Determine the values of the Lamé coefficients for which the operator L is strongly elliptic. Assume in the followings that the Lamé coefficients satisfy the conditions just found.

b) Prove that the bilinear form a corresponding to L is not only coercive, but also *strictly coercive* in $H_0^1(\Omega)^n$.

c) Conclude that the equation $Lu = f$ has a unique solution $u \in H_0^1(\Omega)^n$ for each $f \in L^2(\Omega)^n$.

7. Let Ω be an n-dimensional bounded domain with smooth boundary, and let λ_k be the k^{th} eigenvalue of the Laplacian on Ω with homogeneous Dirichlet boundary conditions. Prove *Weyl's law*:

$$\lim_{k\to\infty} \frac{|\lambda_k|^{n/2}}{k} = \frac{c_n}{\text{vol}(\Omega)},$$

where the constant c_n depends only on n.