# Wave Maps

Morgane HENRY

16 April 2012

## 1 Introduction

In this project we will study non linear equations of the form

$$\Box u = f(t, x, u, Du) \tag{1}$$

where $\Box$ is called the d'Alembertian, the d'Alembert operator or the wave operator, $\Box = (-\partial_t^2 + \nabla)$ on $\mathbb{R} \times \mathbb{R}^m$ and $Du = (\partial_t u, \Delta u)$ is the vector space-time derivatives of $u$ . A wave map is a map $u : \mathbb{R} \times \mathbb{R}^m \to N$ from an n+1-dimensional Minkowski space to a Riemannian manifold $N$ that satisfies the wave equation with partial derivatives replaced by covariant derivatives. Wave equations can be viewed as wave maps specialized to the case when the manifold $N$ is Euclidean and the prototypes of geometrics wave equations are wave maps.

We can think of wave maps as describing the free motion of an n-dimensional surface in a non-Euclidean space; for instance the motion of a string that is constrained to lie on a sphere would be given as a wave map. They are one of the fundamental equations used to describe geometric motion, although they are not as well understood as other geometric flows as Ricci flows. There are still major gaps in our understanding of wave maps, notably in knowing when they can form singularities and if so what types of singularities. A major difficulty is that they are non linear, what is not obvious in equation (1) but will be shown in an example. One contrast between the wave equations and the other flows is that the wave map equation is time reversible and non dissipative; a surface never gains or looses energy.

The natural problem when we study wave maps is to consider initial data

$$(u_0, u_1) = (u(o, x), \partial_t u(0, x)) \in H^s \times H^{s-1}(\mathbb{R}^m; TN) \tag{2}$$

and study:
-the local well-posedness of the Cauchy problem: For what values of $s$ does initial value problem (1), (2), admit a unique local solution $u \in H^s$?
-the global well-posedness: For what values of $s$ does this solution extend for all time ?
-the global regularity: Does this solution preserve the regularity properties of the initial data?

The goal of this project is to introduce wave maps and to prove the local well-posedness for $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; TN)$ when $u$ of class $H^2$ and $m \leq 3$ and the global well-posedness for $m = 1$.

## 2 Some preliminaries

First, we have to introduce some notations and some concepts from geometry.

-An arbitrary point in the $(m+1)$-dimensional Minkowski space $\mathbb{R}\times\mathbb{R}^m$ will be denoted by $z = (t, x) = (x^\alpha)_{0\leq\alpha\leq m}$.

-The space-time derivatives of a function $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ are denoted by

$$Du = (\partial_t u, \nabla u) = (\partial_\alpha u)_{0\leq\alpha\leq m} = (u_t, (u_i)_{1\leq i\leq m}).$$

-On the Minkowski space $\mathbb{R} \times \mathbb{R}^m$ we have the pseudo-Riemannian metric

$$\eta = (\eta_{\alpha\beta}) = \eta^{-1} = (\eta^{\alpha\beta}) = diag(-1, 1, ..., 1).$$

We use the summation convention so we can rewrite the wave equation in the form

$$-\partial^\alpha\partial_\alpha u = 0$$

where $\partial^t = -\partial_t$ and $\partial^x = \partial_x$.

-On a $k$-dimensional Riemannian manifold $N$ with metric $g$ we let $\langle ., . \rangle_g$ denote the scalar product. In local coordinates we write $g = (g_{ab})$ and

$$|v|_g^2 = \sum_{j,k=1}^{n} g_{jk}(x)x^j x^k$$

.

-For a map $u : \mathbb{R} \times \mathbb{R}^m \to N$ we consider vector fields $V$ along $u$, which are vector fields having the property $V(z) \in T_{u(z)N}$ for all $z$, where $TpN$ is the tangent space of $N$ at any point $p \in N$. We also say that $V$ is a section of the pullback bundle $u^{-1}TN$. Any vector field $X$ on $N$ by composition with $u$ induces a vector field $V = X \circ u$. The components $\partial_\alpha u$ of $Du$ may be interpreted in this way.

