Let A strongly elliptic with smooth coefficients of order $2m$ and

$$Au = f \quad \text{in } \Omega, \qquad u \in H^m(\Omega).$$

We want to prove if $f \in H^s \implies u \in H^{s+2m}$.

0.1 Interior Regularity

$$f \in H^s_{loc}(\Omega) \implies H^{s+2m}_{loc}(\Omega).$$

Definition 1. *$v \in H^s_{loc}(\Omega) \iff \varphi v \in H^s$ for all $\varphi \in \mathscr{D}(\Omega)$.*

Simplified picture:

$$(-\Delta + 1)u = f, \quad u \in L^2(\Omega),\ f \in H^s_0(\Omega).$$

$$\implies (|\xi|^2 + 1)\widehat{u}(\xi) = \widehat{f}(\xi)$$

$$\|u\|_{H^{s+2}} = \| \langle\xi\rangle^{s+2} \widehat{u}\|_{L^2} = \| \langle\xi\rangle^{s+2-2} \widehat{f}\| = \|f\|_{H^2}.$$

Now suppose

$$(-\Delta + 1 - a)u = f, \quad u \in L^2_{loc}(\Omega),\ f \in H^s_{loc}(\Omega),$$

$\varphi \in \mathscr{D}(\Omega)$,

$$\Delta(\varphi u) = \varphi\Delta u + [\Delta, \varphi]u = \varphi\Delta u + 2\nabla\varphi \cdot \nabla u + u\Delta\varphi$$

$$(-\Delta + 1)(\varphi u) = \varphi f - a\varphi u - 2\nabla\varphi \cdot \nabla u - u\Delta\varphi = g$$

$$\begin{aligned} g \in H^1_0(\Omega)' &\implies \varphi\varphi u \in H^1\ \forall\varphi \\ &\implies u \in H^1_{loc}(\Omega) \\ &\implies g \in L^2_{loc} \\ &\implies \varphi u \in H^2\ \forall\varphi \\ &\implies u \in H^2_{loc} \\ &\quad \text{... iteratively.} \end{aligned}$$

$$\implies u \in H^{s+2}_{loc}(\Omega)$$

This procedure is called **boot strapping**. More generally, define

$$\delta^h_j u(x) = \frac{u(x + e_j h) - u(x)}{h}$$

Lemma 1. *$u \in H^s \implies \delta^h_j u \to \partial_j u$ in H^{s-1}*

Proof.

$$\widehat{\delta^h_j u}(\xi) = h^{-1}(e^{i\xi_j h} - 1)\widehat{u}(\xi)$$

($e^{i\xi_j h}$) Taylor series) $\to i\xi_j\widehat{u}(\xi) = \widehat{\partial_j u}(\xi)$ *a.e* ξ.

$$\int \langle\xi\rangle^{s-1} \left|h^{-1}(e^{i\xi_j h} - 1) - i\xi_j\right| |\widehat{u}(\xi)|^2\, d\xi \to 0$$

since $\left|h^{-1}(e^{i\xi_j h}-1)-i\xi_j\right| = \underbrace{|\xi_j||\theta^{-1}(e^{i\theta}-1)-i|}_{<\infty} \le C|\xi|$, which follows from

$$\frac{e^{i\theta}-1}{\theta} \to 1 \quad as \quad \theta \to 0.$$

□

Lemma 2. *A is elliptic of order q, with smooth coefficients.*

$$\|u\|_{H^s} \lesssim \|Au\|_{H^{s-q}} + \|u\|_{L^2}$$

$s \ge 0,\ u \in \mathscr{D}(\Omega)$.

Proof. Constant coefficients. $A = a_q(D)$. Ellipticity implies

$$|a_q(\xi)| \ge c|\xi|^q$$

$$\implies \|u\|^2_{H^s} \lesssim \int (1+|\xi|^{2s})|\widehat{u}(\xi)|^2\ d\xi \lesssim \|u\|^2_{L^2} + \int |\xi|^{2s-2q}|a_q(\xi)|^2|\widehat{u}(\xi)|^2\ d\xi$$

(This proof is similar to that of Garding)

$$= \|u\|^2_{L^2} + \|Au\|^2_{H^{s-q}}$$

For general case, localize and freeze coefficients. □

Theorem 1. *A elliptic of order q with smooth coefficients.*

$$Au = f \quad in\ \Omega, \quad u \in L^2_{loc}(\Omega),\ f \in H^s_{loc}(\Omega).$$

Then, $u \in H^{s+q}_{loc}(\Omega)$.

