MATH 581 ASSIGNMENT 1

DUE WEDNESDAY JANUARY 25

Note: Give fairly detailed proofs.

1. Recall that in a topological space X, a *neighbourhood* of $x \in X$ is a set which contains an open set containing x. Prove the followings.
   a) A set is open iff it is a neighbourhood of each one of its points.
   b) A mapping $f : X \to Y$ between two topological spaces is continuous iff for each $x \in X$, the preimage under f of any neighbourhood of $f(x)$ is a neighbourhood of x.
2. Let X be a set. A *filter* $\mathcal{F}$ in X is a family of subsets of X satisfying
   i) $\varnothing \notin \mathcal{F}$,
   ii) $a, b \in \mathcal{F} \Rightarrow a \cap b \in \mathcal{F}$,
   iii) If $A \subset X$ is a set containing some $a \in \mathcal{F}$, then $A \in \mathcal{F}$.
   An example of a filter is the collection of all subsets of X which contain a given nonempty set $A \subset X$. This filter is called the *principal filter for A*. Prove the following.

   Lemma. *In a topological space, the set of all neighbourhoods of any given point $x \in X$ defines a filter in X (called the* neighbourhood filter *at x). Conversely, let X be a set and let a filter $\mathcal{N}(x)$ be given for each x, satisfying $A \in \mathcal{N}(x) \Rightarrow x \in A$. Then a topology on X can be defined as the collection of subsets $A \subset X$ with the property that $x \in A \Rightarrow A \in \mathcal{N}(x)$, i.e., the property characterizing open sets considered in 1a).*

3. We say that a filter $\mathcal{F}$ in X *converges* to $x \in X$ and write $\mathcal{F} \to x$ if $\mathcal{N}(x) \subset \mathcal{F}$ (read: $\mathcal{F}$ is finer than $\mathcal{N}(x)$), where $\mathcal{N}(x)$ is the neighbourhood filter at x. A *filter base* $\mathcal{B}$ in X is a family of nonempty subsets of X satisfying
   bf) For any $a, b \in \mathcal{B}$, there is $c \in \mathcal{B}$ such that $c \subset a \cap b$.
   Given a base $\mathcal{B}$, one can generate a unique filter, defined as the family of subsets of X which contains some subset belonging to $\mathcal{B}$ (in other words, the union of the principal filters for the elements of $\mathcal{B}$).
   a) Show that the family so generated indeed forms a filter.
   b) For a sequence $\{x_k\}$ in X, its *associated filter* is defined as the filter generated by the base consisting of the "tails" $T_n = \{x_n, x_{n+1}, \ldots\}$. Show that a sequence converges to x iff its associated filter converges to x.
   c) Let $g : X \to Y$ be a mapping between two sets. Then the *image* $g\mathcal{F}$ of a filter $\mathcal{F}$ in X under g is defined as the filter generated by the base $\{g(U) : U \in \mathcal{F}\}$. Show that the latter is indeed a filter base, and prove that if X and Y are topological spaces, g is continuous at $x \in X$ iff $\mathcal{F} \to x$ implies $g\mathcal{F} \to g(x)$. Recall that f is said to

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be *continuous at* $x \in X$ if the preimage under f of any neighbourhood of $f(x)$ is a neighbourhood of x.

4. Let $\Omega \in \mathbb{R}^n$ be a nonempty open set. Show that $C_0^\infty(\Omega)$ is infinite dimensional. Show also that $C(\Omega)$ is not locally bounded (i.e., there is no bounded neighbourhood of 0).
5. Let X and Y be two metric spaces, B a dense subset of X, and $f : B \to Y$ a uniformly continuous function. If Y is complete, there is a unique continuous function $\bar{f} : X \to Y$ which extends f, i.e., such that $\bar{f}(x) = f(x)$ for $x \in B$. Moreover, $\bar{f}$ is uniformly continuous, and if all given are linear, so is $\bar{f}$.