-We denote $\mathcal{D}V = (\mathcal{D}_\alpha V)_{0\leq\alpha\leq m}$ as the covariant derivatives of any $V \in u^{-1}TN$. In local coordinates on $N$ we have:

$$\mathcal{D}_\alpha V^a = \partial_\alpha V^a + \Gamma^a_{bc}(u)(V^b, \partial_\alpha u^c)$$

where $\Gamma^a_{bc}$ are the Christoffel symbols of the metric $g$:

$$\Gamma^a_{bc} := \frac{1}{2}\sum_{l=1}^{n} g^{al}(\frac{\partial g_{bl}}{\partial x_c} + \frac{\partial g_{cl}}{\partial x_b} - \frac{\partial g_{bc}}{\partial x_l})$$

and $g^{al}$ is the inverse matrix to $g_{al}$.

-In general, covariant derivatives do not commute. Their commutator is measured the Riemann curvature tensor $R$

$$\mathcal{D}_\alpha \mathcal{D}_\beta V = \mathcal{D}_\beta \mathcal{D}_\alpha V + R(\partial_\alpha u, \partial_\beta u)V.$$

-Nash's embedding theorem states that any manifold may be isometrically embedded in some Euclidean space $\mathbb{R}^n$.

-If the manifold $N$ is isometrically embedded in some Euclidean space $\mathbb{R}^n$, the covariant derivative of a section $V \in u^{-1}TN$ at a point $z_0 \in \mathbb{R} \times \mathbb{R}^m$ is the standard derivative of $V$, viewed as a map $V : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ projected orthogonally into the tangent space $T_{u(z_0)}N$.

-In the following, for simplicity, we will consider that $N$ is compact.

# 3 Extrinsic and Intrinsic descriptions

## 3.1 Intrinsic description

We said earlier that wave maps are prototypes of geometric wave equations, satisfying the wave equation with partial derivatives replaced by covariant derivatives. So, in local coordinates, we obtain:

$$D_\alpha \partial^\alpha u^a = \partial^\alpha \partial_\alpha u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0 \tag{3}$$

$$u(0,x) = u_0(x) \;\partial_t u(0,x) = u_1(x)$$

where $u_0$ and $u_1$ are given. The equations (3) are the intrinsic description of the wave maps.

## 3.2 Extrinsic description

Wave maps can also be derived extrinsically by isometrically embedding $N \hookrightarrow \mathbb{R}^n$.

Indeed, thanks to Nash's theorem, $N$ is isometrically embedded in some Euclidean space $\mathbb{R}^n$ and the covariant derivative of a vector field $W$ at a point $p \in N$ is the standard derivative of $W$ in the ambient space, projected orthogonally into the tangent space $T_pN$. At a point $p \in N$, let $T_pN \subset T_p\mathbb{R}^n \simeq \mathbb{R}^n$. Let $\langle ., . \rangle$ denote the inner product in $\mathbb{R}^n$.

Then $u = (u^1, ..., u^n) : \mathbb{R} \times \mathbb{R}^m \to N \hookrightarrow \mathbb{R}^n$ is a wave map if at every point $z = (t, x)$ we have

$$\Box u(z) \perp T_{u(z)}N$$

where $\Box u(z) = (\Box u^1(z), ..., \Box u^n(z))$.

To understand this orthogonality relation in more explicit terms, we fix a point $z_0 \in \mathbb{R} \times \mathbb{R}^n$ and let $\bar{\nu}_{k+1}, ..., \bar{\nu}_n$ be an orthonormal frame for $T_p^\perp N$ depending smoothly on $p \in N$ for $p$ near $p_0 = u(z_0)$. Then we can find scalar functions $\lambda^l : \mathbb{R} \times M \to \mathbb{R},\; k < l \leq n$, such that near $z = z_0$

$$\Box u = \lambda^l \nu^l$$

holds, where $\nu^l = \bar{\nu}_l \circ u$.
Then, since $\langle \partial_\alpha u, \nu_l \rangle = 0$, we have

$$\begin{aligned} \lambda^l &= \langle \Box u, \nu_l \rangle \\ &= -\partial^\alpha \langle \partial_\alpha u, \nu_l \rangle + \langle \partial_\alpha u, \partial^\alpha (\nu_l) \rangle \\ &= \langle \partial_\alpha u, d\bar{\nu}_l(u).\partial^\alpha u \rangle \\ &= A^l(u)(\partial_\alpha u, \partial^\alpha u) \end{aligned}$$

where $A^l$ is the second fundamental form of $N$ with respect to $\bar{\nu}_l$. So, if we use the extrinsic concept to derive the wave map equation, we obtain the following form

$$\Box u = A(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N, \tag{4}$$

where $A = A^l \bar{\nu}_l$ is the second fundamental form of $N$.

