An Introduction to Hamilton's Ricci Flow

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Abstract

In this project we study the Ricci flow equation introduced by Richard Hamilton in 1982. The Ricci flow exhibits many similarities with the heat equation: it gives manifolds more uniform geometry and smooths out irregularities. The Ricci flow has proven to be a very useful tool in understanding the topology of arbitrary Riemannian manifolds. In particular, it was a primary tool in Grigory Perelman's proof of Thurston's geometrization conjecture, of which Poincaré conjecture is a special case.

Keywords: Hamilton's Ricci flow, manifold, Riemannian metric, soliton

1. Introduction

Geometric flows, as a class of important geometric partial differential equations, have been highlighted in many fields of theoretical research and practical applications. They have been around at least since Mullin's paper [1] in 1956, which proposed the curve shortening flow to model the motion of idealized grain boundaries. In 1964 Eells and Sampson [2] introduced the harmonic map heat flow and used it to prove the existence of harmonic maps into targets with nonpositive sectional curvature. Motivated by the work of Eells and Sampson, in 1975 Richard Hamilton [3] continued the study of harmonic map heat flow by considering manifolds with boundary. [4]

In the late seventies, William Thurston suggested a classification of three-dimensional manifolds, which became known as the geometrization conjecture. Thurston's conjecture stated that any compact 3-manifold can be decomposed into one or more types (out of 8) of components with homogeneous geometry, including a spherical type, and thus implied the Poincaré conjecture. In 1982 Thurston won a Fields Medal for his contributions to topology. That year Hamilton [5] introduced the so-called Ricci equation, which he suspected could be relevant for solving Thurston's conjecture. The Ricci flow equation has been called the heat equation for metrics, due to its property of making metrics "better". A large number of innovations that originated in Hamilton's 1982 and subsequent papers have had a profound influence on modern geometric analysis. The most recent result is Perelman's proof of Thurston's conjecture.

The structure of the project is as follows. In Section 2 we recall some basic facts in Riemannian geometry. In section 3 we introduce the Ricci flow equation and prove the short-time existence for the Ricci flow with an arbitrary smooth initial metric. In section 4, we describe Ricci solitons.

2. Notational Preliminaries

2.1. Metrics and Connections

Throughout this project, we adopt Einstein's summation convention on repeated indices.

Let $\mathcal{M}$ be a manifold and let p be a point of $\mathcal{M}$. Then $T\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$ and $T_p\mathcal{M}$ is the tangent space at p. Similarly, $T^*\mathcal{M}$ denotes the cotangent bundle of $\mathcal{M}$ and $T_p^*\mathcal{M}$ is the cotangent space at p.

Definition 2.1. Let $\mathcal{M}$ be an n-dimensional smooth manifold. A *Riemannian metric* g on $\mathcal{M}$ is a smooth section of $T^*\mathcal{M} \otimes T^*\mathcal{M}$ defining a positive definite symmetric bilinear form on $T_p\mathcal{M}$ for each $p \in \mathcal{M}$.

Let $\{x^i\}_{i=1}^n$ be local coordinates in a neighbourhood U of some point of $\mathcal{M}$. In U the vector fields $\{\partial_i\}_{i=1}^n$ form a local basis for $T\mathcal{M}$ and the 1-forms $\{dx^i\}_{i=1}^n$ form a dual basis for $T^*\mathcal{M}$, that is,

$$dx^i(\partial_j) = \delta_{ij}. \tag{2.1}$$

The metric g may then be written in local coordinates as

$$g = g_{ij} dx^i \otimes dx^j, \tag{2.2}$$

where $g_{ij} := g(\partial_i, \partial_j)$. We denote by $g^{-1} = (g^{ij})$ the inverse of the positive definite matrix (g_{ij}). The pair $(\mathcal{M}, g)$ is called a *Riemannian manifold.*

Proposition 2.1. *Any smooth manifold admits a Riemannian metric.*

Proof. Take a covering of $\mathcal{M}$ by coordinate neighbourhoods $\{U_\alpha\}$ and a partition of unity $\{\varphi_i\}$ subordinate to the covering. On each open set U_α we have a metric

$$g_\alpha = dx^i \otimes dx^i \tag{2.3}$$

in the local coordinates. Define

$$g = \sum \varphi_i g_{\alpha(i)}. \tag{2.4}$$

This sum is well-defined because the supports of φ_i are locally finite. Since $\varphi_i \geq 0$ at each point, every term in the sum is positive definite or zero, but at least one is positive definite so the sum is positive definite. □

A Riemannian metric on $\mathcal{M}$ allows us to measure lengths of smooth paths in $\mathcal{M}$. We define a distance function on $\mathcal{M}$ by setting $d(p, q)$ equal to the infimum of the lengths of smooth paths from p to q.

Definition 2.2. Given a smooth map $F : \mathcal{N} \to \mathcal{M}$ and a metric g on $\mathcal{M}$, we can pull back g to a metric on $\mathcal{N}$

$$(F^*g)(V, W) := g(F_*V, F_*W), \tag{2.5}$$

where $F_* : T\mathcal{N} \to T\mathcal{M}$ is the tangent map. If F is a diffeomorphism, then the pull-back of contravariant tensors is defined as the push forward by F^{-1}.

Theorem 2.2. *Given a Riemannian metric g on $\mathcal{M}$, there exists a unique torsion-free connection on $T\mathcal{M}$ making g parallel, i.e., there is a unique $\mathbb{R}$-linear mapping $\nabla : C^\infty(T\mathcal{M}) \to C^\infty(T^*\mathcal{M} \otimes T\mathcal{M})$ satisfying the Leibniz formula*

$$\nabla(fX) = df \otimes X + f\nabla X, \tag{2.6}$$

and the following conditions for all vector fields X and Y:

(i) g orthogonal

$$d(g(X,Y) = g(\nabla X, Y) + g(X, \nabla Y); \tag{2.7}$$

(ii) torsion-free

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0, \tag{2.8}$$

where $\nabla_X Y := \nabla Y(X)$ is the covariant derivative of Y and $[X,Y]$ is the Lie bracket acting of functions.

