More on Solvability: Proof Revisited

Consider the Cauchy Problem:

$$\partial_t u = P(D_x)u + f \quad \text{in} \quad [0,T]\times\mathbb{R}^n, \qquad u(0) = g. \tag{1}$$

The last theorem from the last class will be proved again in more detail.

Theorem 1. *Assume* $\left|e^{tP(\xi)}\right| \le \alpha(t)$ *for all* $\xi \in \mathbb{R}^n$ $t \ge 0$, $\alpha \in C\left(\overline{\mathbb{R}}_+\right)$. *Let* $g \in H^s$, $f \in L^1([0,T], H^s)$. *Then the Cauchy Problem has a unique solution* $u \in C^0([0,T], H^s)$, *which satisfies*

$$\|u(t)\|_{H^s} \le \alpha(t)\|g\|_{H^s} + \int_0^t \alpha(t-\tau)\|f(\tau)\|_{H^s}\, d\tau.$$

Some clarification:

- $u \in L^1([0,T], H^s) \iff u(t) \in H^s$ for almost every $t \in [0,T]$ and $\int_0^T \|u(t)\|_{H^s}\, dt < \infty$.
- So $L^1_{\text{loc}}((0,T), H^s)$ can be defined.
- $u \in C^0((0,T), H^s) \iff u : (0,T) \to H^s$ is continuous.
- $u \in C^0([0,T), H^s) \iff u \in C^0((0,T), H^s)$ and $\lim_{t\searrow 0} u(t)$ exists in H^s.

Claim:

$$L^1_{loc}((0,T), H^s) \subset \mathcal{D}'((0,T)\times\mathbb{R}^n).$$

The reason for that requires some technical results. Define

$$\langle u, \varphi\rangle := \int_0^T \langle u(t), \varphi(\cdot,t)\rangle\, dt, \text{for } u \in L^1_{loc}((0,T), H^s),\ \varphi \in \mathcal{D}'((0,T)\times\mathbb{R}^n).$$

It follows from Parseval that $\langle u(t), \varphi(\cdot,t)\rangle = C\langle \widehat{u}(t), \widehat{\varphi}(t)\rangle$ for almost every $t \in (0,T)$ where $\widehat{\varphi}(t) := \widehat{\varphi}(\cdot,t)$ and $\widehat{u}(\xi,t) := \widehat{u}(t)(\xi)$. Explicitly this is equal to

$$= \int \widehat{u}(\xi,t)\langle\xi\rangle^s \langle\xi\rangle^{-s}\widehat{\varphi}(\xi,t)\, d\xi \le \|u(t)\|_{H^s}\|\varphi(t)\|_{H^{-s}}. \tag{2}$$

by Cauchy-Schwartz. Recalling that under the Fourier transform

$$\widehat{\partial^\alpha\phi}(\xi) = (i\xi)^\alpha\widehat{\phi}(\xi) \implies |\widehat{\partial^\alpha\phi}(\xi)| = |\xi_1|^{\alpha_1}\cdots|\xi_n|^{\alpha_n}\cdot|\widehat{\phi}(\xi)|.$$

Moreover,

$$\|\phi\|^2_{H^1} = \int (1+|\xi|^2)|\widehat{\phi}(\xi)|^2\, d\xi = \int |\widehat{\phi}|^2 + |\widehat{\partial_1\phi}|^2 + \cdots + |\widehat{\partial_n\phi}|^2,$$

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$$= \int |\phi|^2 + |\partial_1\phi|^2 + \cdots + |\partial_n\phi|^2 = \int |\phi|^2 + |\nabla\phi|^2. \tag{3}$$

More generally, using elementary inequalities such as $2ab \le a^2 + b^2$ we have

$$\|\phi\|^2_{H^k} = \int (1+|\xi|^2)^k|\widehat{\phi}(\xi)|^2\, d\xi \sim \int \sum_{|\alpha|\le k} |\xi_1|^{2\alpha_1}\cdots|\xi_n|^{2\alpha_n}|\widehat{\phi}(\xi)|^2\, d\xi$$

$$= \sum_{|\alpha|\le k} \|\partial^\alpha \phi\|_{L^2}^2 \sim \|\phi\|_{L^2}^2 + \sum_{|\alpha|\le k} \|\partial^\alpha \phi\|_{L^2}^2.$$

$$\implies \|\varphi\|_{H^\sigma}^2 \lesssim \sum_{|\alpha|\le k} \|\partial^\alpha \varphi\|_{L^2}^2 \le C\|\varphi\|_{C^k}^2, C = C(vol(supp\ \varphi)).$$

