0.1 Overview of Elliptic Theory

0.2 Short remarks on wave equations

- $$u_{tt} = \Delta u, v_0 := \partial_t u, v_k = \partial_k u$$
  $$\implies \begin{cases} \partial_t v_0 = \partial_1 + \cdots + \partial_n v_n \\ \partial_t v_k = \partial_k v_0 \end{cases}$$

  In matrix form:

  $$\partial_t \begin{pmatrix} v_0 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 & \partial_1 & \cdots & \partial_n \\ \partial_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_n & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ \vdots \\ v_n \end{pmatrix} \quad \text{symmetric hyperpolic.}$$
  $$= A_1 \partial_1 v + \cdots + A_n \partial_n v$$

- Cartoon for YM:
  $$\partial_t^2 u = \Delta u + u\partial u + u^3 \iff \Box u = u\partial u + u^3.$$

- Cartoon for WM:
  $$\Box u = u\partial u \cdot \partial u.$$

0.3 Overview of Elliptic Theory

$$A = \sum_{|\alpha| \le q} a_\alpha(x) D^\alpha, \quad \underbrace{a_q(x,\xi) = \textstyle\sum_{|\alpha|=q} a_\alpha(x)\xi^\alpha}_{\text{principal symbol}}$$

Definition 1. *A is **elliptic** at x if $a_q(x,\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$. A is elliptic in Ω if it is elliptic at all $x \in \Omega$. For systems, i.e, where $a_\alpha(x)$ are matrices, we say A is Petrowsky elliptic at x, if $a_q(x,\xi)$ has maximal rank.*

Remark: For determined systems (α_q are square), there is more general notion called Douglis-Nirenberg ellipticity.

$$\text{Example: Stokes Equation :} \qquad \begin{aligned} -\Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}$$

Remark: The definitions are in the spirit of Petrowsky parabolically and strong hyperbolicity.

Lemma 1. *Let A be elliptic, and $n \ge 3$ or $n = 2$ and a_α are real. Then q is even. Moreover, for all $\xi, \eta \in \mathbb{R}^n$ linearly independent, the equation $a_q(x, \xi + \lambda\eta) = 0$ has exactly $q/2$ roots λ with Im $\lambda > 0$.*

Proof. Consider the case where $n = 2$, a_α real. Assume q is odd. We achieve a contradiction due to the continuity of $a_\alpha(x)$.

Consider now $n \geq 3$. If q is odd, we have

$$q_q(x, \lambda\xi) = \lambda^q a_q(x, \xi).$$

$(n = 2)$ The roots of $a_q(x, \xi + \lambda\eta)$ come in conjugate pairs.
$(n \geq 3)$, $a_q(x, \xi + \lambda\eta) = 0 \implies a_q(x, -\xi - \lambda\eta) = 0$. If the trajectory crosses the real axis, that would contradict the ellipticity of the problem, i.e $a_q(x) = 0$ for $x \in \mathbb{R}$. ☐

Definition 2. *A is **properly elliptic** if $q = 2m$ and $a_q(x, \xi + \lambda\eta)$ for $\xi, \eta \in \mathbb{R}^n$ linearly independent, has exactly m roots with Im $\lambda > 0$ and m roots with Im $\lambda < 0$.*

Suppose A is properly elliptic in Ω an consider the boundary valued problem

$$(BVP) \qquad \begin{cases} Au = f & \text{in } \Omega \\ B_j u = g & \text{on } \partial\Omega, \qquad j = 1...m = q/2 \end{cases} \tag{1}$$

where $B_j = \sum_{|\alpha| \leq m_j} b_\alpha(x) D^\alpha$. From considerations of BVP's in half space, we need to to impose some conditions on B_j. Let $x^* \in \partial\Omega$, $x = (y, t)$ and let A^* be the principal part of A frozen at x^*. Similarly define B_j^* in the same way.
Ellipticity Condition for the BVP: For any $x^* \in \partial\Omega$, and $\eta \in \mathbb{R}^{n-1} \backslash \{0\}$, the problem

$$A^*(\eta, D_t) v(t) = 0$$
$$B_j^*(\eta, D_t) v(t)\big|_{t=0} = 0 \quad j = 1...m$$

admits no nontrivial solution $v_b \in C_b(\overline{\mathbb{R}}_+)$.

