We showed the spectrum of a strongly elliptic operator with homogeneous Dirichlet boundary conditions is discrete.

$$(A-\lambda I)u = f \iff u-(t+\lambda)R_t u = R_t f, \quad t \text{ large}.$$

If $\{\mu_k\}$ is the spectrum of R_t, then

$$\lambda_k = \frac{1}{\mu_k} - t.$$

0.1 Self-Adjointness

$$A : \text{Dom } A \subset X \to X$$

$$\langle Au, v\rangle = \langle u, v^*\rangle \quad \forall u \in \text{Dom}(A). \tag{$*$}$$

If $\text{Dom}(A)$ is dense in X, then v^* is unique (for given v).

$$v \in \text{Dom } A^* \iff \exists v^* \; s.t \; (*) \text{ holds}, \; A^*v = v^*.$$

Go back to differential operators:

$$\int Au \cdot \overline{v} = \int \sum a_{\alpha,\beta} D^\alpha u \cdot \overline{D^\beta v} = \int u \cdot \overline{A^* v}, \quad u \in \text{Dom } A, \; v \in \text{Dom } A^*.$$

If $a_{\alpha,\beta} = \overline{a_{\beta,\alpha}}$, $A \subset A^*$ (symmetric) and

$$\int u \cdot \overline{A^* v} = \int u \cdot \overline{Av},$$

then,

$$A^* = A \implies R_t = R_t^*.$$

Theorem 1. *Suppose A strongly elliptic and its Friedrich Extension is self-adjoint. Then A has a complete orthonormal system of eigenfunctions $\{v_k\}$ in L^2 and corresponding to real eigenvalues $\{\mu_k\}$ satisfying $\lambda_k \to \infty$.*
$\{v_k\}$ are also complete in $H_0(\Omega)$, and orthonormal with respect to innerproduct

$$a(\cdot,\cdot) + t(\cdot,\cdot)_{L^2}.$$

Moreover, $v_k \in C^\infty(\Omega)$ and $v_k \in C^\infty(\overline{\Omega})$ if $\partial\Omega \in C^\infty$. Also, $v \in C^\omega(\overline{\Omega})$ if $\partial\Omega \in C^\omega$ and coeff A are in $C^\omega(\overline{\Omega})$.

Proof.

$$Av_k = \lambda_k v_k \implies \lambda_k \langle v_k, v_k\rangle = \langle Av_k, Av_k\rangle \geq c\|v_k\|^2_{H^m} - c_1\|v_k\|_{L^2},$$

$\lambda_k \to +\infty$.

$$a(v_k, v_j) = \lambda_k(v_k, v_j) = \lambda\delta_{k,j}.$$

$$v \in H_0^m(\Omega), \; a(v, v_k) = \langle v, Av_k\rangle = \overline{\lambda_k}(v, v_k)$$

implies if $a(v, v_k) + t(v, v_k) = 0$, then $(v, v_k) = 0$.

$$(A - I\lambda_k)v_k = 0 \implies \textit{regularity}.$$

∎

0.2 Functional Calculus

X Hilbert, $A : \mathrm{Dom}(A) \subset X \to X$ self-adjoint, having a complete orthonormal set of eigenfunctions.

$$Av_k = \lambda v_k.$$

Lemma 1. $Au = \sum_k \lambda_k \langle u, v_k \rangle v_k, \quad u \in Dom(A)$

$$u \in Dom(A) \iff \sum \lambda_k \langle u, v_k \rangle v_k$$

converges if and only if

$$\sum |\lambda_k \langle u, v_k \rangle|^2 < \infty.$$

Proof. $u \in \mathrm{Dom}(A)$:

$$Au = \sum \langle Au, v_k \rangle v_k = \sum_k \lambda_k \langle u, v_k \rangle v_k.$$

$(\Leftarrow)$

$$Cu := \sum \lambda_k \langle u, v_k \rangle v_k,$$

$$u \in Dom(C) \iff \text{series converges.}$$

$u \in Dom(A)$

$$\implies Cu = Au,$$

$$\implies A \subset C, \tag{1}$$

$$\implies C^* \subset A^* = A, \ (C \subset C^*) \tag{2}$$

therefore $A = C$ i.e $Dom(A) = Dom(C)$,

$$\langle Cu, v \rangle = \sum \lambda_k \langle u, v_k \rangle \langle v_k, v \rangle = \langle u, Cv \rangle .$$

□

Let $\overline{f} : \mathbb{R} \to \mathbb{R}$. Define

$$f(A)u = \sum_k f(\lambda_k) \langle u, v_k \rangle v_k$$

$u \in Dom\ f(A)$ if and only if $\sum |f(\lambda_k) \langle u, v_k \rangle|^2 < \infty$.

