0.1 Friedrichs Extension

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let

$$A = \sum_{|\alpha| \leq 2\alpha} a_\alpha D^\alpha, \quad a_\alpha \in C^\infty(\overline{\Omega})$$

be strongly elliptic. Then,

$$A : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$$

and

$$A : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

are both continuous. For $u, v \in \mathscr{D}(\Omega)$,

$$\begin{aligned} a(u,v) &= (Au)(\overline{v}) && \text{sesquilinear} \\ &= \int Au \cdot \overline{v} && L^2\text{-inner product} \\ &= \int \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} D^\alpha u D^\beta \overline{v} && \text{via integration by parts.} \end{aligned}$$

This implies

$$|a(u,v)| \lesssim \|u\|_{H^m} \|v\|_{H^m},$$

which extends $a(\cdot,\cdot)$ to continuous sesquilinear functional : $H_o^m(\Omega) \times H_o^m(\Omega) \to \mathbb{C}$.

We have proved $a(\cdot,\cdot)$ is coercive, i.e for any $u \in \mathscr{D}$,

$$Re\ a(u,u) \geq c\|u\|_{H^m}^2 - c_1\|u\|_{L^2}^2 \qquad (c > 0)$$

which may be extended to $u \in H_o^m$ by a density argument. Given $w \in H_o^m(\Omega)$ where for simplicity we denote $H_o^m(\Omega) = X$,

$$A' : \ w \mapsto a(\cdot, w) \in X'$$
$$\widetilde{A} : \ w \mapsto [v \mapsto a(w, \overline{v})] \in X'$$

This defines $A' : X \to X'$ and $\widetilde{A} : X \to X'$ bounded and linear.

Theorem 1 (Lax-Milgram)**.** *If a is strictly coersive in X, then A' and $\widetilde{A}$ are invertible.*

We have

$$(\widetilde{A}u)(v) = a(u, \overline{v}) = (Au)(v),$$

therefore $\widetilde{A}$ extends $A : \mathscr{D} \to \mathscr{D}$ and restricts $A : \mathscr{D}' \to \mathscr{D}'$ to $H_o^m(\Omega)$. $\widetilde{A}$ is called the *energy extension* of A.
Under the assumption that a is strictly coercive,

$$Au = f$$

is uniquely solvable on $H_o^m(\Omega)$ for $f \in H_o^m(\Omega)'$.
Friedrichs' Extension: $A : L^2(\Omega) \to L^2(\Omega)$ can be defined with domain

$$Dom(A) = \{u \in L^2(\Omega) : \tilde{A}u \in L^2(\Omega)\}.$$

Under similar assumptions, there exists $A^{-1} : L^2(\Omega) \to L^2(\Omega)$ since $L^2(\Omega) \hookrightarrow H_o^m(\Omega)'$ continuously embedded: If $f' \in L^2$, define $f(v) = \int f'v$, then

$$|f(v)| = \left| \int f'v \right| \leq \|f'\|_{L^2} \|v\|_{L^2}.$$

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If A is strongly elliptic, the Garding Inequality guarantees solvability of

$$(A + tI)u = f$$

for large t. We may consider $-\Delta u + tu = f$ as an example.

0.2 Friedrichs/Poincaré Inequality

Theorem 2. *Let Ω bounded, and $s \geq 0$. Then for all $u \in \mathscr{D}(\Omega)$*

$$\|u\|_{L^2}^2 \leq c \int |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi =: c\|u\|_{\dot{H}^s}^2 =: c|u|_{H^s}^2 \quad \textit{semi-norm.} \tag{1}$$

Proof. FIG
FIG

$$\int\limits_{|\xi| \geq R} |\widehat{u}(\xi)|^2 \leq R^{-2s} \int\limits_{|\xi| \geq R} |\xi|^{2s} |\widehat{u}(\xi)|^2 \qquad \left(\frac{|\xi|^{2s}}{R^{2s}} \geq 1 \right)$$

$$E = \int\limits_{|\xi| \leq R} (1 - |\xi|^{2s}) |\widehat{u}(\xi)|^2 \leq \int\limits_{|\xi| \leq R} |\widehat{u}(\xi)|^2$$

Without loss of generality, take $R < 1$,

$$|\widehat{u}(\xi)| \leq \int |u| \leq \sqrt{Vol(\Omega)} \|u\|_{L^2} \qquad (c := \sqrt{\cdot})$$

$$\implies E \leq c\|u\|_{L^2}^2 R^n.$$

We have

$$\|u\|_{L^2}^2 \leq R^{-2s} \int\limits_{|\xi| \geq R} \cdots + \int |\xi|^{2s} |\widehat{u}|^2 + E \leq R^{-2s} |u|_{H^s}^2 + cR^n \|u\|_{L^2}^2.$$

To obtain an estimate for R,

$$\begin{aligned} R^n < \frac{1}{c} \quad &\sim \quad \frac{1}{Vol(\Omega)} \\ R \quad &\sim \quad \frac{1}{diam(\Omega)} \\ R^{-2s} \quad &\sim \quad diam(\Omega)^{2s} \end{aligned}$$

□

Corollary 1. *$A = a_{2m}(D)$ strongly elliptic, Ω bounded. Then*

$$\forall f \in L^2(\Omega) \; \exists! \; u \in H_o^m(\Omega) \; s.t \; Au = f.$$

Proof.

$$Re(Au)(u) \geq \int |\xi|^{2m}|\widehat{u}(\xi)|^2 = |u|^2_{H^m} \gtrsim \underbrace{|u|^2_{H^m} + \|u\|^2_{L^2}}_{\sim\|u\|^2_{H^m}}$$

Example: For $m = 1$:

$$\left(\int |\nabla u|^2 \geq \frac{1}{2}\int |\nabla u|^2 + c\int |u|^2\right).$$

□

0.3 Rellich-Kondrashov Theorem

Ω bounded, $s > t$, then the embedding $H^s_o(\Omega) \hookrightarrow H^t_o(\Omega)$ is compact.

