0.1 Gronwall's Inequalities

This section will complete the proof of the theorem from last lecture where we had left omitted asserting solutions agreement on intersections. For us to do this, we first need to establish a technical lemma.

Lemma 1.

a Let $y \in AC([0,T], \overline{\mathbb{R}}_+)$, $B \in C([0,T], \mathbb{R})$ *with* $y'(t) \leq B(t)A(t)$ *for almost every* $t \in [0,T]$*. Then*

$$y(t) \leq y(0) \cdot \exp\left(\int_0^t B(s)\ ds\right), \quad \forall t \in [0,T]. \tag{1}$$

b $y, B \in C([0,T], \overline{\mathbb{R}}_+)$ *with* $A \geq 0$ *and* $y(t) \leq A + \int_0^t B(s)y(s)\ ds$ *for* $t \in [0,T]$*. Then,*

$$y(t) \leq A\exp\left(\int_0^t B(s)\ ds\right), \quad \forall t \in [0,T]. \tag{2}$$

Proof. a Define

$$z(t) = y(t)\exp\left(-\int_0^t B(s)\ ds\right) \implies z \in AC$$

$$z'(t) = y'(t)\exp\left(-\int_0^t B(s)\ ds\right) - y(t)B(t)\exp\left(-\int_0^t B(s)\ ds\right) \leq 0 \ a.e.$$

$$\implies z(t) \leq z(0).$$

b

$$\frac{d}{dt}\left(A + \int_0^t B(s)y(s)\ ds\right) = B(t)y(t) \leq B(t)\left(A + \int_0^t B(s)y(s)\ ds\right)$$

$$y(t) \leq A + \int_0^t B(s)y(s)\ ds \leq A\exp\left(\int_0^t B(s)y(s)\ ds\right).$$

□

We now proceed to completing the proof from last time.

Lemma 2. *Suppose under the conditions of the last theorem,*

$$\partial_t u_1 = p(D_x)u_1 + f(u_1) \quad on \quad [0,\tau_1]$$
$$\partial_y u_2 = p(D_x)u_2 + f(u_2) \quad on \quad [0,\tau_2]$$

Then,

$$u_1 = u_2 \qquad on\ [0,\tau_1] \cap [0,\tau_2].$$

Proof. Consider the subtraction of both equations,

$$\partial_t v = \partial_t(u_1 - u_2) = p(D_x)(u_1 - u_2) + f(u_1) - f(u_2)$$

and consider the iteration

$$u_k(t) = e^{tp(D_x)}g + \int_0^t e^{t-\tau} p(D_x) f(u_k(\tau))\ d\tau.$$

We have

$$\|v\|_{H^s} \le \int_0^t \alpha(t-\tau)\|f(u_1) - f(u_2)\|_{H^s}\ d\tau$$

where f is assumed to be locally Lipchitz, so it follows that

$$\le C\int_0^t \alpha(t-\tau)\|v(\tau)\|_{H^s}\ d\tau.$$

By Gronwall's lemma, $\|v(t)\|_{H^s} = 0$ for all $t \in [0, \min\{\tau_k\}]$. □

0.2 Classical Solutions

Theorem 1. *Let $k \ge 0$ be an integer. Suppose $s > \frac{n}{2} + k$, then $H^s \hookrightarrow C^k$ continuously embedded and*

$$\|u\|_{C^k} \lesssim \|u\|_{H^s}, \qquad \forall u \in H^s. \tag{3}$$

Proof. $\big(k = 0\big)$. Suppose $u \in \mathcal{S}$, then

$$|u(x)| \le C\int |\widehat{u}(\xi)|\ d\xi = C\int |\widehat{u}(\xi)| \langle\xi\rangle^s \langle\xi\rangle^{-s}\ d\xi$$
$$\le C\|u\|_{H^s}\left(\int (1+|\xi|^2)^{-s}\ d\xi\right)^{1/2} \le CC_s\|u\|_{H^s}$$

where integrand $(1+|\xi|^2)^{-s}$ is integrable for $2s > n$. Now more generally, for $u \in H^s$, take $\{v_k\} \subset \mathcal{S}$ such that $\widehat{v}_k \to \widehat{u}$ in L^1. We use a density argument.

$$\|v_k - v)j\|_{C^0} \lesssim \|\widehat{v}_k - \widehat{v}_j\|_{L^1},$$

thus $\{v_k\}$ defines a Cauchy sequence in $C_b(\mathbb{R}^n)$; that is, there exists $v \in C_b$ such that $v_k \to v$ in C_b. For all $\varphi \in \mathcal{S}$,

$$\langle v_k - v, \widehat{\varphi}\rangle = \langle \widehat{v_k} - \widehat{u}, \varphi\rangle \to 0$$

and therefore $u = v$ as a distribution and thus $u = v$ almost everywhere; equivalently $H^s \hookrightarrow C^k$. □

Concequences:

- $u \in C^0([0,T], H^s)$ with $s > \frac{n}{2} \implies u \in C^0([0,T] \times \mathbb{R}^n)$.
- $u \in C^0([0,T], H^s)$ with $s > \frac{n}{2} + q$, where q is order of $p(D_x) \implies u \in C^0([0,T], C^q),\ u \in C^0C^q$.
- If $\partial_t u = p(D_x)u$ in the distributional sense $\implies \partial_t u \in C^0([0,T] \times \mathbb{R}^n),\ u \in C^1C^q$.

0.3 Multiplication in Sobolev Space

Theorem 2. *Suppose $s > \frac{n}{2}$, then the following estimate holds for all $u, v \in H^s$*

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{H^s} \tag{4}$$

Proof. Consider the identity

$$\langle\xi\rangle^s (\widehat{uv})(\xi) = \langle\xi\rangle^s \int \widehat{u}(\xi-\eta)\widehat{v}(\eta)\ d\eta. \tag{5}$$

Using the fact that $(a+b)^p \leq 2^q(a^p+b^p)$, where $q = \max\{0, p-1\}$, we have $\langle\xi\rangle^s \leq 2^s\langle\xi-\eta\rangle^s + 2^s\langle\eta\rangle^s$ and thus equality (5)

$$\begin{aligned}\langle\xi\rangle^s (\widehat{uv})(\xi) &\leq 2^s \int \langle\xi-\eta\rangle^s |\widehat{u}(\xi-\eta)\widehat{v}(\eta)|\ d\eta + 2^s \int |\widehat{u}(\xi-\eta)|\langle\eta\rangle^s |\widehat{v(\eta)}\ d\eta \\ &\leq 2^s|\langle\cdot\rangle^s\widehat{u}| * |\widehat{v}| + 2^s|\widehat{u}| * |\langle\cdot\rangle^s\widehat{v}|\end{aligned}$$

and thus

$$\|uv\|_{H^s} \leq \|u\|_{H^s}\|\widehat{v}\|_{L^1} + \|\widehat{u}\|_{L^1}\|v\|_{H^s}$$

where $\|\widehat{\cdot}\|_{L^1} \lesssim \|\cdot\|_{H^s}$ if $s > \frac{n}{2}$. □

An example were the theorem applied is $f(u) = u^p$. For $p \geq 0$ integers, the function f is locally Lipschitz. Results may also extend to smooth enough functions.

0.4 Derivative Nonlinearities

Consider the Navier-Stokes term $f(u) = u \cdot \nabla u$. We have

$$\underbrace{f : H^{s+1} \to H^s}_{\text{loss of regularity}} \qquad \left(s > \frac{n}{2} + 1\right) \text{ continuous.}$$

We revisit the iteration:

$$u_{k+1}(t) = e^{tp(D_x)}g + \underbrace{\int_0^t e^{(t-\tau)p(D_x)} f(u_k(\tau))\ d\tau.}_{N(u_k)}$$

What we really need is $N : H^s \to H^s$ contractive on B_R.

$$N(u) = \int_0^t e^{(t-\tau)p(D_x)} f(u(\tau))\ d\tau.$$

Assume that $f : H^s \to H^{s-\sigma}$ is locally Lipschitz, and that the operator $p(D_x)$ is Shilov h-parabolic:

$$|e^{tp(D_x)}| \leq Ce^{(c_1 - \delta|\xi|^h)t}, \qquad (\delta > 0,\ h > 0).$$

Petrowsky q-parabolic implies Shilov q-parabolic.

$$\widehat{N(u)(t)}(\xi) = \int_0^t e^{(t-\tau)p(\xi)} f(u(\tau))\ d\tau \lesssim \int_0^t \int e^{(c_1-\delta|\xi|^h)(t-\tau)} \widehat{f(u(\tau))}(\xi) d\xi d\tau \tag{6}$$

for large $|\xi|$. The propagator $e^{tp(D_x)}$ has strong smoothing effect for $t>0$, but this effect degenerates as $t\to 0$. Multiply both sides of (6) by $\langle\xi\rangle^s$ to obtain

$$\underbrace{\langle\xi\rangle^s\cdots}_{I}=\int\cdots\langle\xi\rangle^{\sigma}\,\langle\xi\rangle^{s-\sigma}\;\;d\xi d\tau.$$

The maximum of $e^{-(t-\tau)\delta|\xi|^h}|\xi|^\sigma$ is located around:

$$\frac{\partial}{\partial|\xi|}\left(e^{-(t-\tau)\delta|\xi|^h}|\xi|^\sigma\right)=\sigma|\xi|^{\sigma-1}e^{-(t-\tau)\delta|\xi|^h}-(t-\tau)\delta h|\xi|^{h-1}|\xi|^\sigma e^{-(t-\tau)\delta|\xi|^h}=0$$

$$|\xi|^h=\frac{\sigma}{\delta h(t-\tau)}\implies M\sim e^{\delta/h}\left(\frac{\sigma}{\delta h(t-\tau)}\right)^{\sigma/h}\sim(t-\tau)^{-\sigma/h}.$$

It follows that

$$I\sim\int_0^t(t-\tau)^{-\sigma/h}\quad\underbrace{\langle\xi\rangle^{s-\sigma}\,\widehat{f}}_{L^2-\text{integrable}}$$

integrability provided $\sigma<h$. We have

$$I\sim t^{1-\sigma/h}\|f\|_{H^{s-\sigma}}.$$

In the particular case of Navier-Stokes equations we have $h=2$ and $\sigma=1$.