0.1 Semilinear Evolution Equations

Last Time

In the last lecture we have proved:

$$\exists! u \in C^0([0,T], H^s) \quad \text{satisfying}$$
$$\partial_t u = p(D_x)u + f, \quad u\big|_{t=0} = g$$

under the assumptions:

(i) $|e^{tP(\xi)}| \leq \alpha(t)\ \forall t \geq 0,\ \xi \in \mathbb{R}^n$ where $\exists\ \alpha \in C(\mathbb{R})$.

(ii) $g \in H^s,\ f \in L^1([0,T], H^s)$.

The solution is given in Fourier space by

$$\widehat{u}(\xi, t) = e^{tP(\xi)}\widehat{g}(\xi) + \int_0^t e^{(t-\tau)P(\xi)}\widehat{f}(\xi,\tau)\ d\tau. \tag{1}$$

whereas in real space the solution is retrieved by means of the Fourier inversion

$$u(t) = e^{tP(D_x)}g(x) + \int_0^t e^{(t-\tau)P(D_x)}f(\tau)\ d\tau. \tag{2}$$

Moreover, the solution satisfies the estimate

$$\|u(t)\|_{H^s} \leq \alpha(t)\|g\|_{H^s} + \int_0^t \alpha(t-\tau)\|f(\tau)\|_{H^s}\ d\tau. \tag{3}$$

Condition (i) is satisfied for the following types of equations:

- **(Shilov) parabolic systems:** $e^{tP(\xi)}| \leq Ce^{(c_1 - \delta|\xi|^h)t}$. In particular, *(Petrowsky) q-parabolic systems.*
- **Strongly hyperbolic systems,** which includes *symmetric-* and *strictly hyperbolic systems.*
- **Dispersive equations** such as the *Shrödinger* and *Airy* equations.

0.2 Semilinear Evolution Equations (SL)

Consider the following problem

$$\begin{cases} \partial_t u = P(D_x) + f(u) \\ u\big|_{t=0} = g \end{cases} \tag{4}$$

where $f : H^s \to H^s$ is a continuous map. The followings make good examples of such systems:

- Semilinear heat equation with derivative nonlinearity:

$$u_t = \Delta u + |\nabla u|^2,$$

where nonlinearity $f(u)(x) = |\nabla u(x)|^2$ is local. We say f is local if

$$f(u)(x) := F(u(x), \partial u(x), ...)$$
$$F : \mathbb{R}^N \to \mathbb{R}^m; \quad \text{with } u : \mathbb{R}^n \to \mathbb{R}^m$$

- Nonlinear Schrödinger equation:
$$u_t = i\Delta u + u^p$$

- Korteweg-de Vries equation:
$$u_t + u_{xxx} = uu_x$$

In the next section will present an important equation of this type.

0.3 The Navier-Stokes Equations

The Navier-Stokes equations for incompressible fluid are

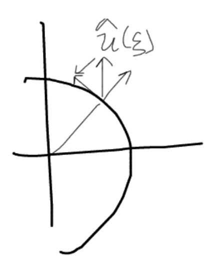
$$\begin{cases} \partial_t u = \Delta u - u \cdot \nabla u - \nabla p \\ \nabla \cdot u = 0 \qquad\qquad\qquad \text{(divergence-free condition)} \end{cases} \tag{5}$$

Here $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is the velocity field and $p : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is the pressure field. We can clearly see that this equations does not have the standard time dependant form expressed above. This is due to the last equation $\nabla \cdot u = 0$. However it is possible to modify the equations by projecting them onto the space $\{\text{div } u = 0\}$ by means of an L^2-orthogonal projection.

