Lecture[1] 8

Let $u \in \mathscr{D}'(\Omega)$. The support of u,

$$supp\ u = \Omega \backslash \cup_\alpha \{\omega \subset \Omega\ open:\ u\big|_\omega = 0\}.$$

- $supp\ u$ is relatively closes.
- $supp\ u = \{x \in \Omega:\ \nexists \omega \in \mathcal{N}(x)\ s.t\ u\big|_\omega = 0\}$.
- agrees with the usual definition when $u \in C(\Omega)$ or $u \in L^1_{loc}(\Omega)$.
- $supp\ u \subset a \cap supp\ u$, where $a \in C^\infty$. Moreover, $< au, \phi >=< u, a\phi >$.
- $supp\ (u + v) \subset supp\ u \cup supp\ v$.
- $supp\ \partial^\alpha u \subset supp\ u$.

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Local Structure of Distributions

Define the following space $\mathscr{E}'(\Omega) = C^\infty(\Omega)'$. Note that $C^\infty(\Omega)' \subset \mathscr{D}'(\Omega)$.

Definition 1. *u is an element of $\mathscr{E}'(\Omega)$ if and only if $u : C^\infty(\Omega) \to \mathbb{R}$ is linear and*

$$\exists K \subset \Omega\ compact,\ \exists m,\ c > 0\ s.t\ | < u, \phi > | \leq C\|\phi\|_{C^m(K)} \qquad \forall \phi \in C^\infty.$$

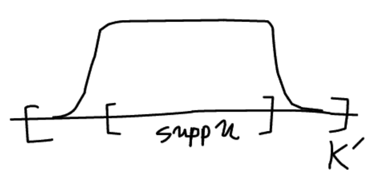
Theorem 1. *The space $\mathscr{E}'(\Omega) = \{u \in \mathscr{D}'(\Omega) : supp\ u\ compact\}$.*

Proof. Let $u \in \mathscr{E}'(\Omega)$.

$$\implies < u, \phi >= 0 \quad if \quad \underbrace{\phi = 0\ on\ K}_{supp\ \phi \subset \Omega \backslash K}, \quad K' \supset U \supset K.$$

$$\implies supp\ u \subset K$$

Conversely, $u \in \mathscr{D}'(\Omega)$, $K = supp\ u$ compact. Want cut off function $\chi \in \mathscr{D}(\Omega)$ satisfying $\chi = 1$ in a neighbourhood of K. Take some $\phi \in \mathscr{D}(\Omega)$. Then supp $(\phi - \chi\phi) \subset \Omega \backslash$ *nbhd* K. We have



$< u, \phi >=< u, \chi\phi >$, hence

$$| < u, \phi > | \leq C\|\chi\phi\|_{C^m(K')} \leq C'\|\phi\|_{C^m(K')}$$

u us continuous in the topology induced by $C_0^\infty(\Omega) \subset C^\infty(\Omega)$. Hence $u \in \mathscr{E}'(\Omega)$. ☐

[1] Notes by Ibrahim Al Balushi

Definition 2. *Define* $\mathscr{D}'(\Omega) = C_0^m(\Omega)'$ *a distribution of order less or equal to* m. $u \in \mathscr{D}'^m(\Omega)$ *if and only if* $u : C_c^m(\Omega) \to \mathbb{R}$ *is linear and*

$$\forall K \subset \Omega \text{ compact}, \; \exists C > 0 \text{ s.t } | < u, \phi > | \leq C\|\phi\|_{C^m(K)}, \qquad \phi \in C_c^m(K).$$

Theorem 2. ω *open* $\overline{\omega} \subset \Omega$ *compact.* $u \in \mathscr{D}'(\Omega)$. *Then*

$$u\big|_\omega \in \mathscr{D}'^m(\omega),$$

for some m.

Proof.

$$| < u, \phi > | \leq C\|\phi\|_{C^m(\overline{\omega})}, \qquad \forall \phi \in \mathscr{D}(\overline{\omega})$$

if $\phi \in \mathscr{D}(\omega) \subset \mathscr{D}(\overline{\omega})$. $u : \mathscr{D}(\omega) \to \mathbb{R}$ is continuous in the topology induced by $\mathscr{D}(\omega) \subset C_c^m(\omega)$. □

Corollary 1.

$$\mathscr{E}'(\Omega) \subset \bigcup_m \mathscr{D}'^m(\Omega) =: \mathscr{D}'_F(\Omega).$$

Theorem 3. $u \in \mathscr{D}'(\Omega)$, $u \in \mathscr{D}'^m(\omega)$, $\overline{\omega}$ *compact, there exists* $f \in L^\infty(\Omega)$ *such that*

$$u\big|_\omega = \partial_1^{m+1}\partial_2^{m+1} \cdots d_n^{m+1} f$$

i.e every distribution is equal locally to a derivative of some function.

Proof.

$$|u(\phi)| \leq C \sup_{\substack{|\alpha| \leq m \\ y \in \omega}} |\partial^\alpha \phi(y)| \qquad \forall \phi \in \mathscr{D}(\omega).$$

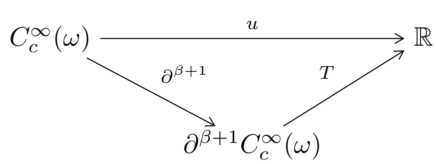
Define multi-index $\beta = (m, m, ..., m)$. We use the following fact for any sufficiently smooth function ψ $\sup|\psi| \leq C \sup|\partial\psi|$, carrying on from above

$$\leq C \sup_{y \in \omega} |\partial^\beta \phi(y)|,$$

meanwhile another fact for $\psi \in C_c^\infty(\Omega) : \psi(x) = \int\limits_{y<x} \partial_1\partial_2 \cdots \partial_n \psi(y) \; dy$,

$$\leq C\|\partial^{\beta+1}\phi\|_{L^1(\omega)}.$$

Now consider $T : \partial^{\beta+1}\phi \mapsto u(\phi) : \partial^{\beta+1}C_c^\infty(\omega) \to \mathbb{R}$,



$$|T(\Phi)| \leq C\|\Phi\|_{L^1(\omega)}$$

implies T can be extended $T \in [L^1(\omega)]' = L^\infty(\omega)$. That is,

$$\exists g \in L^\infty(\omega)\ s.t\ \underbrace{T(\partial^{\beta+1}\phi)}_{u(\phi)} = \int g\partial^{\beta+1}\phi \implies u = (-1)^{|\beta+1|}g.$$

□

Corollary 2. *Under the same setting, there exists $g \in C(\mathbb{R}^n)$ such that*

$$u\Big|_\omega = \partial_1^{m+2}\partial_2^{m+2}\cdots\partial_n^{m+2}g.$$

Corollary 3. *$u \in \mathscr{E}'(\Omega)$. There exists $g_\alpha \in C(\mathbb{R}^n)$ such that for some α finite,*

$$u = \sum_\alpha \partial^\alpha g_\alpha.$$

Proof.

$$< u, \phi > = < u, \chi\phi > = \int g\partial^{\beta+2}(\chi\phi).$$

□