Other names: Lopatinsky-Shapiro, Covering-, or Complementarity Condition.

Remark: The condition does not depend on the choice of coordinates systems (y, t). Also it is easy to give algebraic characterization.

Definition 3. *(BVP) is called elliptic if A is properly elliptic and B_j satisfies the covering condition.*

"Standard elliptic theory" includes the following:

One has to choose:

- Scale X^s of functions space on Ω, and
- Scale Y^s of functions spaces on $\partial\Omega$ $(Y = Y_1^s \times \cdots \times Y_m^s)$, satisfying

$$A : X^s \to X^{s-2m} \textit{ bounded}$$
$$B_j : X^s \to Y_j^s \textit{ bounded.}$$

Then one proves (for some range of s):

- Elliptic estimates:

$$\|u\|_{X^s} \lesssim \|Au\|_{X^{s-2m}} + \|B_1\|_{Y_1^s} + \cdots + \|B_m u\|_{Y_m^s} + \|u\|_{X^0}$$

– Elliptic regularity:

$$Au \in X^{s-2m},\ B_j u \in Y^s \iff u \in X^s.$$

– Fredholm property:

$$u \mapsto (Au, \{B_j u\}): \quad X^s \to X^{s-2m} \times Y$$

is Fredholm.

i.e it has dim $Ker < \infty$, range closed, and co-dim Range $< \infty$.

Remark: If coefficients of A and B_j are not smooth, s will have limited range.

A prototypical example is Schauder's theory for second order elliptic equations in Hölder spaces.

Such a theory for very general elliptic systems in Hölder and Sobolev type spaces was established in 50'-60's. cf. Agmon-Douglis-Nirenberg 59, 64.

Tools:

- Hölder: Potential theory, singular integrals.
- L^2-based Sobolev spaces ($H^s, W^{k,2}$): Fourier transform, partition of unity.
- L^p-based Sobolev ($H^{s,p}, W^{k,p}$): Calderon-Zygmund theory, ΨDO, Littlewood-Paley theory, interpolation.

0.4 Gårding Inequality

There is a simplified and stronger version (hence with stronger hypothesis) of SET that is based on Gårding Inequality. This covers the so-called strongly elliptic systems.

$$a_q(x, \lambda\xi) = \lambda^q a_q(x, \xi). \text{ If } a_q \text{ is elliptic,}$$

$$|a_q(x, \xi)| \geq c|\xi|^q \qquad (c > 0)\ \forall \xi \in \mathbb{R}^n.$$

If a_q real: $a_q(x, \xi) \geq c|\xi|^q$.

Definition 4. *A is called strongly elliptic if*

$$Re\ a_q(x, \xi) \geq c|\xi|^q, \qquad (c > 0)$$

Definition 5. *A is* ***uniformly elliptic*** *if*

$$c|\xi|^q \leq |a_q(x, \xi)| \leq C|\xi|^q, \qquad (c > 0) \quad \forall x \in \overline{\Omega}.$$

Suppose $A = a_q(D),\ u \in \mathscr{D}$. $q = 2m$,

$$\langle Au, u \rangle = c \int a_q(\xi)|\widehat{u}(\xi)|^2\ d\xi \qquad \text{by Parseval}$$

$$Re\,\langle Au, u \rangle \geq c \int |\xi|^q |\widehat{u}(\xi)|^2\ d\xi \geq c\|u\|^2_{H^m} - c_1\|u\|^2_{L^2} \qquad (c > 0).$$