## 3.3 Example

Let $N = S^k \subset \mathbb{R}^{k+1}$, we want to derive the extrinsic wave map equation. At any point $p \in N$ the outer unit normal is given by $\bar{\nu}(p) = p$. So, if $u$ is a wave map, there exists a scalar function $\lambda : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\Box u = \lambda u.$$

Using the relation $\langle u, u \rangle \equiv 1$, we obtain

$$\begin{aligned} \lambda &= \langle \Box u, u \rangle \\ &= -\partial^\alpha \langle \partial_\alpha u, u \rangle + \langle \partial_\alpha u, \partial^\alpha u \rangle \\ &= \langle \partial_\alpha u, \partial^\alpha u \rangle \\ &= |\nabla u|^2 - |u_t|^2 \end{aligned}$$

and equation (4) simplifies to

$$\Box u = (|\nabla u|^2 - |u_t|^2) u.$$

Thanks to this equation we can easily see that the wave map equation is nonlinear.

$\Box$

## 3.4 Equivalence of the two descriptions

The wave map system is a "geometric" wave equation in the sense that it is invariant under coordinate changes.
In fact, let $\phi : N \to \tilde{N}$ be totally geodesic, that is, $\phi$ satisfies $\mathcal{D} d\phi = 0$, and let $u : \mathbb{R} \times \mathbb{R}^m \to N$ be a wave map. Then $\tilde{u} = \phi \circ u : \mathbb{R} \times \mathbb{R}^m \to \tilde{N}$satisfies

$$\begin{aligned} \mathcal{D}^\alpha \partial_\alpha \tilde{u} &= \mathcal{D}^\alpha (d\phi \circ u) \partial_\alpha u \\ &= \mathcal{D} d\phi(u)(\partial^\alpha u, \partial_\alpha u) + d\phi(u) \mathcal{D}^\alpha \partial_\alpha u \\ &= 0. \end{aligned}$$

So we proved that $\tilde{u}$ is also a wave map.
In particular, these considerations apply if $\phi : N \to \tilde{N}$ is an isometry. So, because $N$ is isometrically embedded in some Euclidean space $\mathbb{R}^n$ thanks to Nash's embedding theorem, the extrinsic and intrinsic concepts are completely equivalent.

# 4 Variational Formulation

Solutions of the equations (3) and (5) can be characterized variationally.

For wave maps $u : \mathbb{R} \times \mathbb{R}^m \to N$ we can formulate the corresponding variational principles either intrinsically, viewing $N$ as an abstract Riemannian manifold with metric $g = (g_{ij})_{1 \le i,j \le k}$, or extrinsically, by isometrically embedding $N \subset \mathbb{R}^n$. Instead of requiring that $u$ be a maximizer or a minimizer for the action $\mathcal{A}$, as we do for geodesics, we require the slightly weaker condition that $u$ be critical point for the action $\mathcal{A}$.

## 4.1 Intrinsic description

Intrinsically, the Lagrangian is given by

$$\mathcal{L}(u) = \frac{1}{2}\langle \partial^\alpha u, \partial_\alpha u\rangle_g$$

and the action by

$$\mathcal{A}(u; Q) = \int_Q \mathcal{L}(u) dz.$$

Compactly supported variations of $u$ are defined in terms of local coordinates $(y^1, ..., y^k)$ on $(N, g)$ by

$$u^a_\varepsilon(z) = u^a(z) + \varepsilon\phi^a(z)$$

where $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^m; \mathbb{R}^k)$, and where we assume that the image $u(supp(\phi))$ is strictly contained in a coordinate chart $U$ on $N$. For $z$ fixed and $\varepsilon$ small, $\varepsilon \to u_\varepsilon(z)$ is a well-defined curve on $N$ and $\phi(.) \in T_{u(.)}N$. The variation of $\mathcal{A}$ is given by

