0.1 Semilinear Evolution Equations

Last Time

In the last lecture we have proved:

$$\exists! u \in C^0([0, T], H^s) \text{ satisfying }$$

$$\partial_t u = p(D_x)u + f, \quad u|_{t=0} = g$$

under the assumptions:

(i) $$|e^{tP(\xi)}| \leq \alpha(t) \forall t \geq 0, \xi \in \mathbb{R}^n$$ where $$\exists \alpha \in C(\mathbb{R})$$.

(ii) $$g \in H^s, f \in L^1([0, T], H^s)$$.

The solution is given in Fourier space by

$$\hat{u}(\xi, t) = e^{tP(\xi)}\hat{g}(\xi) + \int_0^t e^{(t-\tau)P(\xi)}\hat{f}(\xi, \tau) d\tau.$$ (1)

whereas in real space the solution is retrieved by means of the Fourier inversion

$$u(t) = e^{tP(D_x)}g(x) + \int_0^t e^{(t-\tau)P(D_x)}f(\tau) d\tau.$$ (2)

Moreover, the solution satisfies the estimate

$$\|u(t)\|_{H^s} \leq \alpha(t)\|g\|_{H^s} + \int_0^t \alpha(t-\tau)\|f(\tau)\|_{H^s} d\tau.$$ (3)

Condition (i) is satisfied for the following types of equations:

- **(Shilov) parabolic systems:** $$e^{tP(\xi)} \leq Ce^{(c_1-\delta)|\xi|^h}t$$. In particular, **(Petrowsky) q-parabolic systems**.

- **Strongly hyperbolic systems**, which includes symmetric- and **strictly hyperbolic systems**.

- **Dispersive equations** such as the **Shrödinger and Airy equations**.

0.2 Semilinear Evolution Equations (SL)

Consider the following problem

$$\begin{cases}
\partial_t u = P(D_x) + f(u) \\
u|_{t=0} = g
\end{cases}$$ (4)

where $$f : H^s \to H^s$$ is a continuous map. The followings make good examples of such systems:

- Semilinear heat equation with derivative nonlinearity:

  $$u_t = \Delta u + |\nabla u|^2,$$

  where nonlinearity $$f(u)(x) = |\nabla u(x)|^2$$ is local. We say $$f$$ is local if

  $$f(u)(x) := F(u(x), \partial u(x), ...)$$

  $$F : \mathbb{R}^N \to \mathbb{R}^m; \quad \text{with } u : \mathbb{R}^n \to \mathbb{R}^m$$
• Nonlinear Schrödinger equation: \[ u_t = i\Delta u + u^p \]

• Korteweg-de Vries equation: \[ u_t + u_{xxx} = uu_x \]

In the next section will present an important equation of this type.

### 0.3 The Navier-Stokes Equations

The Navier-Stokes equations for incompressible fluid are

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - u \cdot \nabla u - \nabla p \\
\nabla \cdot u &= 0
\end{aligned}
\]  

(divergence-free condition) \hspace{5cm} (5)

Here \( u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n \) is the velocity field and \( p : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R} \) is the pressure field. We can clearly see that this equations does not have the standard time dependant form expressed above. This is due to the last equation \( \nabla \cdot u = 0 \). However it is possible to modify the equations by projecting them onto the space \( \{ \text{div } u = 0 \} \) by means of an \( L^2 \)-orthogonal projection.

In order to define such a projector, observe that in Fourier space

\[ \nabla \cdot u = 0 \iff \xi \cdot \hat{u}(\xi) = 0 \]

which is equivalent to the no-radial component condition on \( \hat{u} \). We define \( P : S \to S \) by

\[ \hat{P}u(\xi) = \hat{u}(\xi) - \left( \hat{u}(\xi) \cdot \frac{\xi}{|\xi|} \right) \frac{\xi}{|\xi|} \]

\[ (Pu)_k(\xi) = \hat{u}_k(\xi) - \frac{\xi_k}{|\xi|^2} \hat{u}_j(\xi) \xi_j = \left( \delta_{kj} - \frac{\xi_k \xi_j}{|\xi|^2} \right) \hat{u}_j(\xi) \]

\( P : H^s \to H^s \) is bounded for all \( s \in \mathbb{R} \). Applying \( P \) to the Navier-Stokes equations we obtain:

\[ \frac{\partial}{\partial t} u = \Delta u - P(u \cdot \nabla u) - P(\nabla p) = 0 \]
where the pressure term $\nabla p$ vanishes under the projection since it is completely radial in Fourier space. Now if $\text{div } u|_{t=0} = 0$ then $\text{div } u = 0$ for all time $t > 0$ and hence
\[
\partial_t u = \Delta u - P(u \cdot \nabla u) \iff u \text{ solves Navier-Stokes Eqn’s.}
\]
Note that in this formulation we have $f(u) = P(u \cdot \nabla u)$, which is nonlocal.

### 0.4 Existence and Uniqueness of SL Equations

If $s$ is large enough, it is easy to establish local well-posedness. It is however more difficult to establish:
- Global well-posedness, or
- Local well-posedness for low values of $s$ (what low means depends on the equation).

