MATH 581 ASSIGNMENT 2

DUE FRIDAY FEBRUARY 3

1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $K \subset \Omega$ compact, and $\mathcal{U}$ an open cover of K. Show that there exists a $\mathcal{D}(\Omega)$-partition of unity over K subordinate to $\mathcal{U}$, i.e., show that there exists a finite set $\{\chi_k\} \subset \mathcal{D}(\Omega)$ satisfying
   i) Each $\chi_k(\Omega) \subset [0, 1]$;
   ii) There is an open set $V \supset K$ such that $\sum_k \chi_k = 1$ on V;
   iii) For every k, there is $U \in \mathcal{U}$ such that $\text{supp}\chi_k \subset U$.
2. Let $\Omega \subset \mathbb{R}^n$ be an open set. Prove that
   a) $\mathcal{D}(\Omega)$ is dense in $C^k(\Omega)$ for $0 \leq k \leq \infty$;
   b) $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.
3. Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\varphi \neq 0$, and $\varphi(0) = 0$. In each of the following cases, decide if $\varphi_j \to 0$ as $j \to \infty$ in $\mathcal{D}(\mathbb{R})$. Does it hold $\varphi_j \to 0$ pointwise or uniformly?
   a) $\varphi_j(x) = j^{-1}\varphi(x - j)$;
   b) $\varphi_j(x) = j^{-n}\varphi(jx)$, where $n > 0$ is an integer.
4. Show that a map $f : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega')$ is continuous if and only if for every compact set $K \subset \Omega$ there exists a compact set $K' \subset \Omega'$ such that $f : \mathcal{D}(K) \to \mathcal{D}(K')$ is continuous.
5. Show that the following operations are continuous
   a) $L : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ where L is a linear differential operator with smooth coefficients;
   b) Pointwise multiplication $(u, v) \mapsto uv : \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$.
6. Show that in each of the following cases, f defines a distribution on $\mathbb{R}^2$, and find its order.
   a) $f(\varphi) = \int_{\mathbb{R}^2} |x|^{-1} e^{|x|^2} \varphi(x)\mathrm{d}x$;
   b) $f(\varphi) = \int_{\mathbb{R}} \varphi(s, 0)\mathrm{d}s$;
   c) $f(\varphi) = \int_0^1 \partial_1\varphi(0, s)\mathrm{d}s$.
7. Compute the derivatives of the following functions in the sense of distributions.
   a) The Heaviside step function $\theta(x)$ (1 if $x \geq 0$ and 0 otherwise);
   b) The sign function $\operatorname{sign} x$ (0 if $x = 0$ and $x/|x|$ otherwise);
   c) The absolute value $|x|$;
   d) $\log |x|$.

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