$$\frac{d}{d\varepsilon}\mathcal{A}(u_\varepsilon; Q) \mid_{\varepsilon=0} = \int_Q \langle \partial^\alpha u, \mathcal{D}_\alpha \phi\rangle_g dz = -\int_Q \langle \mathcal{D}_\alpha \partial^\alpha u, \phi\rangle_g dz,$$

where we used the identity

$$\partial_\alpha \langle \phi, \psi\rangle_g = \langle \mathcal{D}_\alpha \phi, \psi\rangle + \langle \phi, \mathcal{D}_\alpha \psi\rangle$$

for any pair of vector fields $\phi$, $\psi$. Therefore, $u$ is a stationary point of $\mathcal{A}$ with respect to compactly supported variations if and only if $u$ solves the wave map equation

$$\mathcal{D}_\alpha \partial^\alpha u = 0.$$

If we introduce local coordinates on $N$, then we can express the Lagrangian as

$$\mathcal{L}(u) = \frac{1}{2} g_{ab}(u) \partial^\alpha u^a \partial_\alpha u^b$$

and the variation of $\mathcal{A}$ as

$$\int_Q (-\partial^\alpha (g_{ab}(u) \partial_\alpha u^b) \phi^a + \frac{1}{2} g_{ab,c}(u) \partial^\alpha u^a \partial_\alpha u^b \phi^c) dz$$

where $g_{ab,c} = \frac{\partial}{\partial x^c} g_{ab}$ This implies that the equations are

$$\begin{aligned} 0 &= -\partial^\alpha (g_{ab}(u) \partial_\alpha u^b) + \tfrac{1}{2} g_{db,a}(u) \partial^\alpha u^d \partial_\alpha u^b \\ &= -g_{ab}(u) \Box u^b - g_{ab,e}(u) \partial_\alpha u^b \partial^\alpha u^e + \tfrac{1}{2} g_{db,a}(u) \partial^\alpha u^d \partial_\alpha u^b \end{aligned}$$

for $1 \leq a \leq k$. Multiplying by $g^{ca}(u)$, and observing that

$$\Gamma^c_{db} = \frac{1}{2} g^{ca} (-g_{db,a} + g_{ba,d} + g_{ad,b})$$

we obtain the intrinsic formulation

$$\Box u^c + \Gamma^c_{db}(u) \partial^\alpha u^d \partial_\alpha u^b = 0$$

of the wave map system.

## 4.2 Extrinsic description

Extrinsically, for $u : \mathbb{R} \times \mathbb{R}^m \to N \hookrightarrow \mathbb{R}^n$ the Lagrangian and the action are given by

$$\mathcal{L}(u) = \frac{1}{2} \langle \partial_\alpha u, \partial^\alpha u \rangle = \frac{1}{2} (|\nabla u|^2 - |u_t|^2)$$

and

$$\mathcal{A}(u, Q) = \int_Q \mathcal{L}(u) dz.$$

We consider compactly supported variations $u_\varepsilon : \mathbb{R} \times \mathbb{R}^m \to N \hookrightarrow \mathbb{R}^n$, for $|\varepsilon| < |\varepsilon_0|$, so that $u_\varepsilon = u$ at $\varepsilon = 0$ as well as away from some compact set for $\varepsilon > 0$. Since $N$ is compact, there exists a tubular neighborhood, which is an open set around $N$ resembling the normal bundle, $U_\delta(N)$ of uniform width $\delta > 0$ and a smooth nearest neighbor projection $\pi_N : U_\delta(N) \to N$. Given $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n)$ , for $|\varepsilon| < \frac{\delta}{||\phi||_\infty}$ we may let $u_\varepsilon = \pi_N(u + \varepsilon\phi)$. Now suppose that $u$ is stationary for $\mathcal{A}$ with respect to such variations then

$$0 = \frac{d}{d\varepsilon} \mathcal{A}(u_\varepsilon, Q)|_{\varepsilon=0} = \int_Q \langle \partial^\alpha [d\pi_N(u) \circ \phi], \partial_\alpha u] \rangle dz$$

for any $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n)$ and $supp(\phi) \subset Q$. By integrating by parts and because $\langle d\pi_N(u).\phi, \Box u \rangle = \langle \phi, d\pi_N(u).\Box u \rangle$ we deduce $d\pi_N(u).\Box u = 0$. So we obtain the extrinsic form $\Box u \perp T_u N$. Conversely, if $u : \mathbb{R} \times \mathbb{R}^m \to N \hookrightarrow \mathbb{R}^n$ solves the above relation, it is immediate that $u$ is stationary for $\mathcal{A}$.