The above connection is called the *Levi-Civita connection* of the metric (or Riemannian covariant derivative).

In local coordinates $\{x^i\}_{i=1}^n$ the Levi-Civita connection is given by $\nabla_{\partial_i}(\partial_j) = \Gamma_{ij}^k \partial_k$, where the Christoffel symbols Γ_{ij}^k are the smooth functions

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{kj} + \partial_j g_{il} - \partial_l g_{ij}). \tag{2.9}$$

2.2. Curvature

Let $(\mathcal{M}, g)$ be a Riemannian manifold. The *Riemannian curvature* $(1,3)$-tensor on $\mathcal{M}$ is given by

$$\mathrm{Rm}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.10}$$

It is easy to check that

$$\mathrm{Rm}(fX,Y)Z = \mathrm{Rm}(X,fY)Z = \mathrm{Rm}(X,Y)(fZ) = f\mathrm{Rm}(X,Y)Z, \tag{2.11}$$

thus Rm is indeed a tensor. To simplify the notations, it is useful to define

$$\nabla^2_{X,Y} Z := \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z, \tag{2.12}$$

so that

$$\mathrm{Rm}(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z. \tag{2.13}$$

Remark. The Lie bracket measures the noncommutativity of the directional derivatives acting on functions, whereas Rm measures the noncommutativity of covariant differentiation acting on vector fields.

The components of the Riemannian curvature tensor Rm are defined by

$$\mathrm{Rm}(\partial_i, \partial_j)\partial_k := R_{ijk}^l \partial_l, \tag{2.14}$$

where

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}. \tag{2.15}$$

Using the metric tensor g we can transform Rm into a $(0,4)$-tensor by

$$\mathrm{Rm}(X,Y,Z,W) = g(\mathrm{Rm}(X,Y)W,Z) \tag{2.16}$$

In local coordinates this gives

$$\mathrm{Rm}(\partial_i,\partial_j,\partial_k,\partial_l) = R_{ijkl} = g_{pl}R^p_{ijk}. \tag{2.17}$$

It can be verified that the Riemannian curvature tensor Rm satisfies the following properties:

(i) anti-symmetry: $\mathrm{Rm}_{ijkl} = -\mathrm{Rm}_{jikl}$, $\mathrm{Rm}_{ijkl} = -\mathrm{Rm}_{ijlk}$, $\mathrm{Rm}_{ijkl} = \mathrm{Rm}_{klij}$;

(ii) first Bianchi identity: $\mathrm{Rm}_{ijkl} + \mathrm{Rm}_{jkil} + \mathrm{Rm}_{kijl} = 0$;

(iii) second Bianchi identity: $\nabla_i \mathrm{Rm}_{jklm} + \nabla_j \mathrm{Rm}_{kilm} + \nabla_k \mathrm{Rm}_{ijlm} = 0$.

There are other important related curvatures. If $P \subset T_p\mathcal{M}$ is a 2-plane, then the *sectional curvature* of P is defined by

$$K(P) := g(\mathrm{Rm}(e_1,e_2)e_2,e_1), \tag{2.18}$$

where $\{e_1,e_2\}$ is an orthonormal basis of P. This definition is independent of the choice of such a basis.

A Riemannian manifold is said to have *constant sectional curvature* if $K(P)$ is the same for all $p \in \mathcal{M}$ and all 2-planes $P \subset T_p\mathcal{M}$. One can show that a manifold $(\mathcal{M}, g)$ has constant sectional curvature λ if and only if

$$R_{ijkl} = \lambda(g_{ik}g^{jl} - g_{il}g_{jk}). \tag{2.19}$$

For instance, the sphere of radius r in $\mathbb{R}^n$ has constant sectional curvature $1/r^2$; $\mathbb{R}^n$ with the Euclidean metric has constant curvature 0; the Poincaré hyperbolic disk $\mathbb{H}^n$, which is given by the unit disk with the metric

$$g = \frac{4dx_1^2 + \cdots + dx_n^2}{(1-|x|^2)^2}, \tag{2.20}$$

has constant sectional curvature -1 [6].

The *Ricci curvature* $(0,2)$-tensor Ric is formed by taking the trace of the Riemannian curvature tensor,

$$\mathrm{Ric}(Y,Z) = tr(\mathrm{Rm}(\cdot,X)Y). \tag{2.21}$$

The components of the Ricci curvature, denoted by $R_{ij} = \mathrm{Ric}(\partial_i,\partial_j)$, are given by

$$R_{ij} = R^k_{kij}. \tag{2.22}$$

The Ricci curvature is a symmetric bilinear form on $T\mathcal{M}$, given in local coordinates by

$$\mathrm{Ric} = R_{ij}\, dx^i \otimes dx^j. \tag{2.23}$$

We say that $\mathrm{Ric} \geq k$ (or $\leq k$) if all the eigenvalues of Ric are $\geq k$ (or $\leq k$).

The *scalar curvature* function $R : \mathcal{M} \to \mathbb{R}$ is given by the metric trace of the Ricci tensor:

$$R = tr(\mathrm{Ric}(\cdot,\cdot)) = g^{ij}R_{ij}. \tag{2.24}$$

3. Ricci Flow Equation

3.1. Definition of Ricci Flow

The Ricci flow is a geometric evolution equation defined on Riemannian manifolds $(\mathcal{M}, g)$. The geometry of $(\mathcal{M}, g)$ is altered by changing the metric g via a second-order nonlinear PDE on symmetric $(0,2)$-tensors:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{3.1}$$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics $g(t)$ on a smooth manifold $\mathcal{M}$, defined on a time interval $I \subset \mathbb{R}$, and satisfying equation (3.1).