In order to define such a projector, observe that in Fourier space

$$\nabla \cdot u = 0 \iff \xi \cdot \widehat{u}(\xi) = 0$$

which is equivalent to the no-radial component condition on $\widehat{u}$. We define $P : \mathcal{S} \to \mathcal{S}$ by



$$\begin{aligned} \widehat{Pu}(\xi) &= \widehat{u}(\xi) - \left(\widehat{u}(\xi) \cdot \frac{\xi}{|\xi|}\right) \frac{\xi}{|\xi|} \\ (\widehat{Pu})_k(\xi) &= \widehat{u}_k(\xi) - \frac{\xi_k}{|\xi|^2} \widehat{u}_j(\xi)\xi_j \\ &= \left(\delta_{kj} - \frac{\xi_k \xi_j}{|\xi|^2}\right) \widehat{u}_j(\xi) \end{aligned}$$

$P : H^s \to H^s$ is bounded for all $s \in \mathbb{R}^n$. Applying P to the Navier-Stokes equations we obtain:

$$\partial_t u = \Delta u - P(u \cdot \nabla u) - \underbrace{P(\nabla p)}_{=0}$$

where the pressure term ∇p vanishes under the projection since it is completely radial in Fourier space. Now if $\operatorname{div} u|_{t=0} = 0$ then $\operatorname{div} u = 0$ for all time $t > 0$ and hence

$$\partial_t u = \Delta u - P(u \cdot \nabla u) \iff u \text{ solves Navier-Stokes Eqn's.}$$

Note that in this formulation we have $f(u) = P(u \cdot \nabla u)$, which is nonlocal.

0.4 Existence and Uniqueness of SL Equations

If s is large enough, it is easy to establish local well-posedness. It is however more difficult to establish:

- Global well-posedness, or
- Local well-posedness for low values of s (what low means depends on the equation).

In order to solve SL, define $u_0(t) = e^{tp(D_x)} g(x)$ and

$$u_{k+1}(t) = u_0(t) + \int_0^t e^{(t-\tau)p(D_x)} f(u_k(\tau))\, d\tau \quad (*)$$

i.e

$$\begin{aligned} \partial_t u_0 &= p(D_x) u_0, \qquad u_0\big|_{t=0} = g \\ \partial_t u_{k+1} &= p(D_x) u_{k+1} + f(u_k), \quad u_{k+1}\big|_{t=0} = g. \end{aligned}$$

We know $u_0 \in C^0([0,T], H^s)$, then $f(u_0) \in C^0([0,T], H^s)$. It follows from the theorem we have established previously that $u_1 \in C^0([0,T], H^s)$ and the rest follows inductively. It is left to show that the sequence defined by $(*)$ converges in $C^0([0,T], H^s)$. This will be done by using the Banach Fixed-Point Theorem. Define $X = C^0([0,T], H^s)$ and write $(*)$ as

$$u_{k+1} = u_0 + N(u_k), \qquad \alpha_T = \|\alpha\|_{C^0[0,T]}.$$

We have

$$\|u_0 + N(u)\|_X \le \alpha_T \|g\|_{H^s} + T\alpha_T \|f(u)\|_X$$

where constant T is the measure of the set we integrate over to be chosen. In order to obtain a contraction, by examining the following estimate

$$\|N(u) - N(v)\|_X \le T\alpha_T \max_{t \in [0,T]} \|f(u(t)) - f(v(t))\|_{H^s}$$

we conclude that f need be locally Lipschitz in time. Suppose for all $R > 0$, f satisfies the Lipschitz property on $B_R(0) \subset H^s$, and suppose $u, v \in B_R(0) \subset X$. We have

$$\|u_0 + N(u)\|_X \le \alpha_T \|g\|_{H^s} + T\alpha_T(\beta_R \|u\|_X + \|f(u)\|_X)$$

where β_R is the Lipchitz constant. It follows that

$$\|N(u) - N(v)\|_X \le T\alpha_T \beta_R \|u - v\|_X.$$

We have

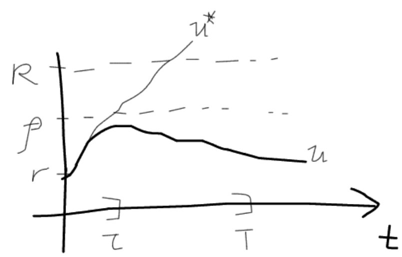
$$\begin{aligned} &\forall r > 0,\ \exists\ R > r \text{ and } \exists\ T > 0 \text{ s.t } \forall q \in B_r(0) \subset H^s \text{ and } u \in B_R(0) \subset X \\ &\implies u_0 + N(u) \subset B_R(0) \subset X \text{ and } N \text{ is a contraction on } B_R(0) \subset X. \end{aligned}$$

