Lecture[1] 4

Fact: X locally convex TVS. Then there exists a separating family of seminorms that defines the topology of X.

Theorem 1. *$(X, \mathcal{P})$, $(Y, \mathcal{Q})$ LCTVS's, where $\mathcal{P}$ and $\mathcal{Q}$ are separating families of seminorms defining the topologies of X, Y. A linear map $f : X \to Y$ is continuous if and only if*

$$\forall q \in Q \ \exists p \in \mathcal{P}, \ \exists C > 0 \ st. \ q(f(x)) \leq Cp(x) \qquad \forall x \in X.$$

Proof. ($\Leftarrow$) Let $x \in X, \ V \in \mathcal{N}(f(x))$. It suffices to show that there exists $\mathcal{U} \in \mathcal{N}(x)$, such that $f(\mathcal{U}) \subset V$. Moreover, $V = \bigcap_{i=1}^{k} V(q_i, n_i)$. Using the assumption that $\exists p_i \in \mathcal{P}, \ c_i > 0 \ s.t \ q_i(f(x)) \leq C_i p_i(x) \ \forall x \in X$, define

$$\mathcal{U} = \bigcap_{i=1}^{k} \left\{ p_i(x) < \frac{1}{c_i n_i} \right\} = \bigcap_{i=1}^{k} \left\{ c_i p_i(x) < \frac{1}{n_i} \right\}.$$

then,

$$f(\mathcal{U}) = \left\{ f(x) : c_i p_i(x) < \frac{1}{n_i} \right\} \subset \left\{ f(x) : q_i(f(x)) < \frac{1}{n_i} \right\} \subset V$$

Conversely, suppose f is continuous on X. Let $x \in X$, $q \in \mathcal{Q}$ and consider $V(q, 1)$. There exists $\mathcal{U} = \bigcap_{i=1}^{k} V(p_i, n_i)$ such that

$$f(\mathcal{U}) = \left\{ f(x) : p_i(x) < \frac{1}{n_i} \right\} \subset V(q, 1) = \{ f(x) : q(f(x)) < 1 \}.$$

i.e $q(f(x)) < 1$ for $x \in X$ satisfying $p_i(x) < \frac{1}{n_i}, \ i = 1, ..., k$. Let $y \in X$ define $x = \frac{y}{t}$ with $t > \max\{n_1 p_1(y), ..., n_k p_k(y)\}$

$$p_i(x) = \frac{1}{t} p_i(y) < \frac{p_i(y)}{\max_j \{n_j p_j(y)\}} \leq 1$$

$q(f(y)) = tq(f(x)) \leq t$, so choose $t = (n_k + 1)p_k(y)$, (where WLOG index k yields max $p_i(y)$) ☐

Corollary 1. *Y normed. $f : X \to Y$ linear map is continuous if and only if*

$$\exists p \in \mathcal{P}, \ \exists C > 0 \ s.t \ \|f(x)\|_Y \leq Cp(x) \qquad \forall x \in X.$$

Example: $\Omega \subset \mathbb{R}^n$ domain. $y \in \Omega$. $\delta_y : C(\Omega) \to \mathbb{R}$. $\delta_y(f) = f(y)$.

$$Want: \ |\delta_f(f)| \leq C \cdot \|f\|_{C^0(k)}.$$

K compactly embedded in Ω such that $y \in K$.

$$|\delta_y(f)| \leq \sup_{x \in K} |f(x)| = \|f\|_{C^0(K)} \leq \|f\|_{C^m(K)}$$

$C(\Omega) \subset C^k(\Omega), \ k = 1, ..., \infty$
$\delta_y \in \mathcal{E}'(\Omega)$

[1]Notes by Ibrahim Al Balushi

LF-spaces [Treves]

X vector space, $X = \bigcup_{n=1}^{\infty} X_n,\ X_n - Frechet,\ X_1 \subset X_2 \subset \cdots$, the topology of X_n is induced by the one on $X_{n+1},\ \forall n$. Define a family of subsets of X, by a convex open sets. $A \subset X$ is in σ if and only if $\forall n,\ A \cap X_n$ is an open neighbourhood of 0 in X_n.

