0.1 Hyperbolicity

In 1951 Gårding proved the following theorem:

Theorem 1 (Gårding Hyperbolicity)**.** *The Cauchy problem for $P(D_x, \partial_t)u = 0$ is well posed in C^∞ if and only if P is Petrovsky well-posed, and $\{t = 0\}$ is noncharacteristic, i.e $P_m(0, 1) \neq 0$ where P_m is the principal part.*

In order to motivate the following theory we first consider this example. Let $a(x)$ be a variable coefficient for the PDE

$$\begin{aligned} u_t &= a(x)u_x \\ u_{xt} &= a_x u_x + a u_{xx} \quad \text{after taking } \partial_x \\ v_t &= a v_x + a_x v \quad \text{setting } v = u_x \end{aligned}$$

v_t the variation of v in time while $a_x v$ are lower order perturbations. Now consider

$$\underbrace{u_t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u_x}_{\text{principal part}} + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} u.$$

With the theory we considered one concludes that $\|u(\cdot, t)\|_{L^2} \leq C(t)\|u(\cdot, 0)\|_{H^1}$ for the principal part. Meanwhile for the whole problem

$$P(\xi) = \begin{pmatrix} i\xi - 1 & i\xi + 1 \\ -1 & i\xi - 1 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_{1,2} = \underbrace{i\xi - 1}_{\text{Re}=1} \pm \underbrace{\sqrt{-(1 + i\xi)}}_{\text{Re}\approx\sqrt{\xi}}.$$

It follows that PWP is not stable under lower order perturbations. In that light, we aim to have a theory depending only on the principal part. Also, heuristically, high frequency components of the solution are controlled by the principal part.

Lemma 1. *If P is Gårding hyperbolic then the roots λ of $P_m(\xi, \lambda) = 0$ satisfies Re $\lambda \leq 0$ for all ξ.*

Lemma 2. *If P_m is Gårding hyperbolic then the roots of P_m are purely imaginary.*

Definition 1.

- *If P_m has the property that all its roots are imaginary, then we say P is* ***weakly hyperbolic****.*
- *If P is hyperbolic and the roots of its principal part P_m are distinct, then P is called* ***strictly hyperbolic****.*

Example: The wave equation

$$\begin{aligned} \text{From } P(D_x, \partial_t) &= \partial_t^2 + D_1^2 + \cdots + D_n^2. \\ P(\xi, \lambda) &= \lambda^2 + \xi_1^2 + \cdots + \xi_n^2 \\ \text{with roots } \lambda_{1,2}(\xi) &= \pm i|\xi|, \quad \text{hence strictly hyperbolic.} \end{aligned}$$

Lemma 3. *P_m is Petrovsky well posed under arbitrary lower order perturbations if and only if P_m is strictly hyperbolic.*

Example: Consider the nonlinear PDE

$$\partial_t^2 u = u\Delta u + u^3,$$

where u is supposed to be positive. Pick some $u_0 \in \mathbb{R}^n \times [0, T]$, and define recursively

$$\partial_t^2 u_{k+1} = u_k \Delta u_{k+1} + u_k^3.$$

Note that a bad example of such an iteration would be $\partial_t^2 u_{k+1} = u_k \Delta u_k + u_k^3$ for it loses regularity. The question is if $u_k \to u$ for some functions u, and if such u would be a solution of the original nonlinear problem. Typically, we have the following estimate

$$\|u_{k+1}\|_{H^s} \le C(u_k)\|u_k^3\|_{H^{s'}}.$$

We want $s \ge s'$, that is, we do not want to loose regularity.

0.2 Strong Hyperbolicity and Parabolicity

Definition 2. *A Cauchy problem is called* ***Strongly Well Posed*** *if it is uniquely solvable for all initial data in L^2, in the class of functions satisfying the estimate*

$$\|u(\cdot, t)\|_{L^2} \le Ce^{\alpha t}, \tag{1}$$

for some α and C.

Consider the system $\partial_t u = P(D_x)u$ with principal part P_q, $P = P_q + Q$. Suppose P_q is fixed and suppose the Cauchy problem for P with arbitrary Q is Strongly Well Posed. Then $q = 1$, or q even.

Proof.

- If q is odd then $P_q(\xi, \lambda) = 0 \implies P_q(t\xi, t\lambda) = 0$ and therefore Re $\lambda = 0$.
- If $q \ge 3$ and odd then take $P(\xi) = P_q(\xi) + \xi_1^2$. It follows that Re $\lambda(\xi) = \xi_1^2$ and therefore unbounded.
- Now suppose q is even. Then with the arrangement $\text{Re}\lambda_1 \le \cdots \le \text{Re}\lambda_m$, the functions $\lambda_k : S^{n-1} \to \mathbb{C}$ are continuous. Suppose Re $\lambda_m(\eta) \ge 0$ for some $\eta \in S^{n-1}$, and take

$$\begin{aligned} P(\xi) &= P_q(\xi) + (\eta \cdot \xi)I \\ P(t\eta) &= |t|^q P_q(\eta) + tI \\ &\implies \text{Re } \lambda_m[P(t\eta)] \ge t \\ \text{so Re } \sigma(P_q(\eta)) &\subset (-\infty, -\delta],\ \delta > 0,\ \eta \in S^{n-1} \\ &\implies \text{Re } \lambda_m(\xi) \le -\delta|\xi|^q. \end{aligned}$$

□

Definition 3. *The system $\partial_t u = P(D_x)u$ is called q-parabolic if Re $\sigma[P_q(\xi)] \subset (-\infty, -\delta|\xi|^q]$ for some $\delta > 0$, for all ξ.*

Now let $q = 1$. We know Re $\lambda = 0$.

$$\|e^{P_1(\xi)t}\| \leq Ce^{\alpha t}$$

not depending on ξ. Take $\xi \mapsto \xi/a$ and let $t \mapsto at$. Then

$$\|e^{P_1(\xi/a)at}\| \leq Ce^{\alpha at}$$

fix t and send $a \to 0$

$$\|e^{P_1(\xi)t}\| \leq Ce^{\alpha at} \implies \|e^{P_1(\xi)}\| \leq C, \ \forall \xi \in \mathbb{R}^n.$$

Definition 4. *$\partial_t u = (P_1(\xi) + Q)u$ is called* ***Strongly Hyperbolic*** *is there exists $C > 0$ such that*

$$\|e^{P_1(\xi)}\| \leq C < \infty, \qquad \forall \xi.$$

Theorem 2. *Consider the system $\partial_t u = P(D_x)u$ with principal part P_q, $P = P_q + Q$. With P_q fixed, the Cauchy problem for P with arbitrary Q is Strongly Well Posed if and only if either*

- *$q = 1$ and P_q is strongly hyperbolic, or*
- *q is even and P_q is q-parabolic.*