Proof. We first consider the case $u \in H^{s+q-1}_0(B_r)$, with $r > 0$ small. Suppose

$$A = \sum a_\alpha(x)\partial^\alpha.$$

Define

$$A^h_j = \sum a_\alpha(x+e_jh)\partial^\alpha$$

$$\implies A^h_j u(x+e_jh) = f(x+e_jh) \qquad (Note\ f \in H^s_0(\Omega)).$$

We have

$$\underbrace{\frac{A^h_j - A}{h}u(x+e_jh)}_{\substack{\text{converges in } H^{s-1} \\ \text{because } \delta^j_k a_\alpha \to \ldots \text{in } C^\infty \\ \text{and } \partial^\alpha u(x+e_jh)\to\ldots\text{in } H^{s-1}}} + \underbrace{A\delta^h_j u}_{\text{converges in } H^{s-1}} = \underbrace{\delta^h_j f}_{\to \partial_j f \text{ in } H^{s-1} \text{ by lemma}}$$

$$\|\delta^h_j u\|_{H^{s+q-1}} \lesssim \|A\delta^h_j u\|_{H^{s-1}} + \underbrace{\|\delta^h_j u\|_{L^2}}_{\lesssim r^{s+q-1}\|\delta^h_j u\|_{H^{s+q-1}}}$$

$$\implies \delta^h_j \to v \text{ in } H^{s+q-1}$$

and

$$\delta_j^h u \to \partial_j u \text{ in } H^{s+q-1}$$

$$\partial_j u = v \in H^{s+q-1},$$

j arbitrary, hence $u \in H^{s+q}$. For the general case, take $\varphi \in \mathscr{D}(B_r)$,

$$\varphi Au = \varphi f \implies g := A(\varphi u) = \varphi f + [A, \varphi]_{\text{differential operator of order } q-1} u$$

$[A, \varphi]$ differential operator of order $q - 1$.

$$\begin{aligned} u \in L^2 &\implies g \in H^{1-q} \\ &\implies \varphi u \in H^1 \\ &\implies u \in H^1_{loc} \\ &\implies ... \end{aligned}$$

□

Remark: Results does not depends on boundary conditions.

Corollary 1. $f \in C^\infty(\Omega) \implies u \in C^\infty(\Omega)$. *In most generality,*

$$f \in W^{s,p} \implies u \in W^{s+q,p}, \quad 1 < p < \infty.$$

0.2 Regularity at the Boundary

Suppose $f \in H_0^s(b_r)$, $u \in H_0^m(B_r^+)$. ($s \geq 0$ integer). We claim that $u \in H_{loc}^{s+2m}(B_\delta^+)$ for A strongly elliptic operators of order $2m$.

Proof. $\varphi \in \mathscr{D}(B_\delta)$: locally, $\varphi \partial_x^\gamma u \in H_0^m(B_\delta)$ $|\gamma| \leq m + s$. By using similar finite difference techniques in differential tangential to boundary.

$$\varphi Au = \varphi f \implies \partial_y^m \Big(a_0 \partial) y^m (\varphi u) + \sum_{\substack{|k| \leq m-1 \\ |\gamma| + k \leq m}} c_{\gamma,k} \partial_x^\gamma \partial_y^k u \Big) + Bu = \varphi f$$

B is order $2m$ operator and order m in y. By ellipticity, $a_0 \geq c > 0$, for simplicity assume $s = 0$,

$$\partial_x^\gamma g \in L^2 \quad |\gamma| \leq m.$$

$$\varphi f, \ Bu \in L^2 \implies \partial_y^m g \in L^2.$$

$$\implies \partial^\alpha g \in L62 \quad |\alpha| \leq m$$

$$a_0 \partial_y^m (\varphi u) + q(\partial^{m-1} u, ...) = g \quad \text{(boot strap)}.$$

□

Lemma 3. $u \in H_0^1(\Omega)$, $\partial\Omega \in C^1$.

$$u \in C(\overline{\Omega}) \implies u|_{\partial\Omega} = 0.$$

Proof. By coordinate transformation,

$$u \in C^1, \quad u|_{x_n=0} = 0$$

$$u(x) = \int_0^{x_n} \partial_n u \; dx_n = 0$$

$$u^2(x) = \left(\int \cdots\right)^2 \leq h \int_0^{x_n} (\partial_n u)^2 \; dx)_n \quad |x_n| < h \text{ in Q}$$

$$\int_Q u^2 \leq \underbrace{h^2}_{Q} \int_Q (\partial_n u)^2 : \quad \text{true for } u \in H_0^1(\Omega) \text{ by density argument}$$

$$\underbrace{\frac{1}{|Q|}\int_Q u^2}_{\to u^2(0) \;\; as \;\; |Q| \to 0} \leq \int_Q |\partial_n u|^2 \to 0 \; as \; |Q| \to 0$$

$$\implies u^2(0) = 0.$$

□

Corollary 2. $u \in H_0^m(\Omega),\ \partial\Omega \in C^m$

$$u \in C^{m-1}(\overline{\Omega}) \implies \partial^\alpha u|_{\partial\Omega} = 0 \quad |\alpha| \leq m-1.$$

Reason: $\partial^\alpha u \in H_0^{m-|\alpha|}(\Omega) \subset H_0^1(\Omega)$**.**