Equipped with the topology τ generated by σ, X is called the **countable strict inductive limit** of $\{X_n\}$, and written as $X = ind\ lim\ X_n$. Moreover X is called an LF-space, and $\{X_n\}$ a **defining sequenc** of X.

Fact: In the above setting, X is a complete Hausdorff LCTVS. Moreover, the topology of each X_n coincides with the subspace topology of X_n induced by the embedding $X_n \subset X$.

Theorem 2. *Let X be an LF-space with the defining sequence $\{X_n\}$, and Y LCTVS, $f : X \to Y$ linear. f is continuous if and only if for $\forall n$,*

$$f|_{X_n} : X_n \to Y$$

is continuous.

Proof. $(\Rightarrow)$ $f : X \to Y$ continuous, $V \in \mathcal{N}(0)$ convex in Y. $\exists \mathcal{U} \in \mathcal{N}(0)$ convex in X such that $f(\mathcal{U}) \subset V$. $\mathcal{U}_n = \mathcal{U} \cap X_n \in \mathcal{N}(0)$ in X_n.

$$f|_{X_n}(\mathcal{U}_n) = f(\mathcal{U}_n) = f(\mathcal{U} \cap X_n) \subset f(\mathcal{U}) \subset V.$$

$(\Leftarrow)$ $f|_{X_n} : X_n \to Y$ continuous. $V \in \mathcal{N}(0)$ convex in Y. $f^{-1}(V) \subset V$ convex. $f^{-1}(V) \cap X_n = (f|_{X_n})^{-1}(V) \in \mathcal{N}(0)$ in X_n. ☐

Corollary 2. *The same as above but Y normed, each X_n is associated to a separating family $\mathcal{P}_n$ of seminorms. f is continuous if and only if*

$$\forall n,\ \exists p_n \in \mathcal{P}_n,\ \exists C_n > 0\ s.t\ \|f(x)\|_Y \leq C_n p_n(X) \qquad \forall x \in X_n.$$

Examples: $\mathcal{D}(\Omega) = C_o^\infty(\Omega)$. $C_o^k(\Omega),\ L_c^p(\Omega)$.

$$f \in L^1_{loc}(\Omega). \qquad T_f(\phi) = \int_\Omega f\phi \qquad \phi \in \mathcal{D}(\Omega)$$

$T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ linear.
$K_1 \subset K_2 \cdots \subset \Omega$. $\Omega = \cup K_n$. $X_n = \mathcal{D}(K_n) \ni \phi$.
$\forall n$

$$|T_f(\phi)| \leq \underbrace{\|\phi\|_{C^0(K_n)}}_{\text{“}p_n(\phi)\text{”}} \cdot \underbrace{\|f\|_{L^1(K_n)}}_{\text{“}C_n''}$$

$\implies T_f \in \mathcal{D}'(\Omega)$.

Consider

$$\partial^\alpha : \mathcal{D} \to \mathcal{D}(\Omega).$$

$\forall n.\ K_n.\ \phi \in \mathcal{D}(K_n)$

$$\|\partial^\alpha \phi\|_{C^m(K_n)} \leq \underbrace{\|\phi\|_{C^{m+|\alpha|}(K_n)}}_{\text{“}p_n(\phi)\text{”}}$$

$\partial^\alpha \phi \in \mathcal{D}(K_n) \subset D(\Omega)$.

X sequentially complete $\implies$ if X is metrizable $\implies$ X is Baire. $X = \cup_n X_n \implies X_n$ has nonempty interior $\implies X = X_n$.
$S_n = \{x_n, x_{n+1}, ...\}$ $x_n \to x \Leftrightarrow \forall \mathcal{U} \in \mathcal{N}(x)\ \exists n\ S_n \subset \mathcal{U}$. x_n cauchy if $\forall \mathcal{U} \in \mathcal{N}(0)\ \exists n\ S_n - S_n \subset \mathcal{U}$.