Theorem 1. *Assume (i) and $f : H^s \to H^s$ locally Lipchitz, then*

(i) $\forall r > 0,\ \exists\, T > 0$ *s.t* $\forall g \in B_r(0) \subset H^s,\ \exists! u \in C^0([0,T], H^s)$*. Moreover the map* $g \mapsto u : B_r(0) \to C^0([0,T], H^s)$ *is Lipschitz.*

(ii) For any $g \in H^s$*, there is a maximal time of existence* $T \in (0,T]$*, and a unique solution* $u \in C^0([0,T], H^s)$*. Moreover if* $T < \infty$ *then* $\|u(t)\|_{H^s} \to \infty$ *as* $t \uparrow T$*.*

Proof. Existence and uniqueness in $B_R(0)$ has been established. For uniqueness in X, suppose $u^* \in X$ is another solution. We have $\rho = \|u\|_X < R$. We claim that $\|u^*\|_X \leq \rho$ so $u^* = u$ by established uniqueness. Let



$$I = \{\tau \geq 0 : \|u^*\|_{C^0([0,\tau),H^s)} \leq \rho\}$$

$0 \in I,\ I$ is closed. Suppose $\tau \in I$, then there exists $\tau^* > \tau$ such that

$$\begin{aligned} &\|u^*\|_{C^0([0,\tau^*],H^s)} \leq R \\ \implies\ & u^* = u \text{ on } [0,\tau^*] \\ \implies\ & \|u^*\|_{C^0([0,\tau^*),H^s)} \leq \rho \\ & \implies I \text{ open.} \end{aligned}$$

For the Lipschitz continuity of the solution map, let

$$u = u_0 + N(u),\ \ v = v_0 + N(u)$$

then

$$\|u - v\|_X \leq \|u_0 - v_0\|_X + \|N(u) - N(v)\|_X \leq \|u_0 - v_0\|_X + k\|u - v\|_X,$$

with $k < 1$.

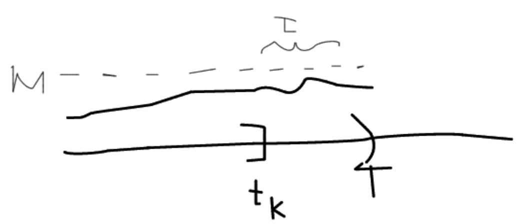
For the proof of (ii), take

$$T = \sup\{\tau > 0 : \exists \text{ sol } u \in C^0([0,\tau), H^s)\}.$$

In other words, $[0,T) = \bigcup_{\exists} [0,\tau)$. Consider solutions u_1 and u_2 such that

$$\begin{aligned} \partial_t u_1 &= p(D_x)u_1 + f(u) \quad \text{on } [0,\tau_1) \\ \partial_t u_2 &= p(D_x)u_2 + f(u_2) \quad \text{on } [0,\tau_2) \end{aligned}$$

The question whether $u_1 = u_2$ on $[0,\tau_1) \cap [0,\tau_2)$ will be discussed in the next lecture. Now suppose $T < \infty$ and $\exists \{t_k\}$ such that $t_k \nearrow T$ and $\|u(t_k)\|_{H^s} \leq M < \infty$ with M independent of k. We can extend u to time $t_k + \tau$ with $\tau > 0$ independently of k. So one can choose k so large that $t_k + \tau > T$ which contradicts the maximality of T. This implies $\|u(t)\|_{H^s} \to \infty$ as $t \nearrow T$. $\square$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