- $u \in L^1_{loc}(0,T;H^s)$: Taking Fourier transform (on spacial coordinates)

$$\begin{aligned}\left\langle \widehat{\partial_t u}, \psi \right\rangle &= \langle \partial_t u, \psi\rangle = -\langle u, \partial_t \psi\rangle \\ &= -\int_0^T \left\langle u(t), \partial_t \widehat{\psi}(t) \right\rangle \, dt \\ &= -\int_0^T \left\langle u(t), \widehat{\partial_t \psi}(t) \right\rangle \, dt \\ &= -\int \langle \widehat{u}(t), \partial_t \psi(t)\rangle \, dt \\ &= -\langle \widehat{u}, \partial_t \psi\rangle = \langle \partial_t \widehat{u}, \psi\rangle \,.\end{aligned}$$

We may now proceed in proving the Theorem.

Proof. To show uniqueness suppose $u \in C^0([0,T), H^s)$ solves the Cauchy problem. Then

$$\begin{cases} \partial_t \widehat{u}(\xi,t) = P(\xi)\widehat{u}(\xi,t) + \widehat{f}(\xi,t) & \text{in the distributional sense} \\ \widehat{u}(\cdot,t) \to \widehat{g} & \text{as } t \searrow 0 \text{ in } H^s. \end{cases} \tag{4}$$

$\langle\, \cdot\, \rangle^s \widehat{u}(\cdot,t) \to \langle\, \cdot\, \rangle^s \widehat{g}$ in L^2 and uniqueness follows. To establish existence, recall that

$$\widehat{u}(\xi,t) = \underbrace{e^{tP(\xi)}\widehat{g}(\xi)}_{\widehat{u}_0(\xi,t)} + \underbrace{\int_0^t e^{t-\tau} P(\xi)\widehat{f}(\xi,t)\, d\tau}_{\widehat{u}_1(\xi,\tau)}$$

solves (4) for almost every $\xi \in \mathbb{R}^n$. Given

$$|\widehat{u}(\xi,t)| \le \alpha(t)|\widehat{g}(\xi)| \implies \|u_0(t)\|_{H^s} \le \alpha(t)\|g\|_{H^s}.$$

The continuity property follows from $|\widehat{u}_0(\xi,\tau) - \widehat{u}(\xi,t)| \to 0$ as $|\tau - t| \to 0$ for almost every ξ so by Lebesgue's Dominated Convergence Thoerem we have

$$\int \langle \xi\rangle^{2s} |\widehat{u_0}(\xi,t) - \widehat{u}_0(\xi,\tau)|^2 \, d\xi \to 0 \text{ as } \tau \to t.$$

It follows that $u_0 \in C^0([0,T), H^s)$ and $u_0(0) = g$. We can show $u_1 \in C^0([0,T), H^s)$ and $u_1(0) = 0$. ☐

Lemma 1. *Let* $\max_k Re\ \lambda_k \le -\delta$ *for some positive* δ *and* λ_k *are eigenvalues of* $P_q(\eta)$, $\eta \in S^{n-1}$. *Then*

$$\left| e^{tP(\xi)} \right| \le Ce^{(c_1 - \delta_1 |\xi|^q)t}, \qquad \text{for some } C, c_1, \delta_1 > 0.$$

Proof. Schur tells us that one may decompose any matrix P_q into

$$SP_qS^{-1} = \underbrace{\begin{pmatrix} \lambda_1 & & b_{ik} \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}}_{=\Lambda+B}$$

with $b_{ik} \leq \epsilon$ for any given $\epsilon > 0$ independently of $\eta \in S^{n-1}$. Define $H = S^*S$. We aim to invoke a previous lemma in the previous lecture by establishing $HP_q + P_q^*H \leq 2\alpha H$.

$$\begin{aligned} HP_q + P_q^*H &= S^*SS^{-1}(\Lambda + B)S + S^*(\Lambda^* + B^*)S^{-*}S^*S \\ &= S^*(\underbrace{2\text{Re }\Lambda}_{<-2\delta} + \underbrace{B + B^*}_{\epsilon \ small})S \leq -\delta I. \end{aligned}$$

Given any y for $Q = Q(\xi)$

$$y^*(HQ + Q^*H)y \leq C(\|Q\| + \|Q^*\|)y^*y \leq (C + c_1|\xi|^{q-1})y^*y$$

which follows from decomposing $Q = Q_{q-1} + Q_{q-2} + \cdots$ into homogeneous parts. $P_q = P_q(\xi)$: $HP_q + P_q^*H \leq -\delta|\xi|^q I$

$$HP + P^*H \leq (-\delta|\xi|^q + C + c_1|\xi|^{q-1})I \leq (C - \delta'|\xi|^q)I.$$

□

General Picture: