MATH 581 ASSIGNMENT 3

DUE FRIDAY FEBRUARY 17

1. Prove
   a) $\partial_j(au) = (\partial_j a)u + a(\partial_j u)$ for $a \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$.
   b) $\partial_j \partial_k u = \partial_k \partial_j u$ for $u \in \mathscr{D}'(\Omega)$.
   c) There is no distribution on $\mathbb{R}$ such that its restriction to $\mathbb{R} \setminus \{0\}$ is $e^{1/x}$.
2. Find the limits of the following sequences in $\mathscr{D}'(\mathbb{R})$.
   a) $\cos nx$.
   b) $n^k \sin nx$, where $k > 0$ is a constant.
   c) $x^{-1} \sin nx$.
3. Let $f, g \in L^1_{\text{loc}}(\mathbb{R})$ and let $u(x,t) = f(x-t) + g(x+t)$. Show that $\partial_t^2 u - \partial_x^2 u = 0$ in the sense of distributions.
4. Prove the following properties of convolution.
   a) $(u * v) * w = u * (v * w)$.
   b) $\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v$.
   c) $\partial^\alpha(u * v) = (\partial^\alpha u) * v = u * (\partial^\alpha v)$.
   d) If $u \in L^1(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$, then $\|u * v\|_{L^p} \leq \|u\|_{L^1} \|v\|_{L^p}$.
5. Show that any continuous function satisfying the Laplace equation in distribution sense must have the mean value property (therefore harmonic in the classical sense).
6. *a)* Find all fundamental solutions of the ordinary differential operators $\dfrac{\mathrm{d}}{\mathrm{d}x} + \lambda$ and $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \lambda$, where $\lambda \in \mathbb{C}$. How many of those are supported in $[0, \infty)$?
   b) Let $A_1, \ldots, A_m$ be $n \times n$ matrices. Describe all fundamental solutions of the ordinary differential operator
   $$\frac{\mathrm{d}^m}{\mathrm{d}x^m} + A_1 \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} + \ldots + A_{m-1} \frac{\mathrm{d}}{\mathrm{d}x} + A_m.$$
7. Write down a fundamental solution of
   a) the heat operator in $\mathbb{R}^n$.
   b) the wave operator in $\mathbb{R}^n$, $n = 1, 2, 3$.
   Are these operators hypoelliptic?

Date: Winter 2012.