Solvability of Hyperbolic and Parabolic PDEs

Last time: Given a system of PDEs of the form

$$\partial_t u = P(D_x)u, \qquad P = P_q + Q$$

we have introduced the following concepts.

- **Strong Well-Possedness**:
$$\|u(t)\|_{L^2} \le Ce^{\alpha t}\|u(0)\|_{L^2}.$$
- **Strong Hyperbolicity**:
$$q = 1, \left|e^{P_1(\xi)}\right| \le C, \forall \xi \in \mathbb{R}^n.$$
- **q-Parabolicity**:
$$\text{Re } \sigma\left[P_q(\xi)\right] \subset (-\infty, \delta], \forall \xi \in S^{n-1},\ \delta < 0.$$

We restate the theorem we concluded last lecture with:

Theorem 1. *The Cauchy problem for $\partial_t u = P(D_x)u$ is* ***Strongly Well-posed*** *for arbitrary Q if and only if either*

- *$q = 1$ and the system is strongly hyperbolic, or*
- *q is even and the system is q-parabolic.*

We have proved last time the forward implication. For us to prove the converse, we will require some additional tools and some technical results. The remainder of this lecture will be dedicated for that purpose.

Remark: It is possible for P_q to be neither strongly hyperbolic nor parabolic and yet still possess Strong Well-Possedness under some special perturbations Q. For example the Shrödinger equation $\partial_t u = \pm i\Delta u$, which is given by

$$P(\xi) = \mp i\xi^2,$$

is Strongly Well-posed. Clearly, this equation is neither Stongly Hyperbolic since $q \neq 1$ nor q-parabolic since we require Re $\lambda(\xi) \le -\delta|\xi|^q$, and the eigenvalues of $P(\xi)$ remain on the imaginary axis.

Definition 1. *Let A and B be square Hermitian matrices. The notation*

$$A \le B \quad \textit{means that} \quad y^*Ay \le y^*y$$

for any vector y.

Lemma 1 (KL 2.1.4)**.** *Suppose there exists a square matrix H satisfying:*

$$c^{-1}I \le H = H^* \le cI, \quad (c > 0)$$
$$\textit{and} \quad HA + A^*H = 2\alpha H, \quad (\alpha \in \mathbb{R}),$$

then

$$|e^{tA}| \le ce^{\alpha t}, \qquad (t \ge 0).$$

Proof. The system of ODEs $\dot{y} = Ay$ has the solution $y(t) = e^{tA}y(0)$. It follows that

$$\begin{aligned} \frac{d}{dt} y^* H y &= (\dot{y})^* H y + y^* H \dot{y} \\ &= y^* A^* H y + y^* H A y \\ &\leq 2\alpha y^* H y \end{aligned}$$

$$\implies \underbrace{(y^* H y)}_{c^{-1} y^* y \leq}(t) \leq e^{2\alpha t} \underbrace{(y^* H y)(0)}_{\leq c (y^* y)(0)} \implies c^{-1} \left| e^{tA} y(0) \right|^2 \leq c e^{2\alpha y} |y(0)|^2.$$

□

Theorem 2. *The followings are equivalent:*

(i) P_1 *is strongly hyperbolic.*

(ii) $\forall \xi \in \mathbb{R}^n$, *the eigenvalues of* $P_1(\xi)$ *are purely imaginary, semi-simple and the spectral projectors of* $P_1(\xi)$ *are uniformly bounded.*

(iii) $\forall \xi \in \mathbb{R}^n$, *there exists a square matrix* H *such that*

$$\begin{aligned} & c^{-1} I \leq H = H^* \leq cI, \quad c > 0 \text{ independent of } \xi \\ \text{and} \quad & H P_1(\xi) + P_1(\xi)^* H \leq 0. \end{aligned}$$

Proof. $(iii) \implies (i)$. Apply the previous lemma to obtain the necessary bounds. As for $(i) \implies (ii)$ we define the *spectral projectors* by the matrices Π_i derived as such if $P_1 = S^{-1} D S$,