In order to solve $SL$, define $u_0(t) = e^{tp(D_x)} g(x)$ and
\[
u_{k+1}(t) = u_0(t) + \int_0^t e^{(t-\tau)p(D_x)} f(u_k(\tau)) \, d\tau \quad (*)
\]
i.e.
\[
\partial_t u_0 = p(D_x) u_0, \quad u_0|_{t=0} = g \\
\partial_t u_{k+1} = p(D_x) u_{k+1} + f(u_k), \quad u_{k+1}|_{t=0} = g.
\]
We know $u_0 \in C^0([0,T], H^s)$, then $f(u_0) \in C^0([0,T], H^s)$. It follows from the theorem we have established previously that $u_1 \in C^0([0,T], H^s)$ and the rest follows inductively. It is left to show that the sequence defined by $(*)$ converges in $C^0([0,T], H^s)$. This will be done by using the Banach Fixed-Point Theorem. Define $X = C^0([0,T], H^s)$ and write $(*)$ as
\[
u_{k+1} = u_0 + N(u_k), \quad \alpha_T = \|u\|_{C^0([0,T]}
\]
We have
\[
\|u_0 + N(u)\|_X \leq \alpha_T \|g\|_{H^s} + T\alpha_T \|f(u)\|_X
\]
where constant $T$ is the measure of the set we integrate over to be chosen. In order to obtain a contraction, by examining the following estimate
\[
\|N(u) - N(v)\|_X \leq T\alpha_T \max_{t \in [0,T]} \|f(u(t)) - f(v(t))\|_{H^s}
\]
we conclude that $f$ need be locally Lipschitz in time. Suppose for all $R > 0$, $f$ satisfies the Lipschitz property on $B_R(0) \subset H^s$, and suppose $u, v \in B_R(0) \subset X$. We have
\[
\|u_0 + N(u)\|_X \leq \alpha_T \|g\|_{H^s} + T\alpha_T (\beta_R \|u\|_X + \|f(u)\|_X)
\]
where $\beta_R$ is the Lipschitz constant. It follows that
\[
\|N(u) - N(v)\|_X \leq T\alpha_T \beta_R \|u - v\|_X.
\]
We have
\[
\forall r > 0, \exists R > r \text{ and } \exists T > 0 \text{ s.t. } \forall q \in B_r(0) \subset H^s \text{ and } u \in B_R(0) \subset X \implies u_0 + N(u) \subset B_R(0) \subset X \text{ and } N \text{ is a contraction on } B_R(0) \subset X.
\]
Theorem 1. Assume (i) and $f : H^s \to H^s$ locally Lipchitz, then

(i) $\forall r > 0, \exists T > 0$ s.t $\forall g \in B_r(0) \subset H^s, \exists u \in C^0([0,T], H^s)$. Moreover the map $g \mapsto u : B_r(0) \to C^0([0,T], H^s)$ is Lipschitz.

(ii) For any $g \in H^s$, there is a maximal time of existence $T \in (0,T]$, and a unique solution $u \in C^0([0,T], H^s)$. Moreover if $T < \infty$ then $\|u(t)\|_{H^s} \to \infty$ as $t \uparrow T$.

Proof. Existence and uniqueness in $B_R(0)$ has been established. For uniqueness in $X$, suppose $u^* \in X$ is another solution. We have $\rho = \|u\|_X < R$. We claim that $\|u^*\|_X \leq \rho$ so $u^* = u$ by established uniqueness. Let

$$I = \{\tau \geq 0 : \|u^*\|_{C^0([0,\tau], H^s)} \leq \rho\}\]$$

$0 \in I$, $I$ is closed. Suppose $\tau \in I$, then there exists $\tau^* > \tau$ such that

$$\|u^*\|_{C^0([0,\tau^*], H^s)} \leq R$$

$$\implies u^* = u \text{ on } [0,\tau^*]$$

$$\implies \|u^*\|_{C^0([0,\tau^*], H^s)} \leq \rho$$

$$\implies I \text{ open.}$$

For the Lipschitz continuity of the solution map, let

$$u = u_0 + N(u), \ v = v_0 + N(u)$$

then

$$\|u - v\|_X \leq \|u_0 - v_0\|_X + \|N(u) - N(v)\|_X \leq \|u_0 - v_0\|_X + k\|u - v\|_X,$$

with $k < 1$.

For the proof of (ii), take

$$T = \sup\{\tau > 0 : \exists u \in C^0([0,\tau], H^s)\}.$$ 

In other words, $[0,T) = \bigcup_\tau [0,\tau)$. Consider solutions $u_1$ and $u_2$ such that

$$\partial_t u_1 = p(D_x)u_1 + f(u) \text{ on } [0,\tau_1)$$

$$\partial_t u_2 = p(D_x)u_2 + f(u_2) \text{ on } [0,\tau_2)$$

The question whether $u_1 = u_2$ on $[0,\tau_1) \cap [0,\tau_2)$ will be discussed in the next lecture. Now suppose $T < \infty$ and $\exists \{t_k\}$ such that $t_k > T$ and $\|u(t_k)\|_{H^s} \leq M < \infty$ with $M$ independent of $k$. We can extend $u$ to time $t_k + \tau$ with $\tau > 0$ independently of $k$. So one can choose $k$ so large that $t_k + \tau > T$ which contradicts the maximality of $T$. This implies $\|u(t)\|_{H^s} \to \infty$ as $t \uparrow T$. 

\[\square\]