MATH 581 ASSIGNMENT 5

DUE FRIDAY MARCH 16

1. Prove the followings. The results in b) and c) are part of the *Paley-Wiener theorem.*
   a) For a compactly supported distribution $u \in \mathscr{E}'$, its Fourier transform is equal to
   $$\hat{u}(\xi) = \langle u(x), e^{-i\xi\cdot x}\rangle,$$
   where the notation $u(x)$ is to indicate that the distribution u acts on $e^{-i\xi\cdot x}$ as a function of x. The above expression also makes sense for $\xi \in \mathbb{C}^n$, defining an entire analytic function $\hat{u}$. This is called the *Fourier-Laplace transform* of u.
   b) In this setting, if u is supported in a ball of radius r centred at the origin, then $\hat{u}$ satisfies the growth estimate
   $$|\hat{u}(\xi)| \leq C(1 + |\xi|)^N e^{r|\mathrm{Im}\,\xi|},$$
   with some constants C and N. Hence the Fourier-Laplace transform of a compactly supported distribution is an entire function of growth order at most 1.
   c) If $u \in \mathscr{D}$ and is supported in a ball of radius r centred at the origin, then for every integer N there is a constant C_N such that
   $$|\hat{u}(\xi)| \leq C_N(1 + |\xi|)^{-N} e^{r|\mathrm{Im}\,\xi|}.$$
   Hence the Fourier-Laplace transform of a compactly supported smooth function is an entire function of growth order at most 1, with rapid decay in the real directions.
   d) If the set of real zeroes of p is bounded, then every tempered distribution solution of $p(D)u = 0$ is an entire function of growth order at most 1.
2. In this exercise, we will construct a fundamental solution for an arbitrary (nontrivial) constant coefficient operator by using a construction known as *Hörmander's staircase.* Let p be a nontrivial polynomial in 2 variables.
   a) Show that without loss of generality we can assume
   $$p(\xi) = \xi_1^m + \sum_{k=0}^{m-1} q_k(\xi_2)\xi_1^k,$$
   where m is the degree of p, and q_k are polynomials of a single variable.
   b) Consider a subdivision of the ξ_2-axis into a countably many disjoint intervals $\{I_k\}$, and consider a sequence $\{\eta_1^{(k)}\}$ of real numbers. Then the set of points $(\xi_1, \xi_2, \eta_1) \in \mathbb{R}^3$ where $(\xi_1, \xi_2) \in \mathbb{R}^2$ and $\eta_1 = \eta_1^{(k)}$ for $\xi_2 \in I_k$, can be visualized as a staircase, with steps of heights $\eta_1^{(k)}$ and widths $|I_k|$ and that are infinitely long in the ξ_1-direction.

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Under the identification of $\mathbb{R}^3$ with $\mathbb{C}\times\mathbb{R}$ by $(\xi_1,\xi_2,\eta_1)\mapsto(\xi_1+i\eta_1,\xi_2)$, let us denote by $H\subset\mathbb{C}\times\mathbb{R}$ the above described staircase. Construct a staircase H (that is, sequences $\{I_k\}$ and $\{\eta_1^{(k)}\}$) such that $|p|\geq\alpha$ on H and $|\eta_1^{(k)}|\leq\beta$ for all k, with some constants $\alpha>0$ and β.

c) Let $E:\mathscr{D}\to\mathbb{C}$ be defined by

$$\langle E,\varphi\rangle=\frac{1}{4\pi^2}\int_H\frac{\tilde{\hat{\varphi}}(\zeta)}{p(\zeta)}\mathrm{d}\zeta\equiv\frac{1}{4\pi^2}\sum_k\int_{I_k}\int_{\mathbb{R}}\frac{\tilde{\hat{\varphi}}(\xi_1+i\eta_1^{(k)},\xi_2)}{p(\xi_1+i\eta_1^{(k)},\xi_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

where $\varphi\in\mathscr{D}$, and the tilde $\tilde{}$ denotes the reflection. Prove that indeed $E\in\mathscr{D}'$ and that E is a fundamental solution of $p(D)$.

d) Extend the construction to n dimensions.

3. Let p be a nonzero polynomial. Show the followings.
   a) The equation $p(D)u=f$ has at least one smooth solution for every $f\in\mathscr{D}$.
   b) If all solutions of $p(D)u=0$ are smooth, then $\operatorname{sing\,supp}u\subset\operatorname{sing\,supp}p(D)u$ for any $u\in\mathscr{D}'$. So hypoelliptic operators can be defined as those $p(D)$ such that all solutions of $p(D)u=0$ are smooth.
4. Recall Hörmander's theorem that $p(D)$ is hypoelliptic if and only if for any $\eta\in\mathbb{R}^n$ one has $p(\xi+i\eta)\neq0$ for all sufficiently large $\xi\in\mathbb{R}$. Apply this criterion to show the followings.
   a) All elliptic operators are hypoelliptic.
   b) The wave operator is not hypoelliptic.
   c) The heat operator is hypoelliptic.
5. Construct a non-hypoelliptic polynomial p in dimension $n>1$ such that $|p(\xi)|\to\infty$ as $|\xi|\to\infty$ for $\xi\in\mathbb{R}^n$.
6. Consider the Cauchy problem

$$\partial_t u=\sum_{k=1}^n A_k\partial_k u,\qquad u|_{t=0}=f,$$

where u is a vector function with m components, each A_k is a (possibly complex) $m\times m$ matrix, and f is a given (vector) function. We say that the problem is *strongly well-posed* if for any $f\in L^2$, there exists a solution $u\in C^0(\overline{\mathbb{R}}_+,L^2)$, which satisfies the estimate

$$\|u(t)\|_{L^2}\leq Ce^{\alpha t}\|f\|_{L^2},\qquad t\geq0,$$

with some constants α and C, and u is the only solution in $C^0(\overline{\mathbb{R}}_+,L^2)$. In each of the following cases, prove that the corresponding Cauchy problem is strongly well-posed.
   a) Symmetric hyperbolic: All A_k are Hermitian.
   b) Strictly hyperbolic: For all nonzero $\xi\in\mathbb{R}^n$, the eigenvalues of $P(\xi)=\sum_{k=1}^n A_k\xi_k$ are real and distinct.
7. Prove the strong well-posedness of the Cauchy problem for the system

$$\partial_t u=P(\partial)u+Q(\partial)v,$$
$$\partial_t v=H(\partial)v+Mu,$$

where u and v are vector functions, $P(\partial)$ is a second order parabolic operator, $Q(\partial)$ is an arbitrary first order operator, $H(\partial)$ is a first order symmetric hyperbolic operator, and M is simply a matrix (i.e., a zeroth order operator). The operators $P(\partial)$, $Q(\partial)$, and $H(\partial)$ may contain lower order terms, and the spatial dimension is n.