Proof. ($t = 0$) Take $\{u_k\} \subset H^s_o(\Omega)$ with $\|u_k\|_{H^s} \leq 1$. We want a subsequence that converges in $H^t_o(\Omega)$. Consider a cutoff function χ in Fourier space, that is equal to 1 for $|\xi| \leq R$ and equal to 0 for $|\xi| > R$, where R is a large constant to be adjusted.
FIG
Define

$$\widehat{u}^{(1)}_k = \chi\widehat{u}_k$$
$$\widehat{u}^{(2)}_k = (1-\chi)\widehat{u}_k$$

$$\|u^{(2)}_k\|_{L^2} \leq R^{-s}\|u^{(2)}_k\|_{H^s} \leq R^{-2s}\|u_k\|_{H^s} \qquad \big(\|u_k\|_{H^s} \leq 1\big). \tag{2}$$

thus

$$\begin{aligned}|u^{(1)}_k(x)| &\lesssim \int\limits_{|\xi|\leq R} |\widehat{u}^{(1)}_k| \leq cR^{n/2}\|\widehat{u}^{(1)}_k\|_{L^2} \\ &\leq cR^{n/2}\|\widehat{u}_k\|_{L^2} \leq R^{n/2}\|\widehat{u}_k\|_{H^s} \leq cR^{n/2}.\end{aligned}$$

Moreover,

$$\begin{aligned}|u^{(1)}_k(x) - u^{(1)}_k(y)| &\leq \int\limits_{|\xi|\leq R} |e^{i\xi x} - e^{i\xi y}||\widehat{u}^{(1)}_k(\xi)|\, d\xi \\ &\leq \int\limits_{|\xi|\leq R} |1 - e^{i\xi(x-y)}||\widehat{u}^{(1)}_k(\xi)|\, d\xi \\ &\leq \sup_{|\xi|\leq R} |1 - e^{i\xi(x-y)}|cR^{n/2}\end{aligned}$$

where constant c comes from the L^1 of $\widehat{u}^{(1)}_k$. Passing to a subsequence:

$$u^{(1)}_k \to u \text{ uniformly in } \overline{\Omega}.$$

$$\implies u^{(1)}_k \text{ is Cauchy in } L^2(\Omega).$$

Define

$$u_k = u^{(1)}_k + u^{(2)}_k$$

and in (2) choose R such that $\|u_k^{(2)}\|_{L^2} \le \epsilon$. Then

$$\epsilon_0 > \epsilon_2 > \cdots \to 0$$

$$\begin{array}{ll} \text{for } \epsilon_0: & u_{0k} = u_{0k}^{(1)} + u_{0k}^{(2)} \\ \text{for } \epsilon_1: & u_{11},\, u_{12},\, u_{13}, \ldots \\ \text{for } \epsilon_2: & u_{21},\, u_{22},\, u_{23}, \ldots \end{array}$$

pick diagonal entries,

$$\Longrightarrow \; u_{kk} = u_{kk}^{(1)} + u_{kk}^{(2)}.$$

Then for $k < j$,

$$\|u_{kk} - u_{jj}\|_{L^2} \le \|u_{kk}^{(1)} - u_{jj}^{(1)}\|_{L^2} + \mathcal{O}(\epsilon_k) \to 0 \text{ as } k \to \infty.$$

□

Let A be strongly elliptic. Then for all sufficiently large t,

$$\exists\, R_t : (A + tI)^{-1} : L^2(\Omega) \to H_o^m(\Omega),$$

is bounded.

$$\underset{\text{Introduce } (+tu - tu)}{Au = f \quad \Longleftrightarrow \quad R_t A u = R_t f} \;\Longleftrightarrow\; u - tR_t u = R_t f$$

and $R_t : L^2(\Omega) \to H_o^m(\Omega) \subset] \subset L^2(\Omega)$ is compact. Hence

$$\underbrace{(I - tR_t)}_{\text{Fredholm}} u = R_t f.$$

The Riesz-Schauder theory implies

$$dim\ Ker(I - tR_t) = CoDim\ Range(I - tR_t).$$

It follows that if $Ker(A) = \{0\}$, then $A : Dom(A) \to L^2(\Omega)$ is surjective.

Moreover, consider the eigenvalue problem

$$Au - \lambda u = f \iff (I - (t + \lambda)R_t)u = R_t f.$$

If $\{\mu_k\}$ are the eigenvalues of R_t, then $\{\lambda_k\}$ eigenvalues of A, related by

$$\lambda_k = \frac{1}{\mu_k} - t.$$