Space-time symmetry and other invariances of $\mathcal{A}$ implies that solutions to wave map equations satisfy conservation laws.

# 5 Local and Global well-posedness

In this section we want to prove the following theorem:

**Theorem 5.1** *Suppose $m \leq 3$. Then for any data*

$$(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; TN)$$

*there exists a unique local solution $u$ of class $H^2$. If $m = 1$, the solution extends uniquely for all time.*

*Proof*
To prove the existence of a solution we'll first prove that if we approximate the initial data by smooth functions, there is a smooth solution to equation (7.1). Then, using an a priori bound, we'll prove that the smooth functions converge to a solution in $H^2$. To find an a priori bound we'll use the following form of the wave map equation

$$\Box u = A(u)(Du, Du) \perp T_u N. \tag{5}$$

which implies the energy momentum conservation

$$0 = \langle \Box u, u_t \rangle = \frac{1}{2}\partial_t |Du|^2 - div\langle \nabla u, u_t \rangle,$$

and the energy identity

$$E(u(t)) := \frac{1}{2}||Du(t)||^2_{L^2(\mathbb{R}^m)} = \frac{1}{2}||Du(0)||^2_{L^2(\mathbb{R}^m)}.$$

Let apply $\partial$, any first-order spatial derivative, to equation (5)

$$\Box(\partial u) = \partial[A(u)(\partial_\alpha u, \partial^\alpha u)] = dA(u)(\partial u, \partial_\alpha u, \partial^\alpha u) + 2A(u)(\partial_\alpha \partial u, \partial^\alpha u)$$

Multiplying the resulting equation by $\partial_t \partial u$ and using that $\langle u_t, A(u)(.,.)\rangle = 0$ by orthogonality, we have

$$\langle \partial u_t, A(u)(\partial_\alpha \partial u, \partial^\alpha u)\rangle = -\langle u_t, dA(u)(\partial u, \partial_\alpha \partial u, \partial^\alpha u)\rangle$$

So we obtain, because $N$ is compact

$$\begin{array}{rcl} \frac{d}{dt}E(\partial u(t)) & = & \int_{\mathbb{R}^m} \langle \Box(\partial u), \partial u_t \rangle dx \\ & \leq & C||dA(u)||_{L^\infty} \int_{\mathbb{R}^m} |Du(t)|^3 |D^2 u(t)| dx. \end{array}$$

For $m = 1, 2$, or 3 space dimensions, we have by Sobolev's embedding

$$\int_{\mathbb{R}^m} |Du(t)|^3 |D^2 u(t)| dx \leq C||Du(t)||^{4-\alpha}_{L^2} ||D^2 u(t)||^\alpha_{L^2}$$

where $\alpha = 2, 3$, or 4 if $m = 1, 2$, or 3, respectively. So we arrive at a Gronwall-type inequality

$$\frac{d}{dt}||Du(t)||^2_{L^2} \leq C||Du(t)||^\alpha_{L^2}, \tag{6}$$

which implies a local in time $H^2$ a priori bound. If $m = 1$, we have $\alpha = 2$, and thus the $H^2$ bound is global.

Now that we have an a priori bound we approximate initial data $(u_0, u_1)$ by smooth data $(u_0^l, u_1^l) \in C^\infty(\mathbb{R}^m, TN)$ such that

$$(u_0^l, u_1^l) \to (u_0, u_1) \ in \ H^2 \times H^1(\mathbb{R}^m; \mathbb{R}^n \times \mathbb{R}^n)$$

as $l \to \infty$.
We consider the system

$$\begin{cases} \Box u^l = f \\ u^l(0) = u_0, \\ u_t^l(0) = u_1. \end{cases}$$

It is symmetric hyperbolic, so by reducing it to a first order system we can easily prove that there is a smooth solution.
Then, we consider the following system

$$\begin{cases} \Box u^{l+1} = f(u^l) \\ u^{l+1}(0) = u_0, \\ u_t^{l+1}(0) = u_1, \end{cases}$$

