0.1 Garding Inequality

We have proved that if $A = a_q(D)$ with $q = 2m$ is strongly elliptic then

$$Re \langle Au, u \rangle \geq c\|u\|_{H^m}^2 - c_1\|u\|_{L^2}^2$$

for some $c > 0$, for all $u \in \mathscr{D}(\Omega)$.

Definition 1. *Let X be a Hilbert space. $a : X \times X \to \mathbb{C}$ is **sesquilinear** form if $a(u, v) = \overline{a(v, u)}$, $a(\cdot, v) : X \to \mathbb{C}$ is linear for any fixed $v \in X$, and*

$$\|a(u, v)\|_{X^*} \lesssim \|u\|_X \|v\|_X.$$

*$a(\cdot, \cdot)$ is called **strictly coercive** (in X) if*

$$Re\ a(u, u) \geq c\|u\|_X^2 \qquad (c > 0).$$

Suppose $X \hookrightarrow Z$. Then $a(\cdot, \cdot)$ is called coercive in X with respect to Z if

$$Re\ a(u, u) \geq c\|u\|_X^2 - c_1\|u\|_Z^2 \qquad (c > 0).$$

This inequality is called Gårding's inequality.

Remark: If a is coercive in X with respect to Z, then for all sufficiently large λ,

$$a + \lambda \langle \cdot, \cdot \rangle_Z$$

is strictly coercive. Take $(\lambda = c_1)$.

Definition 2. *$H_0^m(\Omega)$ denotes the closure of $\mathscr{D}(\Omega)$ in $H^m(\mathbb{R}^n)-$norm.*

Remark: We will make it more accurate, but for now, think of $H_0^m(\Omega)$ as consisting of functions $u \in W^{m,2}(\Omega)$ with

$$u\big|_{\partial\Omega} = 0,\ \partial_n u\big|_{\partial\Omega} = 0, \dots\ \partial_n u^{m-1} u\big|_{\partial\Omega} = 0.$$

Theorem 1. *Let $A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$ uniformly strongly elliptic in $\overline{\Omega}$, $a_\alpha \in C^k(\overline{\Omega})$ with $k = \max\{|\alpha| - m, 0\}$. Then,*

$$a(u, v) = \langle Au, v \rangle$$

is coercive in $H_o^m(\Omega)$ with respect to $L^2(\Omega)$.

Remark: It is enough to prove for $u \in \mathscr{D}$,

$$Re\ a(u, v) \geq c\|u\|_{H^m}^2 - c_1\|u\|_{L^2}^2 - c_2\|u\|_{H^m}\|u\|_{H^{m-1}}.$$

This is because for any $\epsilon > 0$ there exists C_ϵ such that $ab \leq \epsilon a^2 + C_\epsilon b^2$ and the following lemma:

Lemma 1. *Suppose $s > t \geq 0$, for any $\epsilon > 0$, there exists C_ϵ such that*

$$\|u\|_{H^t} \leq \epsilon\|u\|_{H^s} + C_\epsilon\|u\|_{L^2}.$$

Proof.

$$\|u\|_{H^t}^2 = \int (1 + |\xi|^2)^t |\widehat{u}(\xi)|^2\, d\xi.$$

$$\exists C_\epsilon : \qquad (1 + |\xi|^2)^t - \epsilon(1 + |\xi|^2)^s \leq C_\epsilon \qquad (s > t).$$

□

Hence we have

$$\|u\|_{H^m}\|u\|_{H^{m-1}} \leq \delta\|u\|_{H^m}^2 + C_\delta\|u\|_{H^{m-1}}^2 \leq (\delta+\epsilon)\|u\|_{H^m}^2 + C_{\delta,\epsilon}\|u\|_{L^2}^2.$$

We now have the essential tools to prove Theorem 1.

Proof.

$$\langle Au, v\rangle = \int \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha u(x)\overline{v}(x)\ dx = \int \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)D^\alpha u(x)D^\beta\overline{v}(x)\ dx, \qquad \big(a_{\alpha\beta}\in C(\overline{\Omega})\big),$$

upon integrating by parts. Now individually,

$$\Big|\int a_{\alpha\beta}D^\alpha u D^\beta u\Big| \lesssim \|D^\alpha u\|_{L^2}\|D^\beta u\|_{L^2} \leq \|u\|_{H^{|\alpha|}}\|v\|_{H^{|\beta|}}.$$