To get the insight into the Ricci flow, let us consider equation (3.1) in harmonic coordinates $\{x^i\}_{i=1}^n$ about p, i.e., in a coordinate system whose coordinate functions x^i are harmonic. By Lemma 3.32 in [7, p. 92], the Ricci tensor in harmonic coordinates is given by

$$R_{ij} = \mathrm{Ric}(\partial_i, \partial_j) = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g), \tag{3.2}$$

where $\Delta(g_{ij})$ denotes the Laplacian of the component g_{ij} of the metric regarded as a scalar function, and Q is a quadratic form in the inverse g^{-1} and the first derivative ∂g. In particular, Q is a lower order term in the derivatives of g. As a corollary, the Ricci flow in harmonic coordinates takes the form of a system of nonlinear heat equations for the components of the metric tensor:

$$\frac{\partial}{\partial t} g_{ij} = \Delta(g_{ij}) + Q_{ij}(g^{-1}, \partial g). \tag{3.3}$$

Because of the minus sign in the front of the Ricci tensor in the equation (3.1), the solution metric to the Ricci flow shrinks in positive Ricci curvature direction and expands in the negative Ricci curvature direction. For example, on the sphere S^2, any metric of positive Gaussian curvature will shrink to a point in finite time.

The Ricci flow does not in general preserve volume, so it is often useful to consider the normalized Ricci flow defined by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n}\sigma_g g_{ij}, \tag{3.4}$$

where σ_g denotes the average scalar curvature

$$\sigma_g = \frac{\int_{\mathcal{M}} R_g \,\mathrm{d}\mu_g}{\int_{\mathcal{M}} \mathrm{d}\mu_g}. \tag{3.5}$$

Under this normalized flow, which is equivalent to the unnormalized Ricci flow (3.1) by reparametrizing in time t and scaling the metric in space by a function of t, the volume of the solution metric is constant in time.

We will prove the short-time existence and uniqueness results for the Ricci flow equation, and it would be nice if we could transfer these results to the normalized Ricci flow equation as well. Hamilton showed in [5, Section 3] that there is a bijection between solutions of the unnormalized and normalized Ricci flow equations.

To see that there is a conversion from (3.1) to (3.4), let $g(t)$ be a solution of the unnormalized equation. For $\psi(t)$ that is yet to be determined, set $\tilde{g}(t) = \psi(t)g(t)$, and let tilde variables represent all of the geometric quantities associated with this metric. By examining the coordinate expressions for the volume form $\mathrm{d}\mu = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n$, we see that

$$\mathrm{d}\tilde{\mu} = \psi(t)^{n/2}\mathrm{d}\mu, \qquad \tilde{R}_{ij} = R_{ij}, \qquad \tilde{R} = \frac{1}{\psi}R, \qquad \tilde{\sigma} = \frac{1}{\psi}\sigma, \tag{3.6}$$

so setting

$$\psi(t) = \left(\int_{\mathcal{M}} \mathrm{d}\mu(t)\right)^{-2/n} \tag{3.7}$$

yields

$$\int_{\mathcal{M}} \mathrm{d}\tilde{\mu}(t) = 1. \tag{3.8}$$

We now choose a new time scale $\tilde{t} = \int \psi(t)\mathrm{d}t$. Since $\tilde{t}$ is a strictly increasing function of t, we can invert it to $t(\tilde{t})$ and change time variables to $\tilde{t}$ on the entire interval of existence of the original solution $g(t)$. Assuming the fact that

$$\frac{\partial}{\partial t}\log\mu = -R \tag{3.9}$$

for solutions of the Ricci flow, we compute

$$\frac{\partial}{\partial t}\log\int \mathrm{d}\mu = -\sigma, \qquad \frac{\partial}{\partial t}\log\psi = \frac{2}{n}\sigma. \tag{3.10}$$

Then it follows that

$$\frac{d}{d\tilde{t}}\tilde{g}_{ij} = \frac{\partial}{\partial t}g_{ij} + \left(\frac{\partial}{\partial t}\log\psi\right)g_{ij} = -2\tilde{R}_{ij} + \frac{2}{n}\tilde{\sigma}\tilde{g}_{ij}. \tag{3.11}$$

3.2. Short-Time Existence

Since the Ricci flow system of equations is only weakly parabolic, the short-time existence does not follow directly from standard parabolic theory. In his seminal paper [5] Hamilton showed that for any C^∞ metric g_0 on a closed 3-manifold $\mathcal{M}$, there exists a unique solution $g(t)$, $t \in [0,\varepsilon)$ for some $\varepsilon > 0$, to the Ricci flow equation (3.1) satisfying $g(0) = g_0$. Hamilton's original proof relied on the sophisticated machinery of the Nash-Moser inverse function theorem. Shortly thereafter, Dennis DeTurck [8] [9] proposed a simplified proof by showing that the Ricci flow is equivalent to a strictly parabolic system.

Theorem 3.1 (Hamilton, DeTurck. Short-time existence)**.** *If $\mathcal{M}^n$ is a closed Riemannian manifold and if g_0 is a C^∞ Riemannian metric, then there exists a unique smooth solution $g(t)$ to the Ricci flow defined on some time interval $[0,\varepsilon)$, $\varepsilon > 0$, with $g(0) = g_0$.*

The proof of Theorem 3.1 (called DeTurck's trick) consists in finding a strictly parabolic flow which is equivalent to the Ricci flow, i.e., where the principal symbol of the second-order operator on the right-hand side is positive definite. As we will soon see, the principal symbol of the nonlinear differential operator $-2\mathrm{Ric}(g)$ of the metric g is positive semidefinite and has a non-trivial kernel, which is due to the diffeomorphism invariance of the Ricci tensor. For this reason the Ricci flow is only weakly parabolic.

Before proving the Theorem 3.1, we need to establish how curvature tensors and connections change as the metric evolves. We assume that

$$\frac{\partial}{\partial t} g_{ij} = h_{ij}, \tag{3.12}$$

where h is a symmetric $(0,2)$-tensor.