Lemma 2. *Dom $f(A)$ os dense in X, $f(A)$ is self-adjoint.*

Proof.

$$f(A)u_k = f(\lambda_k)v_k \implies v_k \in Dom\ f(A) \implies \textit{denseness.}$$

$C = f(A),\ C \subset C^*$. $u \in Dom\ C^*$.

$$\langle u, C^* v \rangle = \langle Cu_k, u \rangle = f(\lambda_k) \langle v_k, u \rangle$$

$$C^* u = \sum \langle v_k, C^* u \rangle v_k = \sum f(\lambda_k) \langle v_k, u \rangle v_k, \quad \textit{convergent.}$$

$$\implies C^* \subset C.$$

□

—
$f : \mathbb{R} \times I \to \mathbb{R},\ I \subset \mathbb{R}$. e.g e^{-tA}.

$$f(x,t) = e^{-xt}$$

$$f(A,t)u = \sum_k f(\lambda_k, t) \langle u, v_k \rangle v_k.$$

Theorem 2. *Let $u_0 \in X$. Assume*

$$\sum |c_k \langle u_0, v_k \rangle|^2 < \infty, \tag{$*$}$$

$$C_k = \|f(\lambda, \cdot)\|_{C^0(I)}, \quad f(\lambda) \in C(I),\ \forall \lambda \in \mathbb{R}.$$

Then,

$$u\ :\ t \mapsto f(A,t)u_0 \text{ is continuous on } I.$$

If in addition, $f(\lambda, \cdot) \in C^1(I),\ \forall \lambda$ and

$$\sum |d_k \langle u_0, v_k \rangle|^2 < \infty,\ d_k = \|f(\lambda_k, \cdot)\|_{C^1(I)}$$

then, $u \in C^1(I, \lambda)$ and $u_t(t) = f_t(A,t)u_0$.

Proof.

$$a_k = \langle u_0, v_k \rangle$$

$$\|f(A,t)u_0 - (A,s)u_0\|^2 = \sum_{k \in \mathbb{N}} |f(\lambda_k, t)a_k - f(\lambda_k, s)a_k|^2$$

$$\leq \sum_{k \leq N} |f(\lambda_k, t) - f(\lambda_k, s)|^2 |a_k|^2 + 2 \sum_{k > N} c_k^2 a_k^2$$

using $(*)$, the expression goes to 0 as t approaches s. □

0.3 Application to Heat Equation

$$u_t + Au = 0, \quad u(0) = u_0$$

assume $\lambda_k \to \infty$. Solution is $u(t) = e^{-tA}u_0$. By definition

$$u(t) = \sum e^{-t\lambda_k} \langle u_0, v_k \rangle v_k$$

(agrees with separation of variables)

$$C_k = \sup_{0 \leq t \leq T} e^{-\lambda_k t} \leq \max\{1, e^{-\lambda_1 T}, \lambda := \textit{smallest } \lambda_k$$

$$\implies u \in C^0([0,T], X)$$

$d_k = \sup_{0 \leq t \leq T} |\lambda_k e^{-\lambda_k t}| \leq C \cdot \lambda_k$. If $u_0 \in Dom(A)$ then $u \in C^1([0,T], X)$.

$$u_t = -\sum \lambda_k e^{-\lambda_k t} \langle u_0, v_k \rangle v_k$$

$$Au = \sum \lambda_k e^{-\lambda_k t} \langle u_0, v_k \rangle v_k$$

$$\implies u_t + Au = 0, \quad u(0) = u_0$$

$$d_{k,\epsilon} = \sup_{0 < \epsilon \leq t \leq T} \left|\lambda_k e^{-\lambda_k t}\right| \lesssim \frac{1}{\epsilon}, \quad u \in C^1([\epsilon, \tau], X),\ \forall u_0 \in X.$$

$$\lambda e^{-\lambda \epsilon} \lesssim \frac{1}{\epsilon}.$$