for $l = 0, 1, ...$ and $f(u^l) = A(u^l)(Du^l, Du^l)$, continuous from $H^s$ to $H^{s-1}$ for $s > \frac{m}{2} + 1$.
Moreover, if $u$ and $v$ are in $H^s$ we have

$$\begin{array}{rcl} ||f(u) - f(v)||_{H^s} & \leq & C \int_0^t ||f(u) - f(v)||_{H^{s-1}} d\tau \\ & \leq & CT||u - v||_{H^s}, \end{array}$$

where $T$ doesn't depend on $l$.
Because $f$ is a contraction, $u^l$ converges to $u$ in $H^s$ by the Banach fixed-point theorem. Besides, thanks the a priori bound (6) we can obtain higher regularity so $u^l$ extends as smooth solution to (5) on a time interval whose length $T$ depends only on the norm of the data in $H^2 \times H^1(\mathbb{R}^m; TN)$.
We know that $u^l$ is in a $L^\infty([0,T], H^2)$-ball, which is $L^2H^2$-weakly compact, of radius $M < \infty$ where $M$ is such that $||u^l||_{([0,T],H^2)} \leq M$. So we can extract a subsequence which converges weakly to $u$ in $L^2H^2$. Furthermore, for $K \subset \mathbb{R}^m$ we can extract a subsequence which converges to $u$ in $H^1(K)$.
Besides, we have

$$\Box u^l = A(u^l)(Du^l, Du^l)$$

and

$$A(u^l)(Du^l, Du^l) \to A(u)(Du, Du)$$

when $l \to \infty$, up to a subsequence, thanks to the a priori estimate (6). Moreover, for $v \in \mathcal{D}([0,T], \mathbb{R}^m)$, we have

$$\begin{array}{rcl} \langle \Box u^l, v \rangle & = & \int \int_{[0,T] \times \mathbb{R}^m} u_t^l v_t - \nabla u^l \nabla v \\ & = & \int \int_{[0,T] \times \mathbb{R}^m} A(u^l)(Du^l, Du^l), \end{array}$$

which is finite because $|A(u^l)(Du^l, Du^l)| < |Du^l|^2$ and $u^l$ is smooth. So, thanks to the Gronwall-type inequality (6), we obtain

$$u_t^l \to u_t$$

and

$$\nabla u^l \to \nabla u$$

in $L^2$, up to a subsequence, and $u$ satisfies the equation (5) in the sense of distribution and $u$ is in $H^2$.
So we proved the existence of a solution in $H^2$.

Uniqueness on $H^2$ solutions follows from the energy inequality. Recalling that $\langle u_t, A(u)(.,.)\rangle = 0$,if $u$ and $v$ are $H^2$ solutions we have

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}||D(u-v)(t)||_{L^2}^2 &\leq \textstyle\int_{\mathbb{R}^m} \langle u_t - v_t, A(u)(Du, Du) - A(v)(Dv, Dv)\rangle dx \\
&= \textstyle\int_{\mathbb{R}^m} (\langle u_t, A(u)(Dv, Dv) - A(v)(Dv, Dv)\rangle \\
&\quad -\langle v_t, A(u)(Du, Du) - A(v)(Du, Du)\rangle) dx \\
&\leq C \textstyle\int_{\mathbb{R}^m} |A(u) - A(v)||Du - Dv|[|Du|^2 + |Dv|^2] dx
\end{aligned}$$

This above inequality implies

$$\frac{d}{dt}||D(u-v)||_{L^2} \leq C||(|u-v|)(|Du|^2 + |Dv|^2)||_{L^2}$$

and by Sobolev's embeding we have

$$\frac{d}{dt}||D(u-v)||_{L^2} \leq C||u-v||_{L^6}(||Du||_{L^6}^2 + ||Dv||_{L^6}^2) \leq C||D(u-v)||_{L^2},$$

and uniqueness follows by Gronwall's inequality, because $u$ and $v$ satisfy the same initial data equations.

□

# References


[1] M. Struwe, J. Shatah, *Geometric Wave Equations*, Courant Institute of Mathematical Sciences, New York, New York, American Mathematical Society, Providence, Rhode Island, (2000).

[2] T. Tao, *Wave Maps*, UCLA, Los Angeles, CA, USA.

[3] D. Tataru, *The Wave Maps Equation*, American Mathematical Society, Providence, Rhode Island, (2004).