MATH 581 ASSIGNMENT 4

DUE FRIDAY MARCH 2

1. Let $\omega \subset \mathbb{R}$ be an open interval, and $u \in \mathscr{D}'(\omega)$. Let $0 \leq k \leq \infty$.
   a) Show that if $u' = 0$ then u is a constant function.
   b) Show that if $u' = f$ with $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
   c) Show that if $u' + au = f$ with $a \in C^\infty(\omega)$ and $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
   d) Extend the above result to n-th order linear ODE's with smooth coefficients, assuming that the leading order coefficient does not vanish on ω. This would prove in particular hypoellipticity of linear ordinary differential operators with smooth coefficients (provided the leading order coefficient is nowhere zero). One can also prove analytic-hypoellipticity of such operators with analytic coefficients.
2. A distribution u is called *nonnegative* if $u(\varphi)$ is nonnegative for every nonnegative test function φ. Show that a distribution is nonnegative if and only if it is a nonnegative Radon measure. Note that this means any Jordan-type decomposition would fail for distributions: Radon measures are the only distributions which can be written as the difference of two nonnegative distributions.
3. Let us define the Fourier transform by

$$\hat{u}(\xi) = \alpha \int e^{i\beta x \cdot \xi} u(x) \mathrm{d}x,$$

   for $u \in \mathscr{S}(\mathbb{R}^n)$, where $\alpha, \beta \in \mathbb{R}$ are constants. Derive a formula for the inverse transformation. List some common and/or convenient choices for the constants α and β. For $u, v \in \mathscr{S}$, prove (or derive a formula for) the followings.
   a) Parseval's formula: $\int u\bar{v} = \gamma \int \hat{u}\bar{\hat{v}}$, where $\gamma = \gamma(\alpha, \beta)$ is a constant.
   b) $\widehat{u * v} = \hat{u}\hat{v}$.
   c) $\widehat{uv} = \gamma \hat{u} * \hat{v}$.
   d) Derive a formula for $\widehat{u \circ A}$, where A is an $n \times n$ invertible matrix.
4. There are (at least) two ways to define the Fourier transform on $L^2(\mathbb{R}^n)$.
   - Extend the Fourier transform from $\mathscr{S}$ to L^2 by using the density of $\mathscr{S}$ in L^2 (as well as the Plancherel bound).
   - First define the Fourier transform on $\mathscr{S}'$ by duality, and then restrict it to L^2.

   Show that these two approaches are consistent with each other.
5. Give an example of $u \in C(\mathbb{R}^n)$ such that $\varphi \mapsto \int u\varphi$ is a tempered distribution and that there is no polynomial p satisfying $|u(x)| \leq |p(x)|$ for all $x \in \mathbb{R}^n$.
6. For each of the following functions, determine if it is a tempered distribution, and if so compute its Fourier transform.

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a) $\sin x$, b) $e^{i|x|^2}$, c) The Heaviside step function, d) The sign function, e) $|x|^s$, where s is a real number.

7. a) Let $a \in \mathscr{E}(\mathbb{R}^n)$. Prove that the pointwise multiplication $u \mapsto au : \mathscr{S}' \to \mathscr{S}'$ is well-defined and continuous if and only if for every multi-index α there is a polynomial p such that $|\partial^\alpha a(x)| \leq p(x)$, $x \in \mathbb{R}^n$.

   b) Let p be a polynomial satisfying $|p(i\xi)| \geq c(1 + |\xi|)^m$ for all $\xi \in \mathbb{R}^n$, with some constants $c > 0$ and m. Operators $p(\partial)$ with p satisfying this condition are called *strictly elliptic*. Show that the equation $p(\partial)u = f$ has a solution for each $f \in \mathscr{S}'$.