Lemma 3.2. *The variation of the Christoffel symbols is given by*

$$\frac{\partial}{\partial t} \Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}). \tag{3.13}$$

Proof. Recall that

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \tag{3.14}$$

hence

$$\begin{aligned} \frac{\partial}{\partial t} \Gamma_{ij}^{k} &= \frac{1}{2} \frac{\partial}{\partial t} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &+ \frac{1}{2} g^{kl} \left(\partial_i \left(\frac{\partial}{\partial t} g_{jl} \right) + \partial_j \left(\frac{\partial}{\partial t} g_{il} \right) - \partial_l \left(\frac{\partial}{\partial t} g_{ij} \right) \right) \end{aligned} \tag{3.15}$$

In geodesic coordinates[1] centered at $p \in \mathcal{M}$, one has $\Gamma_{ij}^{k}(p) = 0$. It follows that $\partial_i A_{jk} = \nabla_i A_{jk}$ at p for any tensor A. In particular, $\partial_i g_{jk}(p) = 0$ for all i, j, k. Thus we obtain

$$\frac{\partial}{\partial t} \Gamma_{ij}^{k}(p) = \frac{1}{2} g^{kl} (\nabla_i h_{lj} + \nabla_j h_{il} + \nabla_l h_{ij})(p) \tag{3.16}$$

Since both sides of this equation are components of tensors, the result holds as a tensor equation, i.e., it is true in any coordinate system and at any point. □

Since the Riemann curvature tensor is defined solely in terms of the Levi-Civita connection, we can readily compute its evolution.

Lemma 3.3. *The evolution of the Riemann curvature tensor* Rm *is given by*

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}^{l} &= \frac{1}{2} g^{lp} (\nabla_i \nabla_k h_{jp} + \nabla_j \nabla_p h_{ik} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_k h_{ip}) \\ &- \frac{1}{2} g^{lp} (R_{ijk}^{q} h_{qp} + R_{ijp}^{q} h_{kq}). \end{aligned} \tag{3.17}$$

Proof. In local coordinates $\{x_i\}$, the components of the Riemann curvature $(3,1)$-tensor are given by the standard formula

$$R_{ijk}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{p} \Gamma_{ip}^{l} - \Gamma_{ik}^{p} \Gamma_{jp}^{l}, \tag{3.18}$$

[1] A coordinate system relative to which the components g_{ij} of the metric tensor g are locally constant in the neighbourhood of an arbitrary point p_0 of $\mathcal{M}$. The point p_0 is called the pole.

so

$$\begin{aligned}\frac{\partial}{\partial t}R^l_{ijk} = \partial_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right) - \partial_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right) + \left(\frac{\partial}{\partial t}\Gamma^p_{jk}\right)\Gamma^l_{ip} + \Gamma^p_{jk}\left(\frac{\partial}{\partial t}\Gamma^l_{ip}\right) \\ - \left(\frac{\partial}{\partial t}\Gamma^p_{ik}\right)\Gamma^l_{jp} - \Gamma^p_{ik}\left(\frac{\partial}{\partial t}\Gamma^l_{jp}\right).\end{aligned} \tag{3.19}$$

As in the preceding lemma, we use geodesic coordinates centered at $p \in \mathcal{M}$ to calculate that

$$\frac{\partial}{\partial t}R^l_{ijk}(p) = \nabla_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right)(p) - \nabla_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right)(p), \tag{3.20}$$

and we observe that the result holds everywhere. Commuting derivatives yields

$$\begin{aligned}\frac{\partial}{\partial t}R^l_{ijk} = \frac{1}{2}g^{lp}(\nabla_i\nabla_k h_{jp} + \nabla_j\nabla_p h_{ik} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_k h_{ip}) \\ - \frac{1}{2}g^{lp}(R^q_{ijk}h_{qp} + R^q_{ijp}h_{kq}).\end{aligned} \tag{3.21}$$

□

Lemma 3.4. *The evolution of the Ricci tensor* Ric *is given by*

$$\frac{\partial}{\partial t}R_{jk} = \frac{1}{2}g^{pq}(\nabla_q\nabla_j h_{kp} + \nabla_q\nabla_k h_{jp} - \nabla_q\nabla_p h_{jk} - \nabla_j\nabla_k h_{qp}). \tag{3.22}$$

Proof. This follows from contracting on $i = l$ in the preceding lemma. □

We shall now define what it means for an evolution equation on the vector bundle $S^+_2T^*\mathcal{M}$ of positive definite symmetric $(0,2)$-tensors to be parabolic. Given

$$\begin{cases}\frac{\partial}{\partial t}h_{ij} = P(h_{ij}) \\ h_{ij}(0) = \alpha_{ij},\end{cases} \tag{3.23}$$

where P is a k-th order differential mapping from $C^\infty(S^+_2T^*\mathcal{M})$ into itself. The linearization of P at a section h_{ij} is the linear bundle map $DP : C^\infty(S^+_2T^*\mathcal{M}) \to C^\infty(S^+_2T^*\mathcal{M})$ given by

$$DP(h)[b]_{ij} = \left.\frac{d}{ds}\right|_{s=0} P(h+sb)_{ij} = \sum_{|p|\leq k} P^{p,lm}_{ij}(h)\partial_p b_{lm}. \tag{3.24}$$

The *principal symbol* of P at h in the direction of a one-form ξ is a linear map $\hat{\sigma}[DP(h)](\xi) : C^\infty(S^+_2T^*\mathcal{M}) \to C^\infty(S^+_2T^*\mathcal{M})$ given by

$$\hat{\sigma}[DP(h)](\xi)(b)_{ij} = \sum_{|p|=k} P^{p,lm}_{ij}(h)\xi_p b_{lm}. \tag{3.25}$$

A linear partial differential operator L is said to be elliptic if its principal symbol $\hat{\sigma}[L](\xi)$ is an isomorphism whenever $\xi \neq 0$. A nonlinear differential operator P is said to be elliptic if its linearization DP is elliptic. An evolution equation of the form (3.23) is said to be parabolic if P is an elliptic operator.