We prove for bounded Ω. It is possible to generalize by taking countable collection of bounded sets that its union is the whole space. By boundedness, we take a finite subcover and consider, $\{\chi_k^2\}$, partition of unity (PoU). For any f, $\int f = \sum_k \int \chi_k^2 f$. Then,

$$\int \chi_k^2 a_{\alpha\beta}D^\alpha u D^\beta\overline{v} = \int \chi_k^2(a_{\alpha\beta}-a_{\alpha\beta}^{(k)})D^\alpha u D^\beta \overline{v} + \int \chi_k^2 a_{\alpha\beta}^{(k)}D^\alpha u D^\beta\overline{v}, \tag{1}$$

where we define the "frozen" coefficients $a_{\alpha\beta}^{(k)} = a_{\alpha\beta}(x_k)$, with $x_k \in supp\ \chi_k$. In other words, we approximate the variable coefficient operator by constant coefficient operators. Now, choose PoU so that $|a_{\alpha\beta}-a_{\alpha\beta}^{(k)}|\leq\epsilon$ on each $supp\ \chi_k$. We have

$$\Big|\int \chi_k^2(a_{\alpha\beta}-a_{\alpha\beta}^{(k)})D^\alpha u D^\beta\overline{v}\Big| \leq \epsilon\int \chi_k^2|D^\alpha u D^\beta \overline{v}|,$$

which bounds the first term in (1). Now consider the second term. The commutator $[\chi_k, D^\alpha]u$, the error in

$$\chi_k D^\alpha u = D^\alpha(\chi_k u) + [\chi_k, D^\alpha]u$$

is bounded by $\|u\|_{H^{m-1}}$ when integrated. Thus

$$\sum_{\alpha,\beta}\int a_{\alpha\beta}^{(k)}D^\alpha(\chi_k u)D^\beta(\chi_k\overline{v}) = \sum_{\alpha,\beta}\int a_\alpha^{(k)}D^\alpha(\chi_k u)\chi_k\overline{v}.$$

If $u=v$, we may invoke the statement we have established last time, which is also indicated above at the beginning of the lecture. We have

$$Re\sum_{\alpha,\beta}\int\cdots \geq c\|\chi_k u\|_{H^m}^2 - c_1\|\chi_k u\|_{L^2}^2.$$

If we sum over k,

$$\sum_k Re\sum_{\alpha,\beta}\int\cdots \geq \sum_k c\|\chi_k u\|_{H^m}^2 - c_1\|u\|_{L^2} \approx \sum_k\int|D^\alpha(\chi_k u)|^2+\cdots$$

$$= \sum_k\int\chi_k^2|D^\alpha u|^2 + Remainder.$$

□

0.2 Lax-Milgram

Theorem 2. *$a : X \times X \to \mathbb{C}$, strictly coercive. $f \in X^*$. Then,*

$$\exists!\ u, w \in X \ s.t \ a(v, u) = \overline{a(w, v)} = f(v), \qquad \forall v \in X.$$

Proof. By Riesz Representation Theorem,

$$\exists!\ x \in X \ s.t \ (v, x) = f(v) \qquad \forall v \in X.$$

Pick $u \in X$, then

$$a(\cdot, u) \in X^* \implies \exists!\ x \in X \ s.t \ (v, x) = a(v, u), \qquad \forall u \in X.$$

Define $B : u \mapsto x$, which is linear and bounded. We claim that B is invertible.

$$c\|u\|_X^2 \le |a(u, u)| = |(u, x)| \le \|u\|_X \|x\|_X$$

$$\implies \|u\|_X \lesssim \|Bu\|_X,$$

thus B is injective and its range is closed. Invertibility now follows from surjectivity. To show surjectivity, let $z \in X$ be orthogonal to the range of B. In other words, for all $u \in X$, $0 = (z, Bu) = a(z, u)$. Take $u = z$,

$$0 = a(z, z) \ge c\|z\|_X^2 \implies z = 0.$$

□

Corollary 1. *Suppose $\|u\|_{H^m}^2 \lesssim Re \langle Au, u \rangle$ for all $u \in \mathscr{D}(\Omega)$. $f' \in L^2(\Omega)$. Then*

$$\exists!\ u \in H_o^m(\Omega) \ s.t \ Au = f'.$$

Proof. Define $f \in H_0^m(\Omega)'$ by

$$f(v) = \int \overline{f'} \cdot v = (v, f')_{L^2}.$$

Indeed

$$|f(v)| = \Big| \int \overline{f'} \cdot v \Big| \le \|f'\|_{L^2} \|v\|_{L^2}, \qquad \text{Note : } \|v\|_{L^2} \le \|v\|_{H^m}.$$

By Lax-Milgram, there is a unique $u \in H_0^m(\Omega)$ such that

$$\overline{a(u, v)} = f(v) \iff a(u, v) = \overline{f(v)}, \qquad \forall v \in H_o^m(\Omega),$$

We have

$$\overline{f(v)} = \overline{(v, f')}_{L^2} = \int f \cdot \overline{v},$$

and

$$a(u, v) = \int \sum_{|\alpha|, |\beta| \le m} a_{\alpha\beta}(x) D^\alpha u(x) D^\beta \overline{v}(x) \ dx.$$

Note that it is straightforward to extend the proof to the more general case $f' \in H_o^m(\Omega)'$. □

If A is strongly elliptic, then there exists γ such that $\forall \lambda \in \mathbb{C}$ with Re $\lambda \ge \gamma$, $(A + \lambda I)u = f$ has a unique solution $u \in H_o^m$ for each $f \in L^2(\Omega)$.