Lecture[1] 9

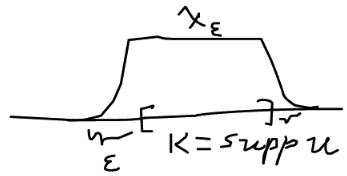
Theorem 1. *Let $u \in \mathscr{E}^m(\Omega)$ be of order m, $\phi \in C^m(\Omega)$ satisfying*

$$\partial^\alpha \phi = 0 \quad \text{on} \quad \text{supp } u, \quad \forall |\alpha| \le m \implies u(\phi) = 0.$$

Proof. Denote $\chi_\epsilon \in \mathscr{D}(\Omega)$ cut off function such

$$\chi_\epsilon \begin{cases} 1 & \text{on nbhd} \quad K \\ 0 & \text{outside} \quad K + B_\epsilon. \end{cases}$$

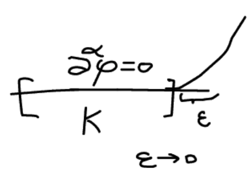
We will use a known property that, $|\partial^\alpha \chi_\epsilon| \le C\epsilon^{-|\alpha|} \quad \forall |\alpha| \le m$.



Now by boundedness of distribution u and $u(\phi) = u(\chi_\epsilon \phi)$ in K,

$$\begin{aligned} |u(\phi)| &\le C \sum_{|\alpha| \le m} \sup |\partial^\alpha (\chi_\epsilon \phi)| && (1) \\ &\le C' \sum_{|\alpha| \le m} \|\partial^\alpha \phi\|_{C(K+B_\epsilon)} \|\chi_\epsilon\|_{C^{m-|\alpha|}} && (2) \\ &\le C'' \sum_{|\alpha| \le m} \epsilon^{|\alpha|-m} \|\partial^\alpha \phi\|_{C(K+B_\epsilon)}. && (3) \end{aligned}$$

It is necessary to check for the nontrivial cases. For $|\alpha| = m$



$$\partial^\alpha \phi(y) = \partial^\alpha \phi(x) + o(|y - x|) = \partial^\alpha \phi(x) + o(\epsilon).$$

For case $|\alpha| < m$, we make use of the Taylor expansion. Let $\partial^\phi = \psi$,

$$\psi(y) = \psi(x) + \sum_{|\alpha| \le m - |\alpha|} (y - x)^\beta \partial^\beta \partial^\beta \psi(x)/\alpha! + R$$

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R denoting the remainder terms. Note that since $x \in supp\ \phi$, then all but the remainder term vanishes in the expansion above.

$$|\psi(y)| = |R| \leq C \sum_{|\beta|=m-|\alpha} |\partial^{\beta}\alpha\phi(\xi)|\epsilon^{m-|\alpha|} \qquad \epsilon \approx |x-y|.$$

□

Corollary 1. *Let $u \in \mathscr{D}'(\Omega)$, with supp $u = \{y\}$ for some $y \in \Omega$. Then,*

$$u(\phi) = \sum_{|\alpha|\leq m} a_{\alpha}\partial^{\alpha}\phi(y)$$

for some m and $\{a_{\alpha}\}$. That is, $u = \sum a_{\alpha}\partial^{\alpha}\delta_y$ a finite sum.

Proof.

$$\phi(x) = \sum_{|\alpha|\leq m} \partial^{\alpha}\phi(y) \underbrace{(x-y)^{\alpha}/\alpha!}_{\xi_{\alpha}(x)} + \psi(x)$$

then,

$$u(\phi) = \sum_{|\alpha|\leq m} u(\xi_{\alpha})\partial^{\alpha}\phi(y) + u(\psi).$$

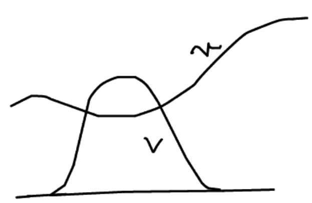
Since $\partial^{\alpha}\psi(y) = 0$ for $|\alpha| \leq m$ then the remainder term is zero. □

Convolutions

The convolution is defined on regular space function as follows:

$$(u * v)(x) = \int u(y)v(x-y)dy = \int u(x-y)v(y)dy = (v * u)(x)$$

There are some conditions which must be imposed on u and v so that the integral above exists. For example this would hold for $v \in L^1_c(\mathbb{R}^n)$ and $u \in L^{\infty}_{loc}(\mathbb{R}^n)$. Another option is when $u, v \in L^1(\mathbb{R}^n)$



which holds since

$$\|u * v\|_{L^1} \leq \|u\|_{L^1}\|v\|_{L^1}.$$

Moreover, the convolution possesses the following property:

$$\partial^{\alpha}(u * v) = (\partial^{\alpha}u) * v,$$

if $|\alpha| \leq m$, $u \in C^m$ and $v \in L^1_c$. The key is to notice that this product rules exists provided *only one* of the functions satisfies regularity conditions in the classical sense.

In order to define the convolution in the distributional sense, we define the following.

Definition 1. *For $v \in L_c^1(\mathbb{R}^n)$, define $C_v : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}\mathbb{R}^n$ by*

$$C_v u : u \mapsto v * u$$

The support of $u * v \subset supp\ u + supp\ v$.

Theorem 2. *The operator $C_v : C^m(\mathbb{R}^n) \to C^m(\mathbb{R}^n)$, for $0 \leq m \leq \infty$ is continuous.*

Proof. $P_k(\phi) = \|\phi\|_{C^l(K)}$ for K compact, l nonnegative integer,

$$\|u * v\|_{C^l(K)} \leq \|v\|_{L^1} \|u\|_{C^l(K - supp\ v)}$$

$$\int u(x-y)v(y)\ dy, \qquad x - y \in K - supp\ v$$

so let $K' = K - supp\ v$. □

Corollary 2. *$C_v : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}^n$ is continuous.*

Proof. Suffices to show for every K compact $C_v : \mathscr{D}_K \to \mathbb{R}^n$ is continuous. By using

$$C_v \mathscr{D}_K = C_v \mathscr{D}(K) \subset \mathscr{D}(K + supp\ u)$$

we have it. □

We now define the convolutions in the sense of distributions. Consider the following:

$$\begin{aligned} < C_v u, \phi > &= \int (u * v)\phi = \int \int u(z)v(x-z)\ dz \cdot \phi(x)\ dx \\ &= \int u(z)\tilde{v}(z-x)\phi(x) \\ &\text{Define reflection } \tilde{v}(y) = v(-y) \\ &= \int (\tilde{v} * \phi) u \end{aligned}$$

Formally,

Definition 2. *Let $v \in L_c^1(\mathbb{R}^n)$. Define for $u \in \mathscr{D}'(\mathbb{R}^n)$*

$$< v * u, \phi > = < C_v u, \phi > := < u, C_{\tilde{v}} \phi > = < u, \tilde{v} * \phi >$$

We have, $C_u : v \mapsto u * v$, for $u \in \mathscr{D}'(\mathbb{R}^n)$.