Rich existence and uniqueness theory for elliptic operators guarantees the short-time existence of a solution to (3.23) on a closed manifold. Hence, to prove the short-time existence for the Ricci flow we have to show that the second order nonlinear differential operator Ric is equivalent to an elliptic operator.

Proof of Theorem 3.1. We have already computed the linearization of the Ricci tensor:

$$D\mathrm{Ric}(g)[h]_{ij} = \frac{1}{2}g^{pq}(\nabla_q\nabla_j h_{kp} + \nabla_q\nabla_k h_{jp} - \nabla_q\nabla_p h_{jk} - \nabla_j\nabla_k h_{qp}). \tag{3.26}$$

This is a linear combination of derivatives of h involving the Christoffel symbols and their derivatives, but the highest-order derivatives of h are simply partial derivatives. Therefore the principal symbol for the Ricci tensor is

$$\hat{\sigma}[D\mathrm{Ric}(g)](h)(\xi)_{jk} = \frac{1}{2}g^{pq}(\xi_q\xi_j h_{kp} + \xi_q\xi_k h_{jp} - \xi_q\xi_p h_{jk} - \xi_j\xi_k h_{qp}). \tag{3.27}$$

The curvature operator is certainly non elliptic, since for any ξ we can define $h_i j = \xi_i\xi_j$ and see that the principal symbol evaluates to zero. The fact that the principal symbol has nontrivial kernel is related to the invariance of the Ricci tensor under diffeomorphism,

$$\mathrm{Ric}(\varphi^* g) = \varphi^*(\mathrm{Ric}(g)). \tag{3.28}$$

However, this is the only thing that goes wrong with the principal symbol. By modifying the flow on the metric by a time-dependent set of diffeomorphisms, we could form a strictly parabolic equation that would allow us to apply short-time existence results. Transforming this solution by another set of diffeomorphisms, we obtain a short-time solution for the Ricci flow equation.

Following DeTurck's proof, we fix a background metric $\tilde{g}$ and mark all geometric quantities related to this metric with a tilde. We then define a vector fields W by

$$W^i = g^{jk}(\Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}). \tag{3.29}$$

Since the difference of two connections is a tensor, W is a globally well-defined vector field (independent of the coordinates used to describe it locally). We will also use the one-form which is metric-dual to W, as usual denoting its components with lowered indices:

$$W_i = g_{ik}g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}). \tag{3.30}$$

We now define *Ricci-DeTurck flow*:

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i. \tag{3.31}$$

We observe that W appears in the equation through the term $\nabla_i W_j + \nabla_j W_i$, which is the Lie derivative $\mathcal{L}_W g_{ij}$ of the metric g with respect to the vector field W. This term that will allow us to modify the Ricci-DeTurck flow by a diffeomorphism to get back the Ricci flow.

To verify that the equation (3.31) is strictly parabolic, we compute the linearization of the extra term on the right-hand side. Let us denote this term by

$$A(g)_{ij} = \nabla_i W_j + \nabla_j W_i = \nabla_i \left(g_{jk} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}) \right) + \nabla_j \left(g_{ik} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}) \right). \tag{3.32}$$

Using geodesic coordinates we again see that we can regard the covariant derivatives as partial derivatives when linearizing, so using Lemmas 3.2 – 3.4 we obtain

$$\begin{aligned} DA(g)(h)_{ij} &= \frac{1}{2} g_{jk} g^{pq} \nabla_i [g^{kl} (\nabla_p h_{lq} + \nabla_q h_{pl} - \nabla_l h_{pq})] \\ &\quad + \frac{1}{2} g_{ik} g^{pq} \nabla_j [g^{kl} (\nabla_p h_{lq} + \nabla_q h_{pl} - \nabla_l h_{pq})] \\ &\quad + (\text{lower-order derivatives of } h) \\ &= g^{pq} (\nabla_i \nabla_p h_{jq} + \nabla_j \nabla_p h_{iq} - \nabla_i \nabla_j h_{pq}) + \text{l.o.d.} \end{aligned} \tag{3.33}$$

Comparing this to the linearized Ricci tensor, we see that

$$D[-2\text{Ric} + A](g)(h)_{ij} = g^{pq} \nabla_p \nabla_q h_{ij} + \text{l.o.d.} = \Delta h_{ij} + \text{l.o.d.} \tag{3.34}$$

Thus $-2\text{Ric} + A$ is an elliptic operator. The parabolic existence results therefore apply and yield a short-time solution to the Ricci-DeTurck ow with any initial metric.

Now given a solution $g(t)$ of the Ricci-DeTurck flow, we construct a family of maps $\varphi_t : \mathcal{M} \to \mathcal{M}$ by solving the ODE

$$\begin{aligned} \frac{\partial}{\partial t} \varphi_t &= -W, \\ \varphi_0 &= \text{id}_{\mathcal{M}}, \end{aligned} \tag{3.35}$$

at each point of $\mathcal{M}$. Pulling back $g(t)$ by the diffeomorphisms φ_t, we obtain a solution

$$\bar{g}(t) := \varphi_t^* g(t) \qquad (0 \le t < \varepsilon) \tag{3.36}$$

of the Ricci flow with $\bar{g}(0) = g_0$. Indeed, we have $\bar{g}(0) = g(0) = g_0$, because $\varphi_0 = \text{id}_{\mathcal{M}}$. We then compute that

$$\begin{aligned} \frac{\partial}{\partial t} \varphi_t^* g(t) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\varphi_{t+s}^* g(t+s) \right) \\ &= \varphi_t^* \left(\frac{\partial}{\partial t} g(t) \right) + \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\varphi_{t+s}^* g(t) \right) \\ &= \varphi_t^* \left(-2\text{Ric}(g(t)) + \mathcal{L}_{W(t)} g(t) \right) + \left.\frac{\partial}{\partial s}\right|_{s=0} \left[(\varphi_t^{-1} \circ \varphi_{t+s})^* \varphi_t^* g(t) \right] \\ &= -2\text{Ric}\left(\varphi_t^* g(t) \right) + \varphi_t^* \left(\mathcal{L}_{W(t)} g(t) \right) - \mathcal{L}_{\left[(\varphi_t^{-1})_* W(t) \right]} \left(\varphi_t^* g(t) \right) \qquad (3.37) \\ &= -2\text{Ric}(\varphi_t^* g(t)) \qquad (3.38) \end{aligned}$$