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \ddots \end{pmatrix} = \lambda_1 \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & O \end{pmatrix}}_{= S \Pi_1 S^{-1}} + \lambda_2 \underbrace{\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & O \end{pmatrix}}_{= S \Pi_2 S^{-1}} + \cdots$$

$$\implies P_1 = \lambda_1 \Pi_1 + \lambda_2 \Pi_2 + \cdots$$

Π_k is the projection onto the Kernel$(P_1 - \lambda_k I)$ along the Range$(P_1 - \lambda_k I)$. We may write $e^{tP_1} = \sum_k e^{t\lambda_k} \Pi_k$. In order to find a bound, we make use of the diagonal decomposition of P_1 so that

$$e^{-t\lambda_l} e^{tP_1} = \sum_k e^{t(\lambda_k - \lambda_l)} \Pi_k, \qquad \lambda_k - \lambda_l = \begin{cases} = 0 & \text{if } k = l \\ \in i\mathbb{R} \backslash \{0\} & \text{if } k \neq l \end{cases}$$

which implies an oscillatory behaviour for $k \neq l$. We integrate

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-t\lambda_l} e^{tP_1} \, dt = \Pi_l + \underbrace{\lim_{T \to \infty} \frac{1}{2T} O(1)}_{\to 0} = \Pi_1,$$

$$\implies |\Pi_1 y| \leq |e^{tP_1 y}| \leq C|y|.$$

Conversely, $(ii) \implies (i)$, $|e^{tP_1}| \leq C|\Pi_k|$.

For $(ii) \implies (iii)$, Take $H = S^*S$. Then for any y

$$y^*Hy = \dot{y}^*S^*SSy = |Sy|^2 \leq |S|^2 y$$

$$HP_1 + P_1^*H = S^*SS^{-1}DS + S^*DS^{-*}S^*S = S^*\underbrace{(D + D^*)}_{=0}S \leq 0.$$

□

Corollary 1. *If P_1 is strongly hyperbolic then there exists α and C such that*

$$\left|e^{(P_1(\xi)+Q)t}\right| \leq Ce^{\alpha t}.$$

Proof. Let $A = P_1(\xi) + Q$ and H from (iii) in the theorem above.

$$HA + A^*H = \underbrace{HP_1 + P_1^*H}_{=0} + HQ + Q^*H \leq \|H\|\left(\|Q\| + \|Q^*\|\right)I \leq \alpha H.$$

□

Examples:

- **Strict hyperbolicity:** $\forall\ \xi \in \mathbb{R}\backslash\{0\}$, $P_1(\xi)$ has distinct and imaginary eigenvalues.
- **Symmetric hyperbolicity**: $P_1(\xi) = -P_1(\xi)^*$ for all $\xi \in \mathbb{R}^n$. Examples like Maxwell's equations and linear elasticity.

$$\begin{aligned} \text{Suppose} \qquad & A_1\partial_1 + A_2\partial_2 + \cdots + A_n\partial_n, \\ \text{then} \quad & P_1(\xi) = i\xi_1 A_1 + i\xi_2 A_2 + \cdots + i\xi_n A_n. \end{aligned}$$

- **Symmetrizable**: $\exists\ S$ such that $SP_1(\xi) = (SP_1(\xi))^*$.
- Further examples: smoothly diagnolizable, systems with constant multiplicities.

Consider the problem:

$$\begin{cases} \partial_t u = P(D_x)u + f & \text{in } [0,T] \times \mathbb{R}^n \\ u(0) = g \end{cases} \tag{1}$$

Theorem 3. *Assuming $\left|e^{tP(\xi)}\right| \leq \alpha(t)$, for all $t \geq 0$, $\xi \in \mathbb{R}^n$ with $\alpha \in C\left(\overline{\mathbb{R}_+}\right)$. Let $g \in H^s$, $f \in L^1\left([0,T], H^s\right)$. Then the problem above (1) has a unique solution $u \in C^0\left([0,T], H^s\right)$ satisfying*

$$\|u(t)\|_{H^s} \leq \alpha(t)\|g\|_{H^s} + \int_0^t \alpha(t-\tau)\|f(\tau)\|_{H^s}\ d\tau.$$

Proof. Taking the Fourier transform of (1) yields the system of ODEs with solution

$$\widehat{u}(\xi,t) = \underbrace{e^{tP(\xi)}\widehat{g}(\xi)}_{=:\widehat{u}_0(\xi,t)} + \underbrace{\int_0^t e^{(t-\tau)P(\xi)}\widehat{f}(\xi,\tau)\ d\tau}_{=:\widehat{u}_1(\xi,t)}.$$

$$\tau \to t \ : \qquad \widehat{u}_0(\xi,\tau) \to \widehat{u}_0(\xi,t),\ a.e\ \xi.$$
$$\int \langle\xi\rangle^{2s} \left|\widehat{u}_0(\xi,t)\right|^2 \ d\xi \leq \alpha(t)^2 \int \langle\xi\rangle^{2s} \left|\widehat{g}(\xi)\right|^2 \ d\xi$$

and so by Lebesgues Dominated Convergence Theorem $u_0 \in C^0\left(\overline{\mathbb{R}_+}, H^s\right)$. □