# Lecture[1] 2

## Topological Space $X$: Basic Notions

**Lemma 1.** *A collections of subsets of $X$, $\sigma \subset 2^X$, is a* base *if and only if:*

*i) $\sigma$ is cover of $X$.*

*ii) $\forall A, B \in \sigma,\ A \cap B$ is the union of elements from $\sigma$.*

*Proof.* Define $\tau = \{union\ of\ elements\ from\ \sigma\}$. Then,

$$(\cup_\alpha A_\alpha) \cap (\cup_\beta B_\beta) = \cup_\alpha \cup_\beta (A_\alpha \cap B_\beta).$$

□

$A \in \tau \Leftrightarrow \forall x \in A,\ \exists B \in \sigma\ s.t\ x \in B \subset A.$

Let $Y \subset X$. **relative topology** $\tau_Y = \{A \cap Y : A \in \tau\}$.

$X_1, X_2$ **product topology** with base $\sigma_{X_1 \times X_2} = \{A_1 \times A_2 : A_1 \in \tau_1, A_2 \in \tau_2\}$

$A \subset X$ is **nbhd of** $x \in X$ if $B \in \tau$ such that $x \in B \subset A$.

$\mathcal{N}(x)$ the set of nbhd's of $x$.

$X$ vector space. $A \subset X$ is **balanced** if $\lambda A \subseteq A,\ \forall \lambda,\ |\lambda| \leq 1$. In other words, $x \in A \implies \lambda x \in A$.

**Theorem 1** (cf. [RUDIN] §1.14, 1.15)**.** *Suppose $X$ topological vector space, $0 < r_1, r_2 < ...,\ r_n \to \infty$. $V \in \mathcal{N}(0)$*

*a) $\bigcup_n r_n V = X$ ($V$ is absorbing).*

*b) $\exists \mathcal{U} \in \mathcal{N}(0)$ open such that $\mathcal{U} + \mathcal{U} \subset V$.*

*c) $\exists \mathcal{U} \in \mathcal{N}(0)$ open balanced such that $\mathcal{U} \subset V$.*

*d) $\exists \mathcal{U} \in \mathcal{N}(0)$ closed such that $\mathcal{U} \subset V$.*

*e) $K \in \mathcal{U}$ compact $\implies K$ bounded.*

*f) $V$ bounded $\implies \{r_n^{-1} V\}$ is local base of $X$.*

*Proof.* a) $x \in X$. Define $\lambda \mapsto \lambda x : \mathbb{R} \to X$ continuous. e

$$\implies A \subset \mathbb{R},\ open,\ A \ni 0\ s.t\ A \cdot x \subset V$$
$$\exists s > 0\ s.t\ |\lambda| < s\ \lambda x \in V \implies x \in \lambda^{-1} V,\ r_n > \lambda^{-1}$$

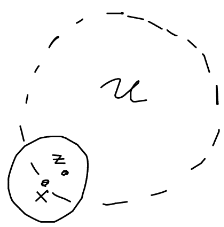
b) $+ : X \times X \to X \exists A, B \subset X$ open such that $A + B \subset V$. $\mathcal{U} = A \cap B$.

c) $\cdot : \mathbb{R} \times X \to X \exists \rho > 0,\ \exists A \in \mathcal{N}(0)\ s.t\ D_\rho \cdot A \subset V,\ D_\rho = \{|\lambda| \leq \rho\}$.

$$\implies \lambda A \subset V \implies \mathcal{U} = \bigcup_{|\lambda| \leq \rho} \lambda A \subset V.$$

d) $\mathcal{U} \in \mathcal{N}(0)$ balanced $\mathcal{U} + \mathcal{U} \subset V$. $\mathcal{U} - \mathcal{U} \subset V$. $x \in \overline{\mathcal{U}}$.

[1] Notes by Ibrahim Al Balushi




$(x+\mathcal{U})\cap\mathcal{U}\neq\emptyset$.

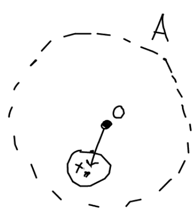
$$\exists y,z\in\mathcal{U}\ s.t\ x+y=z \implies x=z-y\in V.$$

e) $\bigcup_n r_nV\supseteq K \implies \exists m\ s.t\ K\subset\bigcup_{n=1}^m r_nV=r_mV.$

Note: bounded means $\forall\mathcal{U}\in\mathcal{N}(0)\ \forall t>0\ s.t\ K\subset t\mathcal{U}$.

f) $\mathcal{U}\in\mathcal{N}(0)$. $\exists s>0\ s.t\ V\subset s\mathcal{U} \implies V\subset r_n\mathcal{U}$ if $r_n>s$.

$$\implies r_n^{-1}V\subset\mathcal{U}$$




□

**Corollary 1.** *X topological vector space, $M\subset X$ open subspace. Then $M=X$.*

*Proof.* $M\in\mathcal{N}(0) \implies x\in X,\ \exists\lambda>0\ s.t\ x\in\lambda M=M.$ □

**Definition 1. Hausdorff** *property*

$$x,y\in X,x\neq y:\ \exists A,B\ open\ s.t\ x\in A,\ y\in B,\ A\cap B=\emptyset.$$

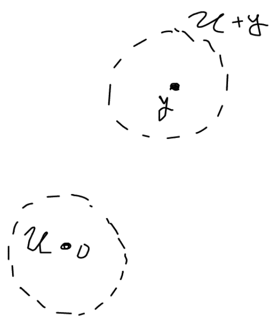
$X$ is Hausdorff TVS: $\{x\}$ is closed $(x\in X)$.

**Lemma 2.** *Suppose $\{0\}$ is closed in TVS $X$. Then $X$ is Hausdorff.*

*Proof.* $X\backslash\{y\}$ is open. $\exists V\in\mathcal{N}(0)\ s.t\ V\subset X\backslash\{y\}$. $y\notin V$. $\exists\mathcal{U}\in\mathcal{N}(0)$ balanced, such that $\mathcal{U}+\mathcal{U}\subset V$. $y\notin\mathcal{U}+\mathcal{U}\subset X\backslash\{y\}$.

$$\begin{aligned} a,b\in\mathcal{U} &\implies a+b\in X\backslash\{y\}. && (1)\\ &\implies a+b\neq y && (2)\\ &\implies a\in\mathcal{U}\neq y-b\in y-\mathcal{U}=y+\mathcal{U} && (3)\\ &\implies \mathcal{U}\cap(y+\mathcal{U})=\emptyset. && (4)\end{aligned}$$

□




**Lemma 3.** *$X$ topological space, $E$ Hausdorff TVS. $f : X \to E$ continuous. If $f = 0$ on dense subset $Y$ of $X$, then $f \equiv 0$ on $X$.*

*Proof.* $f^{-1}(\{0\})$ closed. $f^{-1}(\{0\}) \supset Y$.

$$\implies f^{-1}(\{0\}) \supset \overline{Y} = X$$

□

**Example**
Riemann integral: $I : C_o(\mathbb{R}) \to \mathbb{R}$ continuous.

$$\|u\|_{L^1} = \int |u|. \qquad L^1(\mathbb{R}) = \overline{C_o(\mathbb{R})}.$$

the extension of $I$ is Lebesgue integral.

**Definition 2.** **seminorm** $p : X \to \mathbb{R}$ *($X$ vector space)*

*i)* $p(x + y) \leq p(x) + p(y)$

*ii)* $p(\lambda x) = |\lambda| p(x)$

**norm** *:*

*iii)* $p(x) = 0 \implies x = 0.$

# References

[RUDIN] Walter Rudin, *Functional Analysis*, McGraw-Hill Inc. Second Edition (1991).