The equality on the line (3.37) follows from the identity

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \left(\varphi_t^{-1} \circ \varphi_{t+s} \right) = (\varphi_t^{-1})_* \left(\left.\frac{\partial}{\partial s}\right|_{s=0} \varphi_{t+s} \right) = (\varphi_t^{-1})_* W(t). \tag{3.39}$$

Hence $\bar{g}(t)$ is a solution of the Ricci flow for $t \in [0, \varepsilon)$.

To complete the proof of Theorem 3.1, it remains to show that $\bar{g}(t)$ is unique. While the Ricci-DeTurck flow is strictly parabolic and hence satisfies the usual uniqueness properties, we cannot use that fact alone to demonstrate that solutions to the Ricci flow are unique. The reason is the following: if we start with two solutions to the Ricci flow with identical initial conditions, we can modify them by diffeomorphisms to get two solutions of the Ricci-DeTurck flow with identical initial conditions. From the theory for strictly parabolic equations we conclude that these two solutions coincide. However, the diffeomorphisms used depend on the solutions themselves, therefore they can be different. Hence we can't decide whether the original solutions are the same.

The uniqueness of solutions of the Ricci flow can be shown using the harmonic map heat flow. This procedure is described in section 4.4 of [7]. □

In the interest of completeness, we will establish the existence of a family of diffeomorphisms $\{\varphi_t\}$ solving the ODE (3.35).

Lemma 3.5. *If $\{X_t : 0 \leq t \leq T \leq \infty\}$ is a continuous time-dependent family of vector fields on a compact manifold $\mathcal{M}$, then there exists a one-parameter family of diffeomorphisms $\{\varphi_t : \mathcal{M} \to \mathcal{M} : 0 \leq t < T \leq \infty\}$ defined on the same interval such that*

$$\begin{aligned} \frac{\partial \varphi_t}{\partial t}(x) &= X_t[\varphi_t(x)] \\ \varphi_0(x) &= x \end{aligned} \tag{3.40}$$

for all $x \in \mathcal{M}$ and $t \in [0, T)$.

Proof. We may assume that there is $t_0 \in [0, T)$ such that $\varphi_s(t)$ exists for all $0 \leq s \leq t_0$ and $y \in \mathcal{M}$. Let $t_1 \in (t_0, T)$ be given. We will show that φ_t exists for all $t \in [t_0, t_1]$. Since t_1 is arbitrary, this will prove the lemma. Given any $x_0 \in \mathcal{M}$, choose local coordinate systems $(\mathcal{U}, \mathbf{x})$ and $(\mathcal{V}, \mathbf{y})$ such that $x_0 \in \mathcal{U}$ and $\varphi_{t_0}(x_0) \in \mathcal{V}$. As long as $x \in \mathcal{U}$ and $\varphi_t(x) \in \mathcal{V}$, the equation

$$\frac{\partial \varphi_t}{\partial t}(x) = X_t[\varphi_t(x)] \tag{3.41}$$

is equivalent to

$$\begin{aligned} \frac{\partial}{\partial t}\left[\mathbf{y} \circ \varphi_t \circ \mathbf{x}^{-1}(p)\right] &= \mathbf{y}_*\left[\frac{\partial \varphi_t}{\partial t}[\mathbf{x}^{-1}(p)]\right] \\ &= \left(\mathbf{y}_* X_t \circ \mathbf{y}^{-1}\right)\left(\mathbf{y} \circ \varphi_t \circ \mathbf{x}^{-1}(p)\right) \end{aligned} \tag{3.42}$$

for $p \in \mathbf{x}(\mathcal{U})$ such that $\varphi_t \circ \mathbf{x}^{-1}(p) \in \mathcal{V}$. Setting $\mathbf{F}_t = \mathbf{y}_* X_t \circ \mathbf{y}^{-1}$ and $\mathbf{z}_t = \mathbf{y} \circ \varphi_t \circ \mathbf{x}^{-1}$, we get

$$\frac{\partial}{\partial t}\mathbf{z}_t = \mathbf{F}_t(\mathbf{z}_t). \tag{3.43}$$

Thus we see that (3.41) is locally equivalent to a nonlinear ODE in $\mathbb{R}^n$. Hence for all $x \in \mathcal{U}$ such that $\varphi_{t_0}(x) \in \mathcal{V}$, there exists a unique solution of (3.41) for a short time $t \in [t_0, t_0 + \varepsilon)$. Since the vector fields X_t are uniformly bounded on the compact set $\mathcal{M} \times [t_0, t_1]$, there is an $\bar{\varepsilon} > 0$ independent of $x \in \mathcal{M}$ and $t \in [t_0, t_1]$ such that a unique solution $\varphi_t(x)$ exists for $t \in [t_0, t_0 + \bar{\varepsilon}]$. Since the same claim holds for the flow starting at $\varphi_{t+\bar{\varepsilon}}(x)$, a simple iteration finishes the proof. □

4. Ricci Solitons

In this section we introduce self-similar solutions to the Ricci flow, often called *Ricci solitons*. The self-similar solutions model the solutions near the singularities. We will give examples of long-existing solutions: eternal solutions (that exist for $-\infty < t < \infty$) and ancient solutions (those defined for $-\infty < t < \omega$, where $\omega \in \mathbb{R} \cup \{\infty\}$).

A Ricci soliton is a solution of the Ricci flow where the metric $g(t)$ changes only by a time dependent scale factor and the pull-back by diffeomorphisms. That is, for any t_1 and t_2 in the time interval of existence, there exists a positive constant $\sigma = \sigma(t_1, t_2)$ and a diffeomorphism $\varphi = \varphi(t_1, t_2)$ such that

$$g(t_2) = \sigma \varphi^* g(t_1). \tag{4.1}$$

Fixing a time t_0 we have for any time t,

$$g(t) = \sigma(t)\varphi(t)^* g(t_0). \tag{4.2}$$

We say $g(t)$ is *expanding*, *shrinking*, or *steady* at a time t_0 if $\sigma(t_0) > 0$, $= 0$, or < 0, respectively.

A gradient Ricci soliton is a Ricci soliton where the vector fields $X(t)$ generated by the 1-parameter family of diffeomorphisms $\varphi(t)$ are the gradients of functions $f(t)$. The single time version of this definition is as follows:

Definition 4.1. A Riemannian manifold $(\mathcal{M}, g_0)$ is called a *gradient Ricci soliton* if there exists a smooth function $f_0 : \mathcal{M} \to \mathbb{R}$ and a constant $\varepsilon \in \mathbb{R}$ such that

$$\mathrm{Ric}_{ij}(g_0) + \nabla_i \nabla_j f_0 + \varepsilon g_0 = 0. \tag{4.3}$$

The function f_0 is called the *potential function* of the Ricci soliton. We say that g_0 is *expanding* if $\varepsilon > 0$, *shrinking* if $\varepsilon < 0$, or *steady* if $\varepsilon = 0$.

4.1. Eternal Solutions

An eternal solution of the Ricci flow is the one that exists for all time. We have seen that the curvatures of a solution to the Ricci flow evolve by reaction-diffusion equations. In this type of equations there is a competition between the diffusion term (which seeks to disperse concentrations of curvature uniformly over the manifold as time moves forward) and the reaction term (which tends to create concentrations of curvature as time mores forward). This means that an eternal solution must be stable, with no concentrations of curvature at any finite time in either past of future.

An important two-dimensional example of a steady soliton is Hamilton's **cigar soliton**, which is given on the Euclidean plane $\mathbb{R}^2$ by the metric

$$g_\Sigma(t) = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}. \tag{4.4}$$

To see that $g_\Sigma(t)$ is a solution to the Ricci flow, we compute its Ricci tensor with respect to coordinate system $z_1 = x$, $z_2 = y$. The Christoffel symbols are computed by the formula

$$\Gamma^k_{ij} = \frac{1}{2} \sum_k g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{4.5}$$

Note that

$$g^{-1} = \begin{bmatrix} \dfrac{e^{4t}+x^2+y^2}{dx^2} & 0 \\ 0 & \dfrac{e^{4t}+x^2+y^2}{dx^2} \end{bmatrix} \tag{4.6}$$

Thus we compute

$$\Gamma^1_{11} = -\frac{x}{e^{4t}+x^2+y^2}, \qquad \Gamma^1_{12} = -\frac{y}{e^{4t}+x^2+y^2}, \qquad \Gamma^1_{22} = \frac{x}{e^{4t}+x^2+y^2} \tag{4.7a}$$

$$\Gamma^2_{11} = \frac{y}{e^{4t}+x^2+y^2}, \qquad \Gamma^2_{12} = -\frac{x}{e^{4t}+x^2+y^2}, \qquad \Gamma^2_{22} = -\frac{y}{e^{4t}+x^2+y^2} \tag{4.7b}$$

In terms of the Christoffel symbols, the Ricci tensor has components

$$\mathrm{Ric}_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{kp}\Gamma^p_{ij} - \Gamma^k_{jp}\Gamma^p_{ik}. \tag{4.8}$$

Using the Christoffel symbols for $g_\Sigma(t)$, we obtain

$$\mathrm{Ric}(t) = \frac{2e^{4t}}{(e^{4t}+x^2+y^2)^2}(dx^2+dy^2). \tag{4.9}$$

On the other hand, we have

$$g'_\Sigma(t) = -\frac{4e^{4t}}{(e^{4t}+x^2+y^2)^2}(dx^2+dy^2). \tag{4.10}$$

Thus $g_\Sigma(t)$ satisfies the Ricci flow equation.

By making the change of variables $x \mapsto e^{-2tx}$ and $y \mapsto e^{-2ty}$ we see that $g_\Sigma(t)$ is isometric to $g_\Sigma = g_\Sigma(0)$. Thus we can define a 1-parameter group of diffeomorphisms $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi_t(x,y) = (e^{-2t}x, e^{-2t}y)$, so that

$$g_\Sigma(t) = \varphi_t^* g_\Sigma(0). \tag{4.11}$$

Hence $g_\Sigma(t)$ is a steady soliton.

Since g_Σ is rotationally symmetric, it is naturally to write it in polar coordinates

$$g_\Sigma = \frac{dr^2 + r^2 d\theta^2}{1+r^2}. \tag{4.12}$$

The scalar curvature of g_Σ is

$$R_\Sigma = \frac{4}{1+r^2} \tag{4.13}$$

Since $r^2/(1+r^2) \to 1$ as $r \to \infty$, equations (4.12) and (4.13) imply that the metric is asymptotic at infinity to a cylinder of radius 1.

The cigar metric on the cylinder may be written as

$$g_{\Sigma^2-0} = \frac{dz^2 + d\theta^2}{e^{-2z}+1}, \tag{4.14}$$

where $z \in \mathbb{R}$ and $\theta \in S^1(1) = \mathbb{R}/2\pi\mathbb{Z}$ are the standard cylindrical coordinates.

Remark (Uniqueness of the cigar)**.** If $(\mathcal{M}^2, g(t))$ is a complete steady Ricci soliton with positive curvature, then $(\mathcal{M}^2, g(t))$ is the cigar soliton.

In Hamilton's program for the Ricci flow on 3-manifolds, via dimension reduction, the cigar soliton is a potential singularity model. However, Perelman's *No Local Collapsing* theorem rules out this possibility.
Higher dimensional examples of rotational symmetric Kähler-Ricci solitons have been found in the late nineties by Cao [10].

4.2. Ancient Solutions

The **shrinking round sphere** is the canonical ancient solution of the Ricci flow. Let g_{can} denote the standard round metric on S^n of radius 1, and consider the 1-parameter family of conformally equivalent metrics

$$g(t) = r(t)^2 g_{\text{can}}, \tag{4.15}$$

where $r(t)$ is to be determined. One can check that $g(t)$ is a solution of the Ricci flow if and only if

$$2r\frac{\partial r}{\partial t} g_{\text{can}} = \frac{\partial}{\partial t} g = -2\text{Ric}[g] = -2\text{Ric}[g_{\text{can}}] = -2(n-1)g_{\text{can}}, \tag{4.16}$$

i.e., if and only if $r(t)$ is a solution of the ODE

$$\frac{\partial r}{\partial t} = -\frac{n-1}{r}. \tag{4.17}$$

Setting

$$r(t) = \sqrt{r_0^2 - 2(n-1)t} = \sqrt{2(n-1)}\sqrt{T-t} \tag{4.18}$$

yields an ancient solution $(S^n, g(t))$ of the Ricci flow that exists for the time $-\infty < t < T$, where $T < \infty$ is the singularity time defined by

$$T := \frac{r_0^2}{n-1}. \tag{4.19}$$

Another example is the Rosenau solution. Let $(\mathbb{R} \times S^1(2), h)$ denote the flat cylinder, where $h = dz^2 + d\theta^2$ and $S^1(2) = \mathbb{R}/4\pi\mathbb{Z}$. The **Rosenau solution** is the solution $g(t) = u(t) \cdot h$ to the Ricci flow defined for $t < 0$ by

$$u(x,t) = \frac{\sinh(-t)}{\cosh z + \cosh t} \tag{4.20}$$

Its scalar curvature is given by

$$R[g(t)] = -\frac{\Delta_h \log u}{u} = \frac{\cosh t \cosh z + 1}{\sinh(-t)(\cosh z + \cosh t)} \tag{4.21}$$

for $t < 0$. In particular, the solution has positive curvature for as long as it exists. One can verify that $g(t)$ is a solution to the Ricci flow. Note that the Rosenau solution is ancient but not eternal, since by equation (4.20)

$$\lim_{t \to 0} u(x,t) = 0. \tag{4.22}$$

The metrics $g(t)$ defined on $\mathbb{R} \times S^1(2)$ extend to smooth metrics on the 2-sphere S^2, which is obtained by compactifying $\mathbb{R} \times S^1(2)$ by adding two points, the north and the south poles.

We now take a *backward limit* of the Rosenau solution to see that we can obtain either the cigar soliton or the cylinder. Consider

$$u(z+t,t) = \frac{1}{-\cosh z \cosh t - \sinh z - \coth t}, \tag{4.23}$$

so that

$$\lim_{t\to\infty} u(z+t,t)h(z) = \frac{h(z)}{-\cosh z - \sinh z + 1} = \frac{h(z)}{e^{-z}+1}. \tag{4.24}$$

By making the change of variables $z \mapsto z/2$ and $\theta \mapsto \theta/2$, we have

$$\lim_{t\to\infty} u(z+t,t)h(z) = \frac{h(z)}{e^{-z}+1} = 4\frac{dz^2+d\theta^2}{e^{-2z}+1}, \tag{4.25}$$

where $z \in \mathbb{R}$ and $\theta \in S^1(1)$. This is the cigar soliton.

Note that $u(z,t) \to 1$ as $t \to \infty$. Therefore

$$\lim_{t\to-\infty} g(z,t) = h(z) \tag{4.26}$$

for all $z \in \mathbb{R}$.

To summarize, at the two tips of the Rosenau solution, as we go back in time toward $-\infty$, the metric looks closer and closer to the cigar soliton metric. If we consider points on the "equator" $z = 0$, then as we go back in time toward $-\infty$, the metric looks closer and closer to a cylinder.

The Rosenau solution is of interest here in part because it could potentially occur as a dimension-reduction limit of a 3-manifold singularity. Recent work of Perelman [11] eliminates this possibility for finite-time singularities.

5. Summary of results and open problems of Ricci solitons

In this section we state some known results and open problems about the properties and the classification of gradient Ricci solitons.

5.1. Gradient Ricci solitons of surfaces

(i) A shrinking soliton has constant positive curvature. In particular, the underlying surface is compact.

(ii) The only two-dimensional shrinking gradient Ricci soliton is the round 2-sphere.

(iii) A steady soliton is either flat (i.e., the Riemannian curvature tensor of the Levi-Civita connection is the zero map) or the cigar.

(iv) A compact expanding soliton has constant negative curvature.

(v) An expanding soliton with positive curvature is rotationally symmetric and unique up to a dilation.

Open problem 1. Are there any other expanding solitons of a surface diffeomorphic to $\mathbb{R}^2$ besides the positively curved rotationally symmetric expander and the Poincaré hyperbolic disk?

Open problem 2. Do all complete two-dimensional gradient Ricci solitons have bounded curvature? Are all complete two-dimensional Ricci solitons gradient?

5.2. Gradient Ricci solitons on 3-manifolds

(i) Any non-flat shrinking soliton with bounded nonnegative sectional curvature is isometric to either a quotient of the 3-sphere or a quotient of $S^2 \times \mathbb{R}$. This result is due to Perelman.

(ii) There exists a rotationally symmetric steady soliton with positive sectional curvature, namely the Bryant soliton.

(iii) There exists a rotationally symmetric expanding soliton with positive sectional curvature.

Open problem 3. Are there any three-dimensional steady gradient solitons besides a flat solution, the Bryant soliton, and a quotient of the produce of the cigar